6.946 Assignment 10

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Exercise 5.1: Rotations

We have $q = F(t, q') = Rq'$ relating the two sets of coordinates. The canonical phase-space transformation is

$$(t, q, p) = C(t, q', p') = (t, F(t, q'), p'(\partial_1 F(t, q'))^{-1})$$
$$= (t, Rq', p'(\partial_1(Rq))^{-1}) = (t, Rq', p'R^{-1})$$

We can be sure that R^{-1} exists and is well-defined, since the inverse of a rotation is simply rotation along the same axis but in the opposite direction. The tranformation equations for the rectangular components of the momenta are $p = p'R^{-1}$, as above. The transformation equations for the rectangular components of the velocities are $v = Dq = D(Rq') = R(Dq') = Rv'$ since R is time-independent.

Thus, unsurprisingly, $pv = (p'R^{-1})(Rv') = p'v'$, as shown in (5.10).

Exercise 5.4: Polar-canonical transformations

We first define this transformation:

```
(define ((p-c-transform alpha beta) H-state)
  (let ((t (time H-state))
        (theta (coordinate H-state))
        (I (momentum H-state)))
    (let ((x (* beta (expt I alpha) (sin theta)))
          (p (* beta (expt I alpha) (cos theta))))
      (up t x p))))
```

Since the transformation is time-independent, we can run the time-independent test on it.

```
(show-expression
  ((time-independent-canonical?
    (p-c-transform 'alpha 'beta)) (up 't 'theta 'I)))
```

$$\boxed{\begin{pmatrix} 0 \\ -\alpha\beta^2 \cdot x1116 \cdot I^{(-1)+\alpha} I^{\alpha} + x1116 \\ \alpha\beta^2 \cdot x1115 \cdot I^{(-1)+\alpha} I^{\alpha} - x1115 \end{pmatrix}}$$

The $x1116$ and $x1115$ symbols represent small, unrelated real numbers, which can be factored out. For the expression above to be a vector of zeros as desired, we require

$$\alpha\beta^2 I^{2\alpha-1} = 1$$

When $\alpha = \frac{1}{2}$, the I term is constant, and we are left with $\frac{1}{2}\beta^2 = 1$. The solution, for arbitrary I, is $\alpha = \frac{1}{2},\ \beta = \pm\sqrt{2}$.

For $\alpha \neq \frac{1}{2}$, the only solution is when I is a constant which can be expressed in terms of α and β:

$$I^{2\alpha-1} = \frac{1}{\alpha\beta^2} \Rightarrow I = \left(\frac{1}{\alpha\beta^2}\right)^{1-2\alpha}$$

But in general, only $x = \pm\sqrt{2I}\,\sin\theta$ and $p = \pm\sqrt{2I}\,\cos\theta$ are the transformations of this form that are canonical.

Exercise 5.5: Standard map

Since the transformation that is the standard map has no explicit time-dependence, we can test that it is a canonical transformation with the implemented `time-independent-canonical?` procedure.

```
(define ((standard-transform K) H-state)
  (let ((t (time H-state))
        (theta (coordinate H-state))
        (I (momentum H-state)))
    (let ((thetaprime (+ theta I (* K (sin theta))))
          (Iprime (+ I (* K (sin theta)))))
      (up t thetaprime Iprime))))

(print-expression
  ((time-independent-canonical?
    (standard-transform 'K)) (up 't 'theta 'I)))
(up 0 0 0)
```

So the transformation is canonical.

Exercise 5.14: Rotating coordinates in extended phase space

An appropriate extended phase space type-2 generating function $F_2^e(\tau; r, \theta, t; p'_r, p'_\theta, p'_t)$ can be constructed using (5.213). This has the same relations between coordinates and momenta as the normal phase space generating function of type-2, but with an extra t term in the coordinates and p_t in the momenta. Namely, let

$$F_2^e(\tau; r, \theta, t; p'_r, p'_\theta, p'_t) = tp'_t + F_2(t; r, \theta; p'_r, p'_\theta) = tp'_t + rp'_r + (\theta - t\Omega)p'_\theta$$

The original momenta and the transformed coordinates are

$$\begin{bmatrix} p_r \\ p_\theta \\ p_t \end{bmatrix} = \partial_1 F_2^e(\tau; r, \theta, t; p'_r, p'_\theta, p'_t)$$

$$\begin{bmatrix} r' \\ \theta' \\ t' \end{bmatrix} = \partial_2 F_2^e(\tau; r, \theta, t; p'_r, p'_\theta, p'_t)$$

Now that t is a coordinate, the momentum conjugate of the time transforms, compensating for the change in the identity of θ':

$$\theta' = (\partial_2 F_2^e)^2 = \theta$$

$$p_t = (\partial_1 F_2^e)_2 = p_t' - \Omega p_\theta'$$

The latter equation also defines the difference in the Hamiltonians

$$H'(t; r', \theta'; p_r', p_\theta') = H(t; r, \theta; p_r, p_\theta) - (\Omega p_\theta')$$

Which is the same as (5.234).

Exercise 5.19: Standard-map generating function

We attempt to find an F_1 type generating function for the standard map. Such a function must satisfy

$$I = \partial_1 F_1(t, \theta, \theta'), \quad I' = -\partial_2 F_1(t, \theta, \theta')$$

Thus, we want to express I and I' in terms of θ and θ' alone. The definition of the standard map is

$$I' = (I + K \sin\theta) \bmod 2\pi$$

$$\theta' = (\theta + I') \bmod 2\pi$$

It immediately follows from the second equality that $I' = \theta' - \theta$. We have

$$\partial_2 F_2(t, \theta, \theta') = -I'$$

$$\frac{d}{d\theta'} F_2 = \theta - \theta'$$

$$F_2 = \theta\theta' - \frac{1}{2}\left[\theta'\right]^2 + \phi_2(t, \theta)$$

Where $\phi_2(t, \theta)$ is constant with respect to θ'. Similarly, $I = I' - K \sin\theta = \theta' - \theta - K \sin\theta$.

$$\partial_1 F_2(t, \theta, \theta') = I$$

$$\frac{d}{d\theta} F_2 = \theta' - \theta - K \sin\theta$$

$$F_2 = \theta\theta' - \frac{1}{2}\theta^2 + K \cos\theta + \phi_1(t, \theta')$$

Again, $\phi_1(t, \theta')$ is constant in θ. Let $\phi_2 = -\frac{1}{2}\theta^2 - K \cos\theta$ and $\phi_1 = -\frac{1}{2}\left[\theta'\right]^2$. Our generating function is therefore

$$F_2(t, \theta, \theta') = \theta\theta' - \frac{1}{2}\left(\theta^2 + \left[\theta'\right]^2\right) - K \cos\theta$$