6.946 Assignment 11

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Exercise 5.26: Uniform acceleration

(a) First, let's the solve the simpler problem of computing the action for a uniformly accelerated particle which is at $q_1(t1 = 0) = 0$ initially, and at q_2 at time t_2 later. A particle with acceleration *alpha* moves a distance of $q_2 = \frac{1}{2}\alpha t_2^2$ in t_2 seconds. Its velocity is time-dependent, and is given by $D(q(t)) = \alpha t$.

For a free particle with Lagrangian $L(t; x; v) = \frac{1}{2}mv^2$, the action on the particle is

$$S[q](0, t_2) = \int_{t=0}^{t=t_2} L \circ \Gamma[q] = \frac{1}{2}m \int_{t=0}^{t=t2} [Dq(t)]^2 dt = \frac{1}{2}m \int_{t=0}^{t=t2} [\alpha t]^2 dt = \frac{1}{2}m\alpha^2 \frac{1}{3}t^3$$

$$\frac{1}{6}mt^3 \left[\frac{2q_2}{t_2^2}\right]^2 = \frac{2}{3}m\frac{q_2^2}{t_2}$$

Using the fact that the action is defined by integrals, we can compute the more general action on the solution path between $q_1(t_1)$ and $q_2(t_2)$.

$$S[q](t_1, t_2) = \int_{t=t_1}^{t=t_2} L \circ \Gamma[q] = \int_{t=0}^{t=t_2} L \circ \Gamma[q] - \int_{t=0}^{t=t_1} L \circ \Gamma[q] = \frac{2}{3}m\left(\frac{q_2^2}{t_2} - \frac{q_1^2}{t_1}\right)$$

Let's quickly double-check this. Let a particle have $m = 1$ and uniform acceleration $\alpha = 2$. The solution path is $q(t) = t^2$. Between $t_1 = 1$ and $t_2 = 5$, the action is $\frac{2}{3}\left(\frac{25^2}{5} - \frac{1^2}{1}\right) = \frac{248}{3} = 82.666\ldots$. Calculating by computer, we get

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(define (min-path t) (up (* t t)))
(Lagrangian-action (L-free-particle 1.0) min-path 1.0 5.0)
;Value: 82.6666666666666
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Which leads to reasonable confidence in our answer. This is the action that generated time-evolution

$$\tilde{F}(t_1, q_1, t_2, q_2) = S[q](t_1, t_2)$$

We can find the momenta by taking the partial derivatives with respect to the momenta:

$$-p_1 = \partial_1\tilde{F} = -\frac{4}{3}m\frac{q_1}{t_1}$$

$$p_2 = \partial_3\tilde{F} = \frac{4}{3}m\frac{q_2}{t_2}$$

These are in fact the average momenta for the particle over the first t_1 and t_2 seconds, respectively. Using our original definitions of the the coordinate and momenta for a uniformly accelerated particle, we can derive equations for p_2 and q_2 given an initial state (t_1, p_1, q_1) as a function of t_2.

$$p_2 = p_1 + \alpha t = p_1 + 2m\frac{q_1}{t_1^2}(t_2 - t_1) = 2m\frac{q_1}{t_1^2}t_2$$

$$q_2 = q_1 + \frac{1}{2}\alpha(t_2^2 - t_1^2) = \frac{1}{2}2\frac{q_1}{t_1^2}(t_2^2) = q_1\left(\frac{t_2}{t_1}\right)^2$$

Exercise 5.27: Binomial series

The Taylor expansion of $(1+x)^n$ around $x = 0$ for abitrary n is given by:

$$(1+\epsilon)^n = \sum_{i=0}^{\infty}\frac{\epsilon^i}{i!}\left[\frac{d^i}{dx^i}(1+x)^n\Big|_{x=0}\right]$$

$$= \sum_{i=0}^{\infty}\frac{\epsilon^i}{i!}n(n-1)\ldots(n-i)(1+0)^{n-i-1}$$

The i^{th} term contains the polynomial in n given by $\prod_{j=0}^{i}(n-j)$ as a factor. However, for all $i > n$, one of the factors is $(n-n) = 0$, and all the terms higher-order than the n^{th} are zero.

Thus, the i^{th} term of this Taylor expansion is zero when $n < i$.

Exercise 5.30: Commutators of Lie derivatives

(a) Consider the action of the commutator of two Lie derivatives on an arbitrary phase-space function F.

$$[L_W, L_{W'}]F = (L_W L_{W'} - L_{W'} L_W)F = L_W(\{F, W'\}) - L_{W'}(\{F, W\})$$

$$= -L_W(\{W', F\}) - L_{W'}(\{F, W\}) = -\{\{W', F\}, W\} - \{\{F, W\}, W'\}$$

$$= \{W, \{W', F\}\} + \{W', \{F, W\}\}$$

$$= (\{W, \{W', F\}\} + \{W', \{F, W\}\} + \{F, \{W, W'\}\}) - \{F, \{W, W'\}\}$$

By Jacobi's cyclic Poisson bracket identity (3.92), the first term is zero.

$$= -\{F, \{W, W'\}\} = -L_{\{W,W'\}}F$$

(b) We compute the structured partial derivatives of the angular momenta functions.

$$\partial_0 J_x = \partial_0 J_y = \partial_0 J_z = 0$$

$$\partial_1 J_x = \begin{pmatrix} 0 \\ p_z \\ -p_y \end{pmatrix},\ \partial_1 J_y = \begin{pmatrix} -p_z \\ 0 \\ p_x \end{pmatrix},\ \partial_1 J_z = \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix}$$

$$\partial_2 J_x = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix},\ \partial_2 J_y = \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix},\ \partial_2 J_z = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

The commutator of the Lie derivative with respect to the first two of these can be expressed as a single Lie derivative, by part (a) above:

$$[L_{J_x}, L_{J_y}] = -L_{\{J_x,J_y\}}$$

The Poisson bracket is

$$\{J_x, J_y\} = \partial_1 J_x \partial_2 J_y - \partial_2 J_x \partial_1 J_y = \begin{pmatrix} 0 \\ p_z \\ -p_y \end{pmatrix} \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} - \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} \begin{pmatrix} -p_z \\ 0 \\ p_x \end{pmatrix}$$

$$= xp_y - yp_x = J_z$$

We arrive at

$$[L_{J_x}, L_{J_y}] + L_{J_z} = 0$$

This is also true if we cyclically permute (J_x, J_y, J_z).

(c) We begin with the Jacobi Poisson-bracket identity, for any three operators F, G and H.

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$$

Thus, the Lie derivative with respect to this is zero. Since the Lie derivative is linear, we can split it up into three Lie derivatives:

$$L_{\{F,\{G,H\}\}+\{H,\{F,G\}\}+\{G,\{H,F\}\}} = L_0 = 0$$

$$L_{\{F,\{G,H\}\}} + L_{\{H,\{F,G\}\}} + L_{\{G,\{H,F\}\}} = 0$$

$$-[L_{\{G,H\}}, L_F] - [L_{\{F,G\}}, L_H] - [L_{\{H,F\}}, L_G] = 0$$

$$-[[L_H, L_G], L_F] - [[L_G, L_F], L_H] - [[L_F, L_H], L_G] = 0$$

$$[L_F, [L_H, L_G]] + [L_H, [L_G, L_F]] + [L_G, [L_F, L_H]] = 0$$

The more general result, which holds for any three operators A, B and C (not just Lie derivatives), is known as the Jacobi identity for operators:

$$[A, [B, C]] + [B, [A, C]] + [C, [A, B]] = 0$$

Exercise 6.2: Resonance width

The value of the original Hamiltonian $H = H_0 + \epsilon H_1$ for a periodically driven pendulum given in (6.54) must remain the same along any contour. Since we are interested in the $+\omega$ resonance phenomenon, we do not consider the 0 or $-\omega$ terms. Our Hamiltonian is

$$H(\tau; \theta, t; p, p_t) = H_0 + \epsilon H_1 = \left(p_t + \frac{1}{2\alpha}p^2\right) + \epsilon\gamma \cos(\theta - \omega t)$$

Let Δ be the resonance half-width. The value of the Hamiltonian at $(\sigma, \Sigma) = (0, \alpha\omega)$ must be the same as the value at $(\pi, \alpha\omega \pm \Delta)$.

Using the transformation given by (6.77), we can translate these coordinates back into our original (θ, t, p, p_t) coordinate frame:

$$\theta - \omega t = \sigma$$

$$p = \Sigma$$

$$p_t = p_t' - \omega\Sigma$$

The original Hamiltonian in terms of these values looks like

$$H(\tau;\sigma,t';\Sigma,p_t') = p_t' - \omega\Sigma + \frac{1}{2\alpha}\Sigma^2 + \epsilon\gamma \cos\sigma$$

We equate the Hamiltonian at the two values

$$H(\tau;\sigma=0,t';\Sigma=\alpha\omega,p_t') = H(\tau;\sigma=\pi,t';\Sigma=\alpha\omega\pm\Delta,p_t')$$

$$p_t' - \alpha\omega^2 + \frac{1}{2}\alpha\omega^2 + \epsilon\gamma = p_t' - \alpha\omega^2 \mp \omega\Delta + \frac{1}{2\alpha}(\alpha^2\omega^2 \pm 2\alpha\omega\Delta + \Delta^2) - \epsilon\gamma$$

$$-\frac{1}{2}\alpha\omega^2 + \epsilon\gamma = -\alpha\omega^2 \mp \omega\Delta + \frac{1}{2}\alpha\omega^2 \pm \omega\Delta + \frac{1}{2\alpha}\Delta^2 - \epsilon\gamma$$

$$2\epsilon\gamma = \frac{1}{2\alpha}\Delta^2$$

$$\Rightarrow \Delta = 2\sqrt{\alpha\epsilon\gamma}$$