6.946 Assignment 2

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Exercise 8.1: Chain rule

(a)

$$\partial_0 F(x,y) = \frac{\partial}{\partial x} x^2 y^3 = 2xy^3$$

$$\partial_1 F(x,y) = \frac{\partial}{\partial y} x^2 y^3 = 3x^2 y^2$$

(b)

$$\partial_0 F(F(x,y),y) = \frac{\partial}{\partial F}(F(x,y))^2 y^3 = 2F(x,y)y^3 = 2(x^2y^3)y^3 = 2x^2y^6$$

$$\partial_1 F(F(x,y),y) = \frac{\partial}{\partial y}(F(x,y))^2 y^3 = F(x,y)^2 3y^2 = 3x^4y^8$$

(c)

$$\partial_0 G(x,y) = \begin{pmatrix} \partial_0 F(x,y) \\ \partial_0 y \end{pmatrix} = \begin{pmatrix} 2xy^3 \\ 0 \end{pmatrix}$$

$$\partial_1 G(x,y) = \begin{pmatrix} \partial_1 F(x,y) \\ \partial_1 y \end{pmatrix} = \begin{pmatrix} 3x^2y^2 \\ 1 \end{pmatrix}$$

(d)

$$DF(a,b) = [\partial_0 F(a,b), \partial_1 F(a,b)] = [\frac{\partial}{\partial a} a^2b^3, \frac{\partial}{\partial b} a^2b^3] = [2ab^3, 3a^2b^2]$$

$$DG(3,5) = [\partial_0 G(x,y), \partial_1 G(x,y)]_{x=3,y=5} = \left[\begin{pmatrix} \partial_0 F(x,y) \\ \partial_0 y \end{pmatrix}, \begin{pmatrix} \partial_1 F(x,y) \\ \partial_1 y \end{pmatrix} \right]_{x=3,y=5}$$

$$= \left[\begin{pmatrix} 2xy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3x^2y^2 \\ 1 \end{pmatrix} \right]_{x=3,y=5} = \left[\begin{pmatrix} 750 \\ 0 \end{pmatrix}, \begin{pmatrix} 675 \\ 1 \end{pmatrix} \right]$$

$$DH(3a^2, 5b^3) = DH(x,y)_{x=3a^2, y=5b^3} = [\partial_0 H(x,y), \partial_i H(x,y)]$$

$$= [DF(G(x,y))\partial_0 G(x,y), DF(G(x,y))\partial_1 G(x,y)]$$

We have already computed parts of the above earlier in the problem

$$DF(G(x,y)) = DF(F(x,y),y) = [\partial_0 F(F(x,y),y), \partial_1 F(F(x,y),y)] = [2x^2y^6, 3x^4y^8]$$

$$DG(x,y) = [\partial_0 G(x,y), \partial_1 G(x,y)] = \left[\begin{pmatrix} 2xy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3x^2y^2 \\ 1 \end{pmatrix} \right]$$

We insert these results, and use a calculator:

$$DH(x,y) = \left[[2x^2y^6, 3x^4y^8] \begin{pmatrix} 2xy^3 \\ 0 \end{pmatrix}, [2x^2y^6, 3x^4y^8] \begin{pmatrix} 3x^2y^2 \\ 1 \end{pmatrix} \right]$$

$$= \left[(2x^2y^6)(2xy^3) + (3x^4y^8)(0), (2x^2y^6)(3x^2y^2) + (3x^4y^8)(1)\right] = \left[4x^3y^9, 6x^4y^8 + 3x^4y^8\right]$$
$$= \left[4x^3y^9, 9x^4y^8\right]_{x=3a^2, y=5b^3} = \left[4(3a^2)^3(5b^3)^9, 9(3a^2)^4(5b^3)^8\right]$$
$$= \left[210937500a^6b^{27},\ 284765625a^8b^{24}\right]$$

Exercise 8.2: Computing derivatives

We repeat **8.1**, using the first version of the definitions:

```
(define (f x y) (* (square x) (cube y)))
(define (g x y) (up (f x y) y))
(define (h x y) (f (f x y) y))
```

We then repeat the computations:

```
(show-expression (up (((partial 0) f) 'x 'y)
                     (((partial 1) f) 'x 'y)))
```

$$\begin{pmatrix} 2xy^3 \\ \\ 3x^2y^2 \end{pmatrix}$$

```
(show-expression (up (((partial 0) f) (f 'x 'y) 'y)
                     (((partial 1) f) (f 'x 'y) 'y)))
```

$$\begin{pmatrix} 2x^2y^6 \\ \\ 3x^4y^8 \end{pmatrix}$$

```
(show-expression (up (((partial 0) g) 'x 'y)
                     (((partial 1) g) 'x 'y)))
```

$$\begin{pmatrix} \begin{pmatrix} 2xy^3 \\ \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3x^2y^2 \\ \\ 1 \end{pmatrix} \end{pmatrix}$$

```
(show-expression ((D f) 'a 'b))
```

$$\begin{bmatrix} 2ab^3 \\ \\ 3a^2b^2 \end{bmatrix}$$

```
(show-expression ((D g) 3 5))
```

$$\begin{bmatrix} \begin{pmatrix} 750 \\ \\ 0 \end{pmatrix} \\ \begin{pmatrix} 675 \\ \\ 1 \end{pmatrix} \end{bmatrix}$$

```
(show-expression ((D h) (* 3 (square 'a)) (* 5 (cube 'b))))
```

$$\begin{bmatrix} 210937500a^6b^{27} \\ \\ 284765625a^8b^{24} \end{bmatrix}$$

We switch to a new set of definitions:

```
(define (f v)
  (let ((x (ref v 0))
        (y (ref v 1)))
    (* (square x) (cube y))))

(define (g v)
  (let ((x (ref v 0))
        (y (ref v 1)))
    (up (f v) y)))

(define (h v)
  (let ((y (ref v 1)))
       (f (up (f v) y))))
```

We repeat the process, with the second set of definitions:

```
(show-expression (up (((partial 0) f) (up 'x 'y))
                     (((partial 1) f) (up 'x 'y))))
```

$$\begin{pmatrix} 2xy^3 \\ \\ 3x^2y^2 \end{pmatrix}$$

```
(show-expression (up (((partial 0) f) (up (f (up 'x 'y)) 'y))
                     (((partial 1) f) (up (f (up 'x 'y)) 'y))))
```

$$\begin{pmatrix} 2x^2y^6 \\ \\ 3x^4y^8 \end{pmatrix}$$

```
(show-expression (up (((partial 0) g) (up 'x 'y))
                     (((partial 1) g) (up 'x 'y))))
```

$$\begin{pmatrix} \begin{pmatrix} 2xy^3 \\ \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3x^2y^2 \\ \\ 1 \end{pmatrix} \end{pmatrix}$$

```
(show-expression ((D f) (up 'a 'b)))
```

$$\begin{bmatrix} 2ab^3 \\ \\ 3a^2b^2 \end{bmatrix}$$

```
(show-expression ((D g) (up 3 5)))
```

$$\begin{bmatrix} \begin{pmatrix} 750 \\ \\ 0 \end{pmatrix} \\ \begin{pmatrix} 675 \\ \\ 1 \end{pmatrix} \end{bmatrix}$$

```
(show-expression ((D h) (up (* 3 (square 'a)) (* 5 (cube 'b)))))
```

$$\begin{bmatrix} 210937500a^6b^{27} \\ \\ 284765625a^8b^{24} \end{bmatrix}$$

We note that both of the results found by computer, and the results derived by hand are the same.

Exercise 1.8: Implementation of δ

(a) We define it below.

```
(define (((delta eta) f) q)
  ((D (make-g f q eta)) 0))

(define (make-g f q eta)
  (lambda (epsilon)
    (f (+ q (* eta epsilon)))))
```

We also define some literal functions, and a nicer looking alias, that we will use later.

```
(define eta (literal-function 'eta))
(define q (literal-function 'q))

(define (f q)
  (compose
   (literal-function 'f (-> (UP Real Real Real) Real))
   (Gamma q)))

(define (f_2 q)
  (compose
   (literal-function 'f_2 (-> (UP Real Real Real) Real))
   (Gamma q)))

(define (d eta f q)
  (((delta eta) f) q))

(define F (literal-function 'F (-> Real Real)))
(define ((h F f) q) (compose F (f q)))
```

(b) We now prove each of the results by evaluating the difference in the expressions.

1.23

```
(show-expression (- ((d eta (* f f_2) q) 't)
                    (+
                     ((* (d eta f q) (f_2 q)) 't)
                     ((* (f q) (d eta f_2 q)) 't)
                     )
                    )
                 )
```

0

1.24

```
(show-expression (- ((d eta (+ f f_2) q) 't)
                    (+ ((d eta f q) 't) ((d eta f_2 q) 't))))
```

0

1.25

```
(show-expression (- ((d eta (* 'c f) q) 't)
                    (* 'c ((d eta f q) 't))))
```

0

1.26

```
(show-expression (-
                  ((d eta (h F f) q) 't)
                  ((* (compose (D F) (f q))
                      (d eta f q)) 't)
                  ))
```

0

1.27

```
(show-expression (- ((D (d eta f q)) 't)
                    ((d eta (compose D f) q) 't)))
```

0

Exercuse 1.9: Lagrange's equations

(a) We have $L(t; x, y; v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) - \frac{1}{2}(x^2 + y^2) - x^2y + y^3/3$. We calculate the partial derivatives:

$$\partial_1 L = [-x - 2xy, -y - x^2 + y^2], \quad \partial_2 L = [mv_x, mv_y]$$

The derivative of the latter is $D\partial_2 L = [mDv_x, mDv_y]$. Equating each component, the Lagrange equations are:

$$-x - 2xy = mDv_x, \quad -y - x^2 + y^2 = mDv_y$$

(b) We have $L(t, \theta, \theta') = \frac{1}{2}ml^2\theta'^2 + mglcos(\theta)$. The partial derivates are:

$$\partial_1 L = -mglsin(\theta), \quad \partial_2 L = ml^2\theta'$$

The derivative of the latter is $D\partial_2 L = ml^2 D\theta'$. The Lagrange equation is:

$$-mglsin(\theta) = ml^2 D\theta'$$

(c) We have $L(t; \theta, \phi; \alpha, \beta) = \frac{1}{2}mR^2(\alpha^2 + (\beta sin\theta)^2)$. The partial derivates:

$$\partial_1 L = [mR^2\beta^2 cos(\theta)sin(\theta), 0], \quad \partial_2 L = [mR^2\alpha, mR^2\beta sin^2(\theta)]$$

The derivative of the latter is $D\partial_2 L = [mR^2 D\alpha, mR^2\left(sin^2(\theta)D\beta + 2\alpha\beta sin\theta cos\theta\right)]$. The Lagrange equations are:

$$\beta^2 cos(\theta)sin(\theta) = D\alpha, \quad sin^2(\theta)D\beta = -2\alpha\beta sin(\theta)cos(\theta)$$

Exercise 1.11:

We repeat **1.9**, using code to calculate the residuals. The Lagrange equations hold when the residuals are zero.

```
(define ((L-part-a m) local)
  (let ((x (ref (coordinate local) 0))
        (y (ref (coordinate local) 1))
        (vx (ref (velocity local) 0))
        (vy (ref (velocity local) 1)))
    (- (/ (* m (+ (square vx) (square vy))) 2)
       (+ (/ (+ (square x) (square y)) 2)
          (- (* (square x) y) (/ (cube y) 3))))))

(show-expression (((Lagrange-equations (L-part-a 'm))
                   (up (literal-function 'x) (literal-function 'y))) 't))
```

$$\begin{bmatrix} mD^2x(t) + 2x(t)\,y(t) + x(t) \\ \\ mD^2y(t) + (x(t))^2 - (y(t))^2 + y(t) \end{bmatrix}$$

```
(define ((L-part-b m g l) local)
  (let ((theta (coordinate local))
        (thetadot (velocity local)))
    (+ (/ (* m (* (square l) (square thetadot))) 2)
       (* (* m g) (* l (cos theta))))))

(show-expression (((Lagrange-equations (L-part-b 'm 'g 'l))
                   (literal-function 'theta)) 't))
```

$$glm\sin(\theta(t)) + l^2mD^2\theta(t)$$

```
(define ((L-part-c m r) local)
  (let ((alpha (ref (velocity local) 0))
        (beta (ref (velocity local) 1))
        (theta (ref (coordinate local) 0))
        )
    (* (/ m 2) (* (square r) (+ (square alpha) (square (* beta (sin theta))))))
    ))

(show-expression (((Lagrange-equations (L-part-c 'm 'r))
                   (up (literal-function 'theta) (literal-function 'phi))) 't))
```

$$\begin{bmatrix} -mr^2\sin(\theta(t))\cos(\theta(t))\,(D\phi(t))^2 + mr^2D^2\theta(t) \\ \\ 2mr^2D\theta(t)\sin(\theta(t))\cos(\theta(t))\,D\phi(t) + mr^2D^2\phi(t)\,(\sin(\theta(t)))^2 \end{bmatrix}$$

We see that in all cases, the expressions are equivalent.

1.21: The dumbbell

(a) Before we begin to balance the forces, we note that we can write θ as a function of Cartesian coordinates. Specifically, $cos\theta = \frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2}} = \frac{(x_1-x_0)}{l}$, and $sin\theta = \frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = \frac{(y_1-y_0)}{l}$. We note that, for the forces to balance, it must be that the force on each mass is equal and opposite the tension in the rod. We thus equate the x and y components of the forces with the x and y components of the tension:

$$F_{0,x} = m_0 D^2 x_0 = F\ cos(\theta) = F\frac{(x_1 - x_0)}{l}$$

$$F_{0,y} = m_0 D^2 y_0 = F\ sin(\theta) = F\frac{(y_1 - y_0)}{l}$$

Similarly, the other mass is governed by:

$$F_{1,x} = m_1 D^2 x_1 = -F\ cos(\theta) = -F\frac{(x_1 - x_0)}{l}$$

$$F_{1,y} = m_1 D^2 y_1 = -F\ sin(\theta) = -F\frac{(y_1 - y_0)}{l}$$

(b) We write the formal Lagrangian $L(t; x_0, y_0, x_1, y_1, F; \dot{x}_0, \dot{y}_0, \dot{x}_1, \dot{y}_1, \dot{F})$ according to the generic formula for a system of constrained particles.

$$L(t; x, F; \dot{x}, \dot{F}) = \sum_{\alpha} \frac{1}{2} m_\alpha \dot{x}_\alpha^2 - V(t, x) - \sum_{\{\alpha,\beta\}} \frac{F_{\alpha\beta}}{2l_{\alpha\beta}} \left[(x_\beta - x_\alpha)^2 - l_{\alpha\beta}^2\right]$$

We evaluate the three terms on the right.

$$\sum_{\alpha} \frac{m_\alpha}{2}\dot{x}_\alpha^2 = \frac{m_0}{2}(\dot{x}_0^2 + \dot{y}_0^2) + \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2)$$

$$V(t, x) = 0$$

$$\sum_{\{\alpha,\beta\}} \frac{F_{\alpha\beta}}{2l_{\alpha\beta}} \left[(x_\beta - x_\alpha)^2 - l_{\alpha\beta}^2\right] = \frac{F}{2l}\left[(x_1 - x_0)^2 + (y_1 - y_0)^2 - l^2\right]$$

Our Lagrangian is

$$L = \frac{m_0}{2}(\dot{x}_0^2 + \dot{y}_0^2) + \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) - \frac{F}{2l}\left[(x_1 - x_0)^2 + (y_1 - y_0)^2 - l^2\right]$$

We derive the Lagrangian equations:

$$\partial_1 L = \begin{bmatrix} \frac{F}{l}(x_1 - x_0) \\ \frac{F}{l}(y_1 - y_0) \\ -\frac{F}{l}(x_1 - x_0) \\ -\frac{F}{l}(y_1 - y_0) \\ -\frac{1}{2l}\left[(x_1 - x_0)^2 + (y_1 - y_0)^2 - l^2\right] \end{bmatrix},\ \partial_2 L = \begin{bmatrix} m_0\dot{x}_0 \\ m_0\dot{y}_0 \\ m_1\dot{x}_1 \\ m_1\dot{y}_1 \\ 0 \end{bmatrix},\ D\partial_2 L = \begin{bmatrix} m_0 D\dot{x}_0 \\ m_0 D\dot{y}_0 \\ m_1 D\dot{x}_1 \\ m_1 D\dot{y}_1 \\ 0 \end{bmatrix}$$

Equating the elements of the first and last, we reproduce the relations found in **(a)**, and an additional one:

$$m_0 D^2 x_0 = F\frac{(x_1 - x_0)}{l},\ m_0 D^2 y_0 = F\frac{(y_1 - y_0)}{l},\ m_1 D^2 x_1 = -F\frac{(x_1 - x_0)}{l},\ m_1 D^2 y_1 = -F\frac{(y_1 - y_0)}{l}$$

$$\frac{1}{2l}\left[(x_1 - x_0)^2 + (y_1 - y_0)^2 - l^2\right] = 0$$

(c) We now use the generalized coordinates

$$x_{cm} = \frac{m_0 x_0 + m_1 x_1}{m_0 + m_1},\ y_{cm} = \frac{m_0 y_0 + m_1 y_1}{m_0 + m_1},\ \theta = Arctan\left(\frac{y_1 - y_0}{x_1 - x_0}\right), c = l$$

We want to find a way to express $(m_0 + m_1)\dot{x}_{cm}^2$ in terms of x_0 and x_1:

$$(m_0 + m_1)\dot{x}_{cm}^2 = (m_0 + m_1)\left(\frac{m_0 x_0 + m_1 x_1}{m_0 + m_1}\right)^2 = \frac{1}{(m_0 + m_1)}\left(m_0^2 x_0^2 + m_1^2 x_1^2 + 2m_0 m_1 x_0 x_1\right)^2$$

We note that $\frac{m_0}{m_0+m_1} = 1 - \frac{m_1}{m_0+m_1}$ and that $\frac{m_0^2}{m_0+m_1} = m_0 - \frac{m_0 m_1}{m_0+m_1}$.

$$= m_0 x_0^2 - \frac{m_0 m_1}{m_0 + m_1}x_0^2 + m_1 x_1^2 + \frac{m_0 m_1}{m_0 + m_1}x_1^2 + 2\frac{m_0 m_1}{m_0 + m_1}x_0 x_1 = m_0 x_0^2 + m_1 x_1^2 - \frac{m_0 m_1}{m_0 + m_1}(x_0 - x_1)^2$$

Similarly,

$$(m_0 + m_1)\dot{y}_{cm}^2 = m_0 y_0^2 + m_1 y_1^2 - \frac{m_0 m_1}{m_0 + m_1}(y_0 - y_1)^2$$

Combining these, we have:

$$(m_0 + m_1)(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) = m_0(x_0^2 + y_0^2) + m_1(x_1^2 + y_1^2) - \frac{m_0 m_1}{m_0 + m_1}\left((x_0 - x_1)^2 + (y_0 - y_1)^2\right)$$

The last term is just $-\frac{m_0 m_1}{m_0+m_1}\dot{c}^2$, so we have:

$$\frac{m_0}{2}(x_0^2 + y_0^2) + \frac{m_1}{2}(x_1^2 + y_1^2) = \frac{1}{2}\frac{m_0 m_1}{m_0 + m_1}\dot{c}^2 + \frac{m_0 + m_1}{2}(\dot{x}_{cm}^2 + \dot{y}_{cm}^2)$$

This replaces the first part of the old Lagrangian, and the second part becomes

$$-\frac{F}{2l}\left[c^2 - l^2\right]$$

Thus the new Lagrangian is

$$L(t; x_{cm}, y_{cm}, \theta, c, F; \dot{x}_{cm}, \dot{y}_{cm}, \dot{\theta}, \dot{c}, \dot{F}) = \frac{1}{2}\frac{m_0 m_1}{m_0 + m_1}\dot{c}^2 + \frac{m_0 + m_1}{2}(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) - \frac{F}{2l}\left[c^2 - l^2\right]$$

We derive the Lagrange equations:

$$\partial_1 L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{Fc}{l} \\ -\frac{c^2 - l^2}{2l} \end{bmatrix},\ \partial_2 L = \begin{bmatrix} (m_0 + m_1)\dot{x}_{cm} \\ (m_0 + m_1)\dot{y}_{cm} \\ 0 \\ \frac{m_0 m_1}{m_0+m_1}\dot{c} \\ 0 \end{bmatrix},\ D\partial_2 L = \begin{bmatrix} (m_0 + m_1)D^2 x_{cm} \\ (m_0 + m_1)D^2 y_{cm} \\ 0 \\ \frac{m_0 m_1}{m_0+m_1}D^2 c \\ 0 \end{bmatrix}$$

(d)
We note that $c(t) = l$ will cause the last equation to be true, which imples that $Dc = 0, D^2c = 0$. The resulting equations of motion are therefore:

$$(m_0 + m_1)D^2x_{cm} = 0,\ (m_0 + m_1)D^2y_{cm} = 0,\ c = l$$

(e)
We construct a new Lagrangian, using the fact that $L = T - V$ is a suitable Lagrangian for a constrained system. There is no potential energy to consider, so the Lagrangian is simply the kinetic energy of the system.

$$L(t; x_{cm}, y_{cm}, \theta; \dot{x}_{cm}, \dot{y}_{cm}, \dot{\theta}) = \frac{m_0 + m_1}{2}\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right)$$

We derive the Lagrange equations:

$$\partial_1 L = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \partial_2 L = \begin{bmatrix} (m_0 + m_1)\dot{x}_{cm} \\ (m_0 + m_1)\dot{y}_{cm} \\ 0 \end{bmatrix},\ D\partial_2 L = \begin{bmatrix} (m_0 + m_1)D^2x_{cm} \\ (m_0 + m_1)D^2y_{cm} \\ 0 \end{bmatrix}$$

These are the same equations we derived in part **(d)**.