Johnson Noise and Shot Noise

Dennis V. Perepelitsa*
MIT Department of Physics
(Dated: November 27, 2006)

Two types of electrical noise fundamental to any circuit and their relation to fundamental constants are investigated. Johnson noise of a controlled-gain system is measured across different resistances and temperatures, leading to a calculation of the Boltzmann constant $k = (1.48 \pm .07) \times 10^{-23} J/K$ and absolute zero $T_0 = -(270 \pm 30)\ ^\circ C$. The shot noise of a variable-current system is measured, leading to a calculation of the electron charge at $e = (1.71 \pm .14) \times 10^{-19}\ C$. Sources of error are discussed.

1. INTRODUCTION

Though the fundamental "noisiness" of electrical conductors had been known for some time, it was not until 1918 that German physicist Walter Schottky identified and formulated a theory of "tube noise" - a fluctuation in the current caused by the granularity of the discrete charges composing it. Ten years later, Johnson and Nyquist similarly analyzed a different type of noise - one caused by the thermal fluctuations of stationary charge carriers.

These are now known as "shot noise" and "Johnson noise", respectively, and it is a startling fact that neither depends on the material or configuration of the electrical circuit in which they are observed. Instead, the expressions governing them are relatively simple, and depend on several fundamental constants. A straightforward measurement of the two types of background noise yields an experimental value for these constants. It is our aim to measure them.

2. THEORY

We present summaries of the theories of Johnson noise and shot noise.

2.1. Johnson Noise and Nyquist's Theorem

The thermal agitation of the charge carriers in any circuit causes a small, yet detectable, current to flow. J.B. Johnson was the first to present a quantitative analysis of this phenomenon, which is unaffected by the geometry and material of the circuit.

H. Nyquist showed in his landmark 1928 paper [1] that the problem of determining the amplitude of the noise was equivalent to summing the energy in the normal modes of electrical oscillation along a shorted transmission line connected to two resistors of resistance R.

By the equipartition law of thermodynamics, every mode of oscillation contributes approximately kT average energy to the oscillation. If we consider an area of the frequency domain such that the normal modes with corresponding frequencies in that range are very close, we can treat the domain as continuous, such that each frequency band df contributes $dP = kTdf$ power.

Evidently, the rms voltage evolved in the circuit is given by $dV^2 = dI^2(2R)^2 df = 4R^2 dP/R = 4RkTdf$. However, if the circuit has any capacitance in series with the resistance, we must replace the resistance with the circuit's characteristic impedance $Z = \frac{1}{1/R+iwC}$, such that $V^2 = I^2 Z^2$, where $w = 2\pi f$ is the angular frequency of the circuit.

We now add a gain term $g(f)$ to account for any amplification or attenuation that our system might have on this frequency. After algebraic manipulation and integration over the frequency spectrum, we arrive at

$$V^2 = 4RkT \int_0^\infty \frac{[g(f)]^2}{1+(2\pi fCR)^2} df \qquad (1)$$

2.2. Shot Noise

The quantization of charge carried by electrons in a circuit also contributes to a small amount of noise. Consider a photoelectric circuit in which current caused by the photoexcitation of electrons flow to the anode. Over a relatively long time T, an average current I_{avg} is observed.

Consider the quantized contribution to the current by any given electron. Its current pulse $I_n(t)$ can be approximated as a delta function centered at some time t_n: $I_n(t) = e\delta(t - t_n)$, where the total current over time is the sum of all the I_n. The Fourier decomposition of this impulse over the time domain $[0, T]$ is given by

$$I_n(t) = \frac{e}{T} + \frac{2e}{T} \sum_m \cos \frac{2\pi m(t - t_n)}{T} \qquad (2)$$

Where the terms on the right are the fluctuating components. Now consider the rms of the fluctuating current

*Electronic address: dvp@mit.edu

$\langle I^2 \rangle$. The contribution to it from any given normal mode is $\langle (\frac{2e}{T})^2 \rangle \langle \cos^2(2\pi m(t - t_n)/T) \rangle$.

Since the phases t_n can be anywhere from 0 to T, we average the cosine squared term over all possible values, which has a value of one-half. Thus, for a given fluctuating component with frequency m, $d\langle I^2 \rangle_m = (2e^2/T^2)$.

To derive the total rms shot noise current, we simply sum over all frequencies and electrons. $(I_{avg}/e)(T)$ have hit the anode over time T, and the relation between the index into the normal modes and frequency is $f = m/T$. Thus, $d\langle I^2 \rangle = 2eI_{avg}df$.

In our experimental setup, we observe the voltage $V = IR_F$ evolved over some resistance R_F instead of the current, and keep in mind that there is some gain $g^2(f)$ imposed by the circuit on any measurement of V^2. Integrating over the frequency spectrum, we arrive at

$$V_0^2 = 2eI_{av}R_F^2 \int_0 [g(f)]^2 \, df + V_A^2 \quad (3)$$

The V_A^2 term represents sources of background noise such as Johnson noise that are constant when the resistance and temperature are constant, as they are in our experimental setup. We see that unlike Johnson noise, shot noise is independent of temperature and resistance, but is only present when a current is flowing.

The derivation of the shot noise expression above is inspired by the method of Seth Dorfman [2].

3. EXPERIMENTAL SETUP

The Johnson noise experimental setup consisted of a resistor-holding assembly connected to both a multimeter and a differential amplifier in series with a band-pass filter and digital oscilloscope, as shown in Figure 1. Either connection of the assembly could be shorted through switches. The DC-coupled amplifier was set to a thousand-fold gain, and the band-pass filter strongly attenuated signals outside the frequency range of $1 - 50$ kHz[4].

The shot noise experimental setup consisted of a variable-intensity light-bulb and photodiode connected to a simple circuit. As can be seen in Figure 2, the multimeter measures the voltage across a resistor R_F in series with the photocircuit. A switch selects whether the photocurrent or the calibration signal is connected to the rest of the circuit. The op-amp on the right in parallel with the small resistor causes the shot noise assembly box to have an approximately tenfold gain. The DC-coupled (since we are only interested in the fluctuating AC components) amplifier was set to a hundredfold gain.

In both cases, the oscilloscope was used to digitally integrate the resultant noise signal. A sinusoidal function generator in series with an attenuator was used to calibrate the gain curve in both experiments.

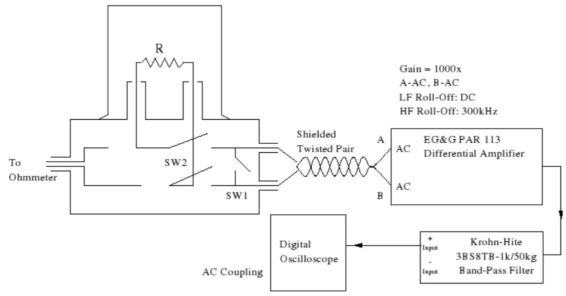

FIG. 1: Johnson noise setup.

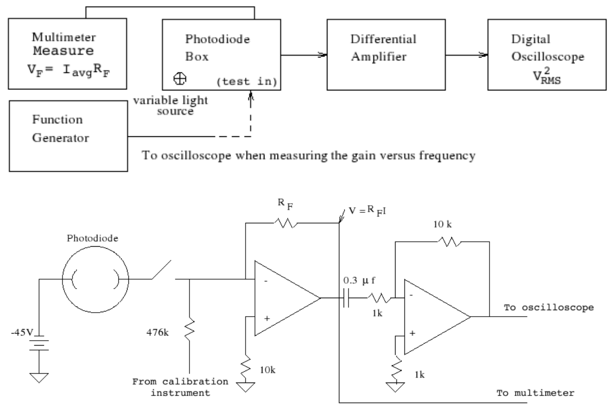

FIG. 2: Shot noise setup with circuit diagram of photoassembly.

3.1. Experimental Procedure

We began both the Johnson noise and Shot noise experiments with a careful calibration of the measurement chain with an attenuated sinusoidal signal from a function generator. We recorded the gain $g(f) = V_{out}(f)/V_{in}(f)$ and the uncertainty therein at as many frequencies as were necessary to model it effectively.

During the Johnson noise experiment, we measured the voltage V of the noise as calculated by the oscilloscope twenty times for ten different values of the resistance, ranging from 5 kΩ to 1 MΩ. We then used the same 500 kΩ resistor to measure the rms voltage across different temperatures by submersing the assembly in liquid nitrogen ($T = 77$ K) and a heated air bath ($T < 100°$ C), measuring the temperature with a mercury thermometer. After each measurement, we shorted the circuit and took a reading of the background noise.

During the shot noise experiment, we recorded the rms voltage V of the noise as calculated by the oscilloscope twenty times for eight different voltages in the light photocircuit V_F. We then broke the photocircuit and recorded the background noise level.

In both cases, the measured noise changed slightly based on the integration time used by the oscilloscope, but this uncertainty ranged on the order of 0.1%, and we ignore it, since it is dwarfed by the uncertainty caused by random fluctuations in V.

4. DATA AND ERROR ANALYSIS

4.1. Calculation of Gain

We are faced with the task of calculating the two integrals in 1) and (3). Our information about either consists of a set of points $g(f)$ for a variety of frequencies. The error in $g^2(f)$, which will be needed later, is given by $\left(\frac{\sigma_{g^2}}{g^2}\right)^2 = \left(2\frac{\sigma_{V_{in}}}{V_{in}}\right)^2 + \left(2\frac{\sigma_{V_{out}}}{V_{out}}\right)^2$.

For the latter integral, we turn to numerical integration through the trapezoidal rule. Given a function f evaluated at some points $x_1, \ldots, x_n$, its integral can be approximated with a sum S

$$\int_{f_1}^{f_n} g^2(f)df \simeq \sum_{i=1}^{n-1} \frac{1}{2}(g^2(f_i) + g^2(f_{i+1}))(f_{i+1} - f_i) = S \tag{4}$$

We propagate error using partial derivatives, as supplied in [3]:

$$\sigma_S^2 = \sum_{i=1}^{n-1} (\sigma^2_{g^2(f_i)} + \sigma^2_{g^2(f_{i+1})})\frac{1}{4}(f_{i+1} - f_i)^2 \tag{5}$$

Using this method, we can immediately calculate the integral in (3): $\int_0^\infty [g(f)]^2\, df \simeq (6.27 \pm .45) \times 10^{10}$ Hz.

The integral in (1) is trickier because it must be calculated for each value of R. Furthermore, it has a relatively complex error expression. After checking the specification of the band-pass filter [4], we decide to fit our gain curve $g^2(f)$ to a Butterworth Gain function [5]. The reduced chi squared for the fit was $\chi^2_\nu = 2.0$, and the error in $g^2(f)$ was propagated by considering the error in the Butterworth Gain curve parameters. We regret that, due to space limitations, we cannot discuss them here in more detail.

We then calculated the integral in (1) for every value of R, using thousand-iteration simulated computer calculation of the relative uncertainty. Typical uncertainties in G were on the order of 2%.

4.2. Determination of Boltzmann's Constant

We consider (1) at room temperature, such that T is constant. We replace V^2 with $V_R^2 - V_S^2$, which is the difference in the measured noise across a given resistor and the background noise present in the shorted circuit. To derive a value for k, we plot $V^2/4TG$ against R and calculate the least-squares-fit line to the data using MATLAB's `fitlin.m` procedure.

There are statistical fluctuations in V^2 and G, and the recorded temperature varied slightly throughout the experiment - $T = 24.4 \pm .2\ ^\circ\ C$. The uncertainty in the variable on the ordinate is determined by simple error propagation [3]:

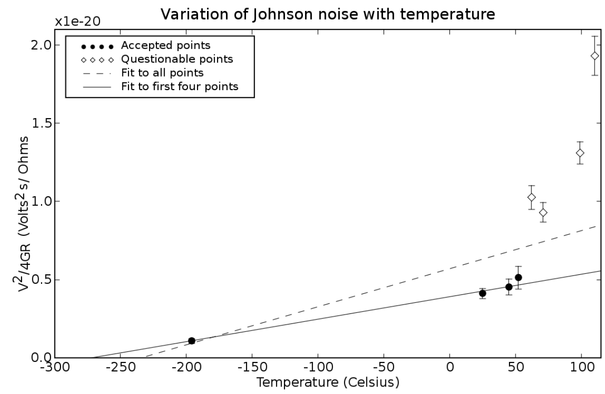

FIG. 3: Plot of $V^2/(4GR)$ versus T, with fitted line.

$$\left(\frac{\sigma_y}{y}\right)^2 = \left(\frac{\sigma_{V^2}}{(V^2)}\right)^2 + \left(\frac{\sigma_T}{T}\right)^2 + \left(\frac{\sigma_G}{G}\right)^2 \tag{6}$$

The slope of the line is Boltzmann's constant. With a reduced-chi-squared of $\chi^2_\nu = 1.8$, we obtain $k = (1.48 \pm .07) \times 10^{-23} J/K$.

4.3. Determination of Absolute Zero

We consider (1) across variable temperature, with a single chosen value of resistance $R = 500 \pm .5\ k\Omega$. We plot $V^2/4GR$ as a function of T in Figure 3, and calculate the least-squares-fit line to the data using MATLAB's `fitlin.m` procedure. The results we obtain are nonsensical.

Re-evaluating the data, we obtain an explanation. In between the measurement of the fourth and fifth points, we lifted the resistor out of the air bath and allowed it to cool to the temperature of the air, which is what we were measuring with the thermometer. We hypothesize that in the heating process, the resistor rose in equilibrium with the metallic surface of the bath, which was significantly hotter than the air measured by the thermometer.

Throwing out these last four points, we fit to the first four, and calculate the temperature-intercept. Our result is $T_0 = (-270 \pm 30)\ ^\circ C$, with $\chi^2_\nu = .3$.

4.4. Determination of the Electron Charge

A salient feature of our tabulated values of V^2 for each V_F is the presence of data points several standard deviations away from the mean. Using Chauvenet's criterion[3], we discarded points whose expected occurrence was less than half an event (for some points, the expected number of occurrences was on the order of .001),

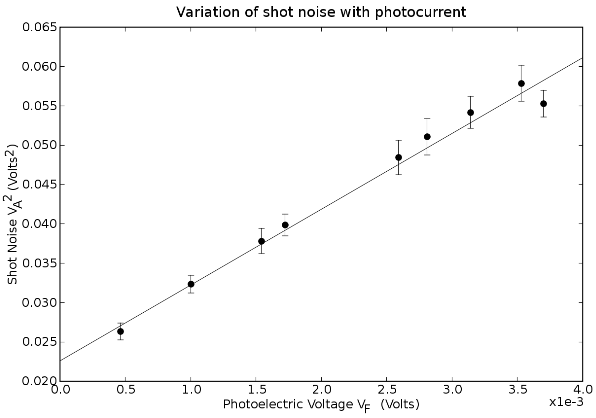

FIG. 4: Plot of V_A^2 versus V_F, with a fitted line.

being sure never to remove more than one point per value of V_F.

This resulted in the loss of about 4% of the most egregiously outlying data. Due to the sensitive nature of the experiment, and the low probability of such events occurring, we felt justified in removing them.

We plot the updated mean value of V^2 for every value of V_F in Figure 4, and calculate the least-squares-fit line to the data using MATLAB's `fitlin.m` procedure.

The slope is a calculation of the coefficient of V_F in (3). With a reduced-chi-squared of $\chi^2_\nu = 0.74$, we obtain $2eR_F\left[\int_0^\infty g(f)^2 df\right] = 9.63 \pm .25$ C · Ω · Hz.

Our derived value of the electron charge is a function of this value, the integral of the square of the gain function as calculated in 3, and value of the resistance R_F, which we found to be $450 \pm 10k\Omega$, with the uncertainty given by the tolerance of the ohmmeter used to measure it. Throwing out the covariant terms, the relative uncertainties add in quadrature.

$$e = \frac{9.63 \text{ C} \cdot \Omega \cdot \text{Hz}}{2(450k\Omega)(6.27 \times 10^{10} \text{ Hz})} = 1.71 \times 10^{-19} \text{ C}$$

$$\frac{\sigma_e}{e} = \sqrt{\left(\frac{.25}{9.63}\right)^2 + \left(\frac{10}{450}\right)^2 + \left(\frac{.46}{6.27}\right)^2} = .14 \times 10^{-19} \text{ C}$$

Our calculated value is $e = (1.71 \pm .14) \times 10^{-19}$ C.

5. CONCLUSIONS

In general, our results are in good, but not great, agreement with the literature values.

Our value of Boltzmann's constant $k = (1.48 \pm .07) \times 10^{-23}$ J/K is a standard deviation and a half away from the established value of 1.38×10^{-23} J/K. Our value of absolute zero $T_0 = -(270 \pm 30)$ $°C$ is within one standard deviation of the established value of -273.15 $°C$. Our value of the electron charge $e = (1.71 \pm .14) \times 10^{-19}$ C is within a standard deviation from the established value of 1.60×10^{-19} C. The tightness of the fits of our lines was decent, with reduced-chi-square values ranging from .74 to 1.8, so we can be reasonably sure of the three linear relations we have been attempting to demonstrate.

Our error bars are noticeably high - 8% in the case of the electron charge. There are several contributors to this fact. Chief among them was the high variance present in any measurements of noise involving the oscilloscope. We observed that a number of outside factors caused the voltage to fluctuate. To combat this, we have attempted to choose an appropriate integrating time, electrically isolate the wires, and periodically stop to calculate whether the values we were observing were sensible.

While we trust that we were justified in discarding every data point we removed by invoking Chauvenet's principle, the large number of such points is disconcerting, and hints at sources of error we have not taken into account. In addition, the fitness of our numerical integration method is a candidate for discussion. An attempted fit of the gain curve to an analytic function as derived from the circuit would have given us another way to calculate $\int g^2(f)df$, but we did not attempt this.

We have tried to choose a consistent and rigorous methodology in our analysis. Several factors impeded this. In particular, we cannot ultimately be sure of the integrity of our temperature-varying Johnson noise data. The variac, air-bath and thermometer setup was relatively precarious, and we are inclined to believe that any number of sources of error negatively impacted our results.

[1] H. Nyquist, Phys. Rev. **23**, 110 (1928).
[2] J. L. Staff, *Johnson noise and shot noise* (2006), jLab E-Library, URL `http://web.mit.edu/8.13/www/JLExperiments/JLExp43.pdf`.
[3] P. Bevington and D. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, 2003).
[4] *Krohn-hite band-pass filter spec.*, jLab E-Library, URL `https://web.mit.edu/8.13/8.13d/manuals/krohn-hite-3BS8TB-1k-50kg.pdf`.
[5] D. A. Neaman, *Electronic Circuit Analysis and Design* (Irwin, 1996).

Acknowledgments

DVP gratefully acknowledges Brian Pepper's equal partnership and the guidance of Prof. Chuang, Dr. Sewell and Scott Sanders throughout the course of the experiment.