Exercise 5.1: Rotations

We have \( q = F(t, q') = Rq' \) relating the two sets of coordinates. The canonical phase-space transformation is

\[
(t, q, p) = C(t, q', p') = (t, F(t, q'), p'(\partial_1 F(t, q'))^{-1})
\]

\[
= (t, Rq', p' (\partial_1 (Rq))^{-1}) = (t, Rq', p'R^{-1})
\]

We can be sure that \( R^{-1} \) exists and is well-defined, since the inverse of a rotation is simply rotation along the same axis but in the opposite direction. The transformation equations for the rectangular components of the momenta are \( p = p' R^{-1} \), as above. The transformation equations for the rectangular components of the velocities are \( v = Dq = D(Rq') = R(Dq') = Rv' \) since \( R \) is time-independent.

Thus, unsurprisingly, \( pv = (p'R^{-1})(Rv') = p'v' \), as shown in (5.10).

Exercise 5.4: Polar-canonical transformations

We first define this transformation:

\[
\text{(define ((p-c-transform alpha beta) H-state)}
\]
\[
\text{(let ((t (time H-state)))}
\]
\[
\text{(theta (coordinate H-state))}
\]
\[
\text{(I (momentum H-state)))}
\]
\[
\text{(let ((x (* beta (expt I alpha) (sin theta)))}
\]
\[
\text{(p (* beta (expt I alpha) (cos theta)))}
\]
\[
\text{(up t x p))))}
\]

Since the transformation is time-independent, we can run the time-independent test on it.

\[
\text{(show-expression}
\]
\[
\text{((time-independent-canonical?}
\]
\[
\text{(p-c-transform ’alpha ’beta)) (up ’t ’theta ’I)))}
\]

\[
\begin{pmatrix}
0 \\
-\alpha \beta^2 \cdot x1116 \cdot I^{(-1)+\alpha} I^\alpha + x1116 \\
\alpha \beta^2 \cdot x1115 \cdot I^{(-1)+\alpha} I^\alpha - x1115
\end{pmatrix}
\]

The \( x1116 \) and \( x1115 \) symbols represent small, unrelated real numbers, which can be factored out. For the expression above to be a vector of zeros as desired, we require

\[
\alpha \beta^2 I^{2\alpha-1} = 1
\]
When \( \alpha = \frac{1}{2} \), the \( I \) term is constant, and we are left with \( \frac{1}{2} \beta^2 = 1 \). The solution, for arbitrary \( I \), is \( \alpha = \frac{1}{2}, \beta = \pm \sqrt{2} \).

For \( \alpha \neq \frac{1}{2} \), the only solution is when \( I \) is a constant which can be expressed in terms of \( \alpha \) and \( \beta \):

\[
I^{2\alpha - 1} = \frac{1}{\alpha \beta^2} \Rightarrow I = \left( \frac{1}{\alpha \beta^2} \right)^{1-2\alpha}
\]

But in general, only \( x = \pm \sqrt{2I} \sin \theta \) and \( p = \pm \sqrt{2I} \cos \theta \) are the transformations of this form that are canonical.

**Exercise 5.5: Standard map**

Since the transformation that is the standard map has no explicit time-dependence, we can test that it is a canonical transformation with the implemented `time-independent-canonical?` procedure.

```scheme
(define ((standard-transform K) H-state)
  (let ((t (time H-state))
        (theta (coordinate H-state))
        (I (momentum H-state)))
    (let ((thetaprime (+ theta I (* K (sin theta))))
          (Iprime (+ I (* K (sin theta))))
          (up t thetaprime Iprime))))

(print-expression
  ((time-independent-canonical?
    (standard-transform 'K)) (up 't 'theta 'I)))
(up 0 0 0)
```

So the transformation is canonical.

**Exercise 5.14: Rotating coordinates in extended phase space**

An appropriate extended phase space type-2 generating function \( F_2^c(\tau; r, \theta, t; p'_r, p'_\theta, p'_t) \) can be constructed using (5.213). This has the same relations between coordinates and momenta as the normal phase space generating function of type-2, but with an extra \( t \) term in the coordinates and \( p_t \) in the momenta. Namely, let

\[
F_2^c(\tau; r, \theta, t; p'_r, p'_\theta, p'_t) = tp'_t + F_2(t; r, \theta; p'_r, p'_\theta) = tp'_t + rp'_r + (\theta - t\Omega)p'_\theta
\]

The original momenta and the transformed coordinates are

\[
\begin{bmatrix}
  p_r \\
  p_\theta \\
  p_t
\end{bmatrix}
= \partial_1 F_2^c(\tau; r, \theta, t; p'_r, p'_\theta, p'_t)
\]

\[
\begin{bmatrix}
  r' \\
  \theta' \\
  t'
\end{bmatrix}
= \partial_2 F_2^c(\tau; r, \theta, t; p'_r, p'_\theta, p'_t)
\]
Now that \( t \) is a coordinate, the momentum conjugate of the time transforms, compensating for the change in the identity of \( \theta' \):

\[
\theta' = (\partial_2 F_2^e)^2 = \theta \\
p_t = (\partial_1 F_2^e)_2 = p'_t - \Omega p'_{\theta}
\]

The latter equation also defines the difference in the Hamiltonians

\[
H'(t; r', \theta'; p'_r, p'_\theta) = H(t; r, \theta; p_r, p_\theta) - (\Omega p'_{\theta})
\]

Which is the same as (5.234).

**Exercise 5.19: Standard-map generating function**

We attempt to find an \( F_1 \) type generating function for the standard map. Such a function must satisfy

\[
I = \partial_1 F_1(t, \theta, \theta'), \quad I' = -\partial_2 F_1(t, \theta, \theta')
\]

Thus, we want to express \( I \) and \( I' \) in terms of \( \theta \) and \( \theta' \) alone. The definition of the standard map is

\[
I' = (I + K \sin \theta) \mod 2\pi \\
\theta' = (\theta + I') \mod 2\pi
\]

It immediately follows from the second equality that \( I' = \theta' - \theta \). We have

\[
\partial_2 F_2(t, \theta, \theta') = -I' \\
\frac{d}{d\theta'} F_2 = \theta - \theta' \\
F_2 = \theta \theta' - \frac{1}{2} [\theta']^2 + \phi_2(t, \theta)
\]

Where \( \phi_2(t, \theta) \) is constant with respect to \( \theta' \). Similarly, \( I = I' - K \sin \theta = \theta' - \theta - K \sin \theta \).

\[
\partial_1 F_2(t, \theta, \theta') = I \\
\frac{d}{d\theta} F_2 = \theta' - \theta - K \sin \theta \\
F_2 = \theta \theta' - \frac{1}{2} \theta^2 + K \cos \theta + \phi_1(t, \theta')
\]

Again, \( \phi_1(t, \theta') \) is constant in \( \theta \). Let \( \phi_2 = -\frac{1}{2} \theta^2 - K \cos \theta \) and \( \phi_1 = -\frac{1}{2} [\theta']^2 \). Our generating function is therefore

\[
F_2(t, \theta, \theta') = \theta \theta' - \frac{1}{2} \left( \theta^2 + [\theta']^2 \right) - K \cos \theta
\]