Johnson Noise and Shot Noise

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We investigate two types of electrical noise fundamental to any circuit, and their relation to fundamental constants. Johnson noise of a controlled-gain system is measured across different resistances and temperatures, leading to a calculation of the Boltzmann constant $k =???\pm???$ and absolute zero $T_0 = -???\pm???$ $^\circ C$. The shot noise of a variable-current system is measured, leading to a calculation of the electron charge at $e =???\pm???$. Sources of error are discussed.

1. INTRODUCTION

Though the fundamental "noisiness" of electrical conductors had been known for some time, it was not until 1918 that German physicist Walter Schottky identified and formulated a theory of "tube noise" - a fluctuation in the current caused by the granularity of the discrete charges composing it. Ten years later, Johnson and Nyquist similarly analyzed a different type of noise - one caused by the thermal fluctuations of stationary charge carriers.

These are now known as "shot noise" and "Johnson noise", respectively, and it is a startling fact that neither depends on the material or configuration of the electrical circuit in which they are observed. Instead, the expressions governing them are relatively simple, and depend on several fundamental constants. A straightforward measurement of the two types of background noise yields an experimental value for these constants. It is our aim to measure them.

2. THEORY

We present a summary of the theories of Johnson noise and shot noise.

2.1. Johnson Noise and Nyquist's Theorem

The thermal agitation of the charge carriers in any circuit causes a small, yet detectable, current to flow. J.B. Johnson was the first to present a quantitative analysis of this phenomenon, which is fundamentally present and is unaffected by the geometry and material of the circuit.

H. Nyquist showed in his landmark 1928 paper [1] that this problem was equivalent to the normal modes of electrical oscillation along a shorted transmission line. By the equipartition law of thermodynamics, every mode of oscillation contributes kT average energy to the oscillation. If we consider an area of the frequency domain such that the normal modes with corresponding frequencies in that range are very close, we can treat the domain as continuous, such that each frequency differential df contributes $kTdf$ energy.

The energy is equal to I^2R, where I is the current in the line caused by this noise, and R is its resistance. The corresponding voltage is $V = (I)(2R)$, since we are considering a round-trip down the line to correspond to one oscillation. With some algebraic manipulation, the differential contribution to the square voltage from a given frequency differential is $dV^2 = 4kRTdf$.

It remains to replace the resistance R with the characteristic impedance in an RC circuit $\frac{R}{1+(2\pi fRC)^2}$, and integrate over the entire frequency spectrum. We arrive at

$$V^2 = 4RkT \int_0^\infty \frac{[g(f)]^2}{1+(2\pi fCR)^2} df \tag{1}$$

2.2. Shot Noise

The quantization of the charge carried by electrons in a circuit also contributes to a small amount of noise in any circuit. Consider a photoelectric circuit in which current caused by the photoexcitation of electrons flow to the anode. Consider a relatively long time T over which an average current I_{avg} is observed. The average number of electrons hitting the anode per second is I_{avg}/e.

If we assume that the inductive response of the circuit to the electron leaving the circuit is much faster than the typical time between events, we can approximate the current pulse $I_n(t)$ as a delta function centered at some time t_n: $I_n(t) = e\delta(t - t_n)$. The Fourier decomposition of this over the time domain $[0, T]$ is given by

$$I_n(t) = \frac{e}{T} + \frac{2e}{T} \sum_m \cos \frac{2\pi m(t-t_n)}{T} \tag{2}$$

Where the terms on the right are the fluctuating components. Consider the contribution to the rms-current $d < I^2 >$ resulting from fluctuating components with that frequency. Since the phases t_n can be anywhere from 0 to T, we average the cosine term over all possible values.

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Thus, for a given fluctuating component with frequency m, $d < I^2 >=< (\frac{2e}{T})^2 >< \cos^2(2\pi mt - \phi) >= \frac{2e^2}{T^2}$.

Since there are $df = dm/T$ such components in any frequency band, and the total number of such pulses over time T is $I_{avg}T/e$, our expression becomes $d < I^2 >= 2eI_{avg}df$. Of course, we are observing the current after any amplification and attenuation from the circuit, so really the contrbution from this frequency is $g(f)^2(dI^2)$. Consider the voltage $V_0 = IR_F$ instead of the current, and integrating over all possible frequencies, we arrive at

$$V_0^2 = 2eI_{av}R_F^2 \int_0 [g(f)]^2 \, df + V_A^2 \tag{3}$$

The V_A^2 term represents sources of background noise such as Johnson noise that are constant when the resistance and temperature are constant, as they are in our experimental setup. We see that unlike Johnson noise, shot noise is independent of temperature and resistance, but is only present when a current is flowing.

The derivation of the shot noise expression above is inspired by one by Seth Dorfman [2].

3. EXPERIMENTAL SETUP

3.1. Experimental Procedure

We began both the Johnson noise and Shot noise experiments with a careful calibration of measurement chain. This was done by measuring the output of a strong sinusoidal signal from a function generator attenuated by 60 dB. Before each experiment, we measured the ratio $g(f) = \frac{V_{\text{out}}(f)}{V_{\text{in}}(f)}$ at enough frequencies to effectively model $g(f)$. We calculated the mean and standard deviation of each $V_{\text{in}}(f)$ and $V_{\text{out}}(f)$.

In the case of the Johnson noise experiment, we measured the rms voltage V_R of the noise as calculated by the oscilloscope twenty times for ten different values of the resistance, ranging from 10 kΩ to 1 MΩ. We then used the same 500 kΩ resistor to measure the rms voltage across different temperatures by submersing the assembly in liquid nitrogen ($T = 77$ K) and a heated air bath ($T < 100°$ C), measuring the temperature with a mercury thermometer. For each measurement at a given resistance and temperature, we shorted the circuit to measure the rms background noise voltage V_S; the difference in the two is the Johnson noise.

In the case of the shot noise experiment, we terminated the calibration input and turned on the internal lamp. We recorded the rms voltage V_0 of the noise as calculated by the oscilloscope twenty times for eight different intensities of the light V_F. We then break the photocircuit and record the constant noise level V_A.

In both cases, the measured noise changed slightly based on the integration time used by the oscilloscope, but this uncertainty ranged on the order of 0.1%, and we ignore it, since it is dwarfed by the uncertainty caused by random fluctuations in V or V_0, respectively.

4. DATA AND ERROR ANALYSIS

4.1. Calculation of Gain

To evaluate the two integrals present in (1) and (3), we turn to numerical integration through the trapezoidal rule. Given a function f evaluated at some points $x_1, \ldots, x_n$, its integral can be approximated with a sum S

$$\int_{x_1}^{x_n} f(x) \text{ dx} \simeq \sum_{i=1}^{n-1} \frac{1}{2}(f(x_i) + f(x_{i+1}))(x_{i+1} - x_i) = S \tag{4}$$

There are two sources of error involved in this calculation. The first is the uncertainty in the values of $f(x_i)$, which add in quadrature with coefficients given by (4):

$$\sigma_S^2 = \sum_{i=1}^{n-1} (\sigma_{f(x_i)}^2 + \sigma_{f(x_{i+1})}^2) \frac{1}{4}(x_{i+1} - x_i)^2 \tag{5}$$

The second is the error inherent in using the trapezoidal rule to estimate the value of a definite integral, and is more difficult to compute.

TODO: SECOND SOURCE OF ERROR

Using the methods described above, we can immediately calculate the integral in (3), since $\left(\frac{\sigma_{g^2}}{g^2}\right)^2 = \left(2\frac{\sigma_{V_{in}}}{V_{in}}\right)^2 + \left(2\frac{\sigma_{V_{out}}}{V_{out}}\right)^2$:

$$\int_0^\infty [g(f)]^2 \, df \simeq (6.27 \pm .45) \times 10^{10} \tag{6}$$

The integral in (1) is trickier. In particular, it varies with R and thus must be calculated for each value thereof. Furthermore, it has a relatively complex error expression. To get around this, we perfomed a thousand-point Monte Carlo simulation for every calculated calculated value of G, and derived an uncertainty. In general, the relative error in G as calculated above was not above ???%.

4.2. Determination of Boltzmann's Constant

We consider (1) at room temperature, such that T is constant. We replace V^2 with $V_R^2 - V_S^2$, which is the difference in the measured noise across a given resistor and the background noise present in the shorted circuit. To derive a value for k, we plot $V^2/4TG$ against R in Figure 1, and calculate the least-squares-fit line to the data using MATLAB's `fitlin.m` procedure.

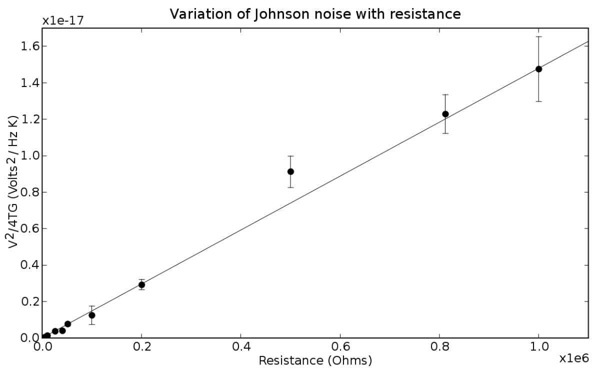

FIG. 1: Plot of $V^2/(4GT)$ versus R, with fitted line.

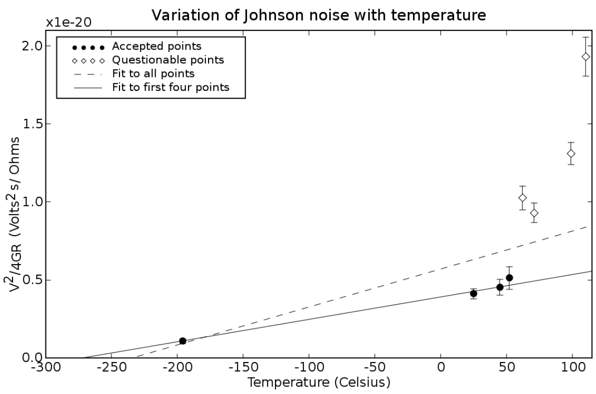

FIG. 2: Plot of $V^2/(4GR)$ versus T, with fitted line.

The are statistical fluctuations in V^2 and G, and the recorded temperature varied slightly throughout the experiment - $T = 24.4 \pm .2\ ^\circ\ C$. The uncertainty in the variable on the ordinate is determined by simple error propagation [3]:

$$\left(\frac{\sigma_y}{y}\right)^2 = 2\frac{\sigma^2_{V_R^2} + \sigma_{V_S^2}}{(V^2)^2} + \left(\frac{\sigma_T}{T}\right)^2 + \left(\frac{\sigma_G}{G}\right)^2 \qquad (7)$$

The slope of the line is simply k. With a reduced-chi-squared of $\chi^2_\nu = 1.8$, we obtain $k = (1.48 \pm .07) \times 10^{-23} J/K$.

4.3. Determination of Absolute Zero

We consider (1) across variable temperature, with a single chosen value of resistance. In this case, G does not vary with R; they are both constant. We plot V^2 as a function of T in Figure 2, and calculate the least-squares-fit line to the data using MATLAB's `fitlin.m` procedure. The slope is a calculation of the coefficient of T. With a reduced-chi-squared value of χ^2_ν

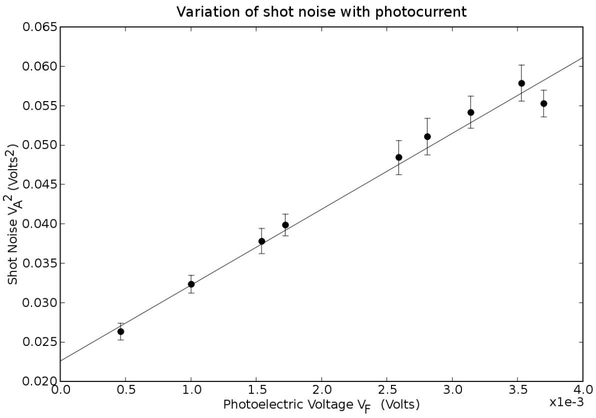

FIG. 3: Plot of V_A^2 versus V_F, with a fitted line.

4.4. Determination of the Electron Charge

We rearrange (3), replacing $I_{av}R_F$ with the photoelectric voltage V_F, and letting V_A be the noise measured when no photelectric current flows.

$$(V_0^2 - V_A^2) = 2eV_F R_F \left[\int_0^\infty g(f)^2 df\right] \qquad (8)$$

The relative error inherent in the left-hand side is given by the formula below. Since we do not expect the random fluctuations in V_A to have any correlations with those in our measurement of V_0, we do not consider any covariant σ_{V_A,V_0} terms.

$$\text{err}(V_0^2 - V_A^2) = \sqrt{\left(2\frac{\sigma_{V_0}}{V_0}\right)^2 + \left(2\frac{\sigma_{V_A}}{V_A}\right)^2} \qquad (9)$$

A salient feature of our tabulated values of V_0 for each V_F is the presence of data points several standard deviations away from the mean. Using Chauvenet's criterion[3], we discarded points whose expected occurence was less than half an event. This resulted in the loss of about 4% of the most egregiously outlying data. We plot this value for every value of V_F in Figure 3, and calculate the least-squares-fit line to the data using MATLAB's `fitlin.m` procedure.

The slope is a calculation of the coefficient of V_F in (8). With a reduced-chi-squared of $\chi^2_\nu = 0.74$, we obtain

$$2eR_F \left[\int_0^\infty g(f)^2 df\right] = 9.63 \pm .25 \text{ C} \cdot \Omega \cdot \text{Hz}$$

Our derived value of the electron charge is a function of this value, the integral of the square of the gain function, as calculated in 8, and value of the resistance R_F, which we found to be $450 \pm 10k\Omega$, with the uncertainty given by the tolerance of the ohmmeter used to measure it. Again, after throwing out the covariant terms, the relative uncertainties add in quadrature.

$$e = \frac{9.63\ \mathrm{C} \cdot \Omega \cdot \mathrm{Hz}}{2(450k\Omega)(6.27 \times 10^{10}\ \mathrm{Hz})} = 1.71 \ \times \ 10^{-16}\ \mathrm{C}$$

$$\frac{\sigma_e}{e} = \sqrt{\left(\frac{.25}{9.63}\right)^2 + \left(\frac{10}{450}\right)^2 + \left(\frac{.46}{6.27}\right)^2} = .14 \ \times \ 10^{-16}\ \mathrm{C}$$

This leads to a value of the electron charge $e = (1.71 \pm .14) \ \times \ 10^{-19}\ C$.

5. CONCLUSIONS

In general, our results are in good, but not great, agreement with the literature values.

Our value of Boltzmann's constant $k = (1.48 \pm .07) \ \times 10^{-23}\ J/K$ is a standard deviation and a half away from the established value of $1.38 \times 10^{-23}\ J/K$. Our value of absolute zero ??? is ??? the established value of $-273.15\ °C$. Our value of the electron charge $e = (1.71 \pm .14) \ \times \ 10^{-19}\ C$ is within a standard deviation from the established value of $1.60 \times 10^{-19}\ C$. The tightness of the fits of our lines was decent, with reduced-chi-square values ranging from .74 to 1.8, so we can be reasonably sure that of the three linear relations we have been attempting to demonstrate.

Our error bars are noticably high - 8% in the case of the electron charge. There are several contributors to this fact. Chief among them was the high variance present in any measurements of noise involving the oscilloscope. We observed that a number of outside factors caused the noise to voltage to fluctuate. To combat this, we have attempted to choose an appropriate integrating time, electrically isolate the wires, and periodically stop to calculate whether the values we were observing were sensible.

Furthermore, while we trust that we were justified in discarding every data point we removed by invoking Chauvenet's principle (some had expected numbers of events on the order of .001), the large number of such points is disconcerting, and hints at sources of error we have not taken into account.

In addition, the fitness of our numerical integration method is a candidate for discussion.

We have tried to choose a consistent and rigorous methodology in our analysis. Several factors impeded this. In particular, we cannot ultimately be sure of the integrity of our temperature-varying Johnson noise data. The variac, air-bath and thermometer setup was relatively precarious, and we are inclined to believe that any number of sources of error negatively impacted our results.

[1] H. Nyquist, Phys. Rev. **23**, 110 (1928).
[2] J. L. Staff, *Johnson noise and shot noise* (2006), jLab E-Library, URL `http://web.mit.edu/8.13/www/JLExperiments/JLExp43.pdf`.
[3] P. Bevington and D. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, 2003).

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