Exercise 2.2: Steiner’s Theorem

The moment of inertia with respect to the first line is given by

\[ I = \sum_{\alpha} m_{\alpha} (\xi_{\alpha}^\perp)^2, \]

where \( m_{\alpha} \) is the mass of the individual particle, and \( \xi_{\alpha} \) is the vector from the reference point to the individual point. Since the axis of rotation goes through the center of mass, the first line goes through the reference point.

Let \( \vec{R} \) be a perpendicular vector from the first line to the second, and let \( \xi_{\alpha}^\perp' = \xi_{\alpha}^\perp + \vec{R} \) be the reference vector from the second line to the individual particle. The new moment of inertia \( I' \) is given by:

\[
I' = \sum_{\alpha} m_{\alpha} (\xi_{\alpha}^\perp')^2 = \sum_{\alpha} m_{\alpha} (\xi_{\alpha}^\perp + \vec{R})^2 = \sum_{\alpha} m_{\alpha} \left( (\xi_{\alpha}^\perp)^2 + 2\xi_{\alpha}^\perp \vec{R} + \vec{R}^2 \right) = \sum_{\alpha} m_{\alpha} (\xi_{\alpha}^\perp)^2 + \sum_{\alpha} m_{\alpha} 2\xi_{\alpha}^\perp \vec{R} + \vec{R}^2 \sum_{\alpha} m_{\alpha} = I + \sum_{\alpha} m_{\alpha} 2\xi_{\alpha}^\perp \vec{R} + \vec{R}^2
\]

By the definition of the center of mass, \( \sum_{\alpha} m_{\alpha} \xi_{\alpha}^\perp = 0 \), and the middle term drops out. So

\[ I' = I + M\vec{R}^2 \]

Exercise 2.3: Some useful moments of inertia

In the continuous case, we can replace the equation for the moment of inertia with

\[ I = \int_M (r_{\perp})^2 dm, \]

where \( r_{\perp} \) is the perpendicular distance from the axis of rotation to the position of \( dm \), and we integrate over the entire mass.

(a) We introduce a density \( \delta = \frac{dm}{dV} = \frac{M}{4\pi R^3} \). Consider a spherical coordinate system, and let the axis of rotation be the z-axis.

\[
I = \int_M r_{\perp}^2 dm = \int_V r_{\perp}^2 \frac{dm}{dV} dV = \delta \int_{\rho=0}^{\rho=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} r_{\perp}^2 (\rho^2 \sin \phi) d\phi d\theta d\rho
\]

Since the axis of rotation is the z-axis, the perpendicular distance is just \( r_{\perp} = \rho \sin \phi \), and we continue the calculation:\n
\[
I = \delta \int_{\rho=0}^{\rho=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (\rho \sin \phi)^2 \rho^2 \sin \phi d\phi d\theta d\rho = \delta 2\pi \int_{\rho=0}^{\rho=R} \int_{\phi=0}^{\phi=\pi} \rho^4 \sin^3 \phi d\phi d\rho
\]

1This only contains the first four of six problems. I’m turning the other two problems in on Tuesday.
2We use the easily verifiable fact that \( \int \sin^3(x)dx = \frac{2}{3}\sin^2(x)\cos(x) + \frac{1}{3}\cos(x) \).
\[ I = \int_{M} r^2 dm = \int_{V} r^2 \frac{dm}{dV} dV = \delta \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} (r^2 \sin \phi \cos \phi + \frac{2}{3} \cos \phi) r^2 \sin \phi \cos \phi d\phi d\theta dr 
\]

Since the axis of rotation is the \( z \)-axis, the perpendicular distance is just \( r_{\perp} = R \sin \phi \), and we continue the calculation:

\[ I = \delta R^2 \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (R \sin \phi)^2 \sin \phi \cos \phi d\phi d\theta = \frac{4}{3} \pi R^4 \int_{\phi=0}^{\phi=\pi} \sin^3 \phi d\phi = \delta R^4 2\pi \int_{\phi=0}^{\phi=\pi} \frac{\sin^2 \phi \cos \phi + \frac{2}{3} \cos \phi}{3} \phi = \frac{8}{3} \pi M R^4 \frac{4 \pi}{12 \pi R^2} = \frac{2}{3} \pi M R^2 \]

**Exercise 2.4: Jupiter**

(a) We can write the rotational inertia of both a sphere with equal density and a planet as \( \int_{V} \frac{dm}{dV} r^2 dV \). In the case of the planet, \( \frac{dm}{dV} \) is a constant density, but in the case of the planet, this term is large when \( r \) is small and small when \( r \) is large. Thus, those mass infinitesimals contribute less to the total value of the integral, since they are integrated at a lower value of \( r \) than in the case of the sphere. Thus, the moment of inertia is smaller for a planet than for a sphere of uniform density of the same mass and radius.

(b) We use a spherical coordinate system, and align the \( z \)-axis with the axis of rotation. The variable density is given by \( \frac{dm}{dV} = \frac{M \sin (\pi r/R)}{4 \pi R^2} \).

\[ I = \int_{M} r^2 dm = \int_{V} r^2 \frac{dm}{dV} dV = \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \left( r^2 \sin \phi \cos \phi \right) r^2 \sin \phi \cos \phi d\phi d\theta dr 
\]

Since the axis of rotation is the \( z \)-axis, the perpendicular distance is just \( r_{\perp} = R \sin \phi \), and we continue the calculation:

\[ I = 2\pi \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=\pi} \left( r^2 \sin \phi \right) r \sin (\pi r/R) \sin \phi \cos \phi d\phi d\theta dr 
\]
\[= 2\pi \frac{M}{4R^2} \int_{r=0}^{r=R} r^3 \sin(\pi r/R) \, dr \int_{\phi=0}^{\phi=\pi} \sin^3 \phi \, d\phi\]

We know from 2.3 (a) that the integral on the right is \(\frac{4}{3}\). We look up the integral on the left on the internet\(^3\).

\[= 2\pi \frac{M}{4R^2} \left[ 3\left(\frac{(\pi)}{R}\right)^2 r^2 - 2)\sin(\pi r/R) - r\left(\frac{(\pi)}{R}\right)^2 r^2 - 6)\cos(\pi r/R) \right]_{r=0}^{r=R} \left[ \frac{4}{3} \right] \]

\[= 2\pi \frac{M}{3R^2} \left[ (0 - \frac{R(\frac{(\pi)}{R})^2 R^2 - 6)\cos(\pi)}{(\frac{(\pi)}{R})^3} - (0 - 0) \right] = \frac{2M\pi}{3R^2} \left[ \frac{R^4}{\pi^2} - \frac{6}{\pi^3} \right] \]

\[= \frac{2MR^2}{3} \frac{\pi^2 - 6}{\pi^2} = \frac{2}{3} \left( 1 - \frac{6}{\pi^2} \right) MR^2 \]

Which is less than the moment of inertia of a sphere with equal, albeit evenly distributed, mass - \(\frac{2}{3} MR^2\).

**Exercise 2.5: A constraint on the moments of inertia**

Let there be some arrangement of particles \(\{\alpha\}\) with mass \(\{m_\alpha\}\) and position \(r_\alpha = r_{\alpha,1}\hat{e}_1 + r_{\alpha,2}\hat{e}_2 + r_{\alpha,3}\hat{e}_3\) with respect to some set of orthonormal axes \(\{\hat{e}_i\}\). Consider the sum of any two moments of inertia, say \(I_{11}\) and \(I_{22}\), and compare it with \(I_{33}\).

\[I_{11} + I_{22} = \sum_\alpha m_\alpha \left( (\hat{e}_1 \times r_\alpha)^2 + (\hat{e}_2 \times r_\alpha)^2 \right) = \sum_\alpha m_\alpha \left( (r_{\alpha,1}^2 + r_{\alpha,2}^2 + r_{\alpha,3}^2) \right) \]

\[I_{33} = \sum_\alpha m_\alpha (\hat{e}_3 \times r_\alpha)^2 = \sum_\alpha m_\alpha (r_{\alpha,1}^2 + r_{\alpha,2}^2) \]

So \(I_{11} + I_{22} \geq I_{33}\). (In particular, there is only equality if \(\sum_\alpha r_{\alpha,3}^2 = 0\).) By symmetry, this is true for any choice of orthonormal axes. The sum of any two moments of inertia are greater than or equal to the third.

**Exercise 2.6: Principal moments of inertia**

(a) We choose a set of axes, and show that they are the principal axes. Let \((0,0,0)\) be the origin and center of mass, and let \(r_1 = (\frac{1}{\sqrt{3}}, 0, -\frac{1}{2}\sqrt{\frac{2}{3}})\), \(r_2, r_3 = (\frac{1}{2\sqrt{3}}, \pm \frac{1}{2}, -\frac{1}{2}\sqrt{\frac{2}{3}})\), and \(p_4 = (0, 0, \frac{3}{4}\sqrt{\frac{2}{3}})\) be the four points. Thus, the distance between any two points is 1. The orthogonal axis we choose is \((\hat{x}, \hat{y}, \hat{z})\). We intend to calculate the components of the moments of inertia for every combination \((i, j)\):

\[I_{ij} = \sum_\alpha m_\alpha (e_i \times r_\alpha) \cdot (e_j \times r_\alpha) \]

Before we do this, however, we determine the cross product of the axes with the different \(\{r_i\}\):

\(^3\)http://integrals.wolfram.com/index.jsp
The principal axes of inertia are the axes for which the moment of inertia is a maximum or minimum. We now find the principal moments of inertia.

(b) Let the cube with side length $R$ centered at $(0,0,0)$ be described by $-\frac{R}{2} \leq x, y, z \leq \frac{R}{2}$. We will show that $(\hat{x}, \hat{y}, \hat{z})$ are the principal axes. Let the cube have a uniform density $\delta = \frac{M}{R^3}$.

First, we compute the off-diagonal inertial moments. Let $r(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$ be the vector from the origin to the point at $(x, y, z)$.

The inertial moment $I_{\alpha\beta}$ is calculated as:

$$I_{\alpha\beta} = \sum m \alpha \beta \alpha \beta$$

where $\alpha, \beta = x, y, z$.

We calculate the off-diagonal components $I_{xy}, I_{yz}$ and $I_{xz}$, and let the mass term be a constant.

$$I_{xy} = m \sum_\alpha (\hat{x} \times r_\alpha) \cdot (\hat{y} \times r_\alpha) = m(0 + \frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} + 0) = 0$$

$$I_{yz} = m \sum_\alpha (\hat{y} \times r_\alpha) \cdot (\hat{z} \times r_\alpha) = m(0 + \frac{1}{8\sqrt{3}} - \frac{1}{8\sqrt{3}} + 0) = 0$$

$$I_{xz} = m \sum_\alpha (\hat{x} \times r_\alpha) \cdot (\hat{z} \times r_\alpha) = m(\frac{\sqrt{3}}{12} - \frac{\sqrt{3}}{24} - \frac{\sqrt{3}}{24} + 0) = 0$$

By symmetry, $I_{yx} = I_{yx} = I_{zx} = 0$, and the off-diagonal elements are zero. Thus, our chosen axes are the principal axes. We now find the principal moments of inertia.

$$I_{xx} = m \sum_\alpha (\hat{x} \times r_\alpha) \cdot (\hat{x} \times r_\alpha) = m \frac{12}{163} + 2m \left(\frac{1}{163} + \frac{1}{4}\right) + m \frac{9}{163} = m$$

$$I_{yy} = m \sum_\alpha (\hat{y} \times r_\alpha) \cdot (\hat{y} \times r_\alpha) = m \left(\frac{12}{163} + \frac{1}{3}\right) + 2m \left(\frac{1}{163} + \frac{1}{12}\right) + m \frac{9}{163} = m$$

$$I_{zz} = m \sum_\alpha (\hat{z} \times r_\alpha) \cdot (\hat{z} \times r_\alpha) = m \frac{1}{3} + 2m \left(\frac{1}{4} + \frac{1}{12}\right) + 0 = m$$

Let $R$ be the distance between the points, or, the scale of the arrangement. We chose $R$ to be 1 earlier, but we can extend this calculation to a general tetrahedral arrangement by multiplying the principal moments of inertia by $R^2$. Thus, the principal axes are $\hat{x}, \hat{y}, \hat{z}$, and the corresponding principal moments of inertia are each $mR^2$, which is what we wanted to show.
\[ I_{xy} = \int_V \delta(\hat{x} \times r) \cdot (\hat{y} \times r) dV = \delta \int_V (-z\hat{y} + y\hat{z}) \cdot (z\hat{x} - x\hat{z}) dV = \delta \int_V -xydV \]

But by symmetry, this is zero, and so are the rest of the off-diagonal components. Thus, the axes we have chosen are the principal axes, and we go to calculate the principal moments.

\[ I_{xx} = \int_V \delta(\hat{x} \times r) \cdot (\hat{x} \times r) dV = \delta \int_V (-z\hat{y} + y\hat{z}) \cdot (-z\hat{y} + y\hat{z}) dV = \delta \int_V z^2 + y^2 dV \]

\[ = \delta \int_V z^2 + y^2 dV = \delta R \int_{-R/2}^{R/2} \int_{-R/2}^{R/2} z^2 + y^2 dz dy = 2\delta R^2 \int_{-R/2}^{R/2} z^2 dz = 2\delta R^2 \frac{1}{12} R^3 = \frac{1}{6} MR^2 \]

Similarly, \( I_{yy} = I_{zz} = \frac{1}{6} MR^2 \).

(c) The center of mass of this arrangement is \((\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + 0 + 0, \frac{0+0+0+0+0}{5}, \frac{0+0+0+0+1}{5}) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5})\).

Our plan of attack is to choose any set of orthogonal axes, and to calculate the tensor matrix \( I \). The eigenvalues of the matrix are the principal moments of inertia, and the eigenvectors are the principal axes. Since it contributes only a multiplicative term, we let the mass \( m_\alpha = 1 \) for all \( \alpha \).

We choose \((\hat{x}, \hat{y}, \hat{z})\) as our axes, and calculate the cross product of each of these with the relative displacement vectors for the points. First we calculate these new vectors:

\[ r_1 = \left(\frac{4}{5}, \frac{1}{5}, \frac{1}{5}\right), \quad r_2 = \left(-\frac{6}{5}, \frac{1}{5}, \frac{1}{5}\right), \quad r_3 = \left(\frac{4}{5}, \frac{4}{5}, -\frac{1}{5}\right) \]

\[ r_4 = \left(-\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), \quad r_5 = \left(-\frac{1}{5}, -\frac{1}{5}, \frac{4}{5}\right) \]

Then the crossproducts.

\[ \hat{x} \times r_1 = \frac{1}{5} \hat{y} - \frac{1}{5} \hat{z}, \quad \hat{y} \times r_1 = \frac{1}{5} \hat{x} - \frac{4}{5} \hat{z}, \quad \hat{z} \times r_1 = \frac{1}{5} \hat{x} + \frac{4}{5} \hat{y} \]

\[ \hat{x} \times r_2 = \frac{1}{5} \hat{y} - \frac{1}{5} \hat{z}, \quad \hat{y} \times r_2 = \frac{1}{5} \hat{x} + \frac{6}{5} \hat{z}, \quad \hat{z} \times r_2 = \frac{1}{5} \hat{x} - \frac{6}{5} \hat{y} \]

\[ \hat{x} \times r_3 = \frac{1}{5} \hat{y} + \frac{4}{5} \hat{z}, \quad \hat{y} \times r_3 = \frac{1}{5} \hat{x} - \frac{4}{5} \hat{z}, \quad \hat{z} \times r_3 = -\frac{4}{5} \hat{x} + \frac{4}{5} \hat{y} \]

\[ \hat{x} \times r_4 = \frac{1}{5} \hat{y} - \frac{1}{5} \hat{z}, \quad \hat{y} \times r_4 = \frac{1}{5} \hat{x} + \frac{1}{5} \hat{z}, \quad \hat{z} \times r_4 = \frac{1}{5} \hat{x} - \frac{1}{5} \hat{y} \]

\[ \hat{x} \times r_5 = -\frac{4}{5} \hat{y} - \frac{1}{5} \hat{z}, \quad \hat{y} \times r_5 = \frac{4}{5} \hat{x} + \frac{1}{5} \hat{z}, \quad \hat{z} \times r_5 = \frac{1}{5} \hat{x} - \frac{1}{5} \hat{y} \]

Now we calculate entries in the matrix \( I \), using \( I_{ij} = \sum_\alpha (\hat{e}_i \times r_\alpha) \cdot (\hat{e}_j \times r_\alpha) \).
\[
I_{xx} = \frac{2}{25} + \frac{2}{25} + \frac{17}{25} + \frac{2}{25} + \frac{17}{25} = \frac{8}{5}
\]

\[
I_{yy} = \frac{17}{25} + \frac{37}{25} + \frac{17}{25} + \frac{2}{25} + \frac{17}{25} = \frac{18}{5}
\]

\[
I_{zz} = \frac{17}{25} + \frac{37}{25} + \frac{32}{25} + \frac{2}{25} + \frac{2}{25} = \frac{18}{5}
\]

And the off-diagonal moments.

\[
I_{xy} = \frac{4}{25} - \frac{6}{25} - \frac{16}{25} - \frac{1}{25} - \frac{1}{25} = \frac{4}{5}
\]

\[
I_{yz} = -\frac{1}{25} - \frac{1}{25} + \frac{4}{25} - \frac{1}{25} + \frac{4}{25} = \frac{1}{5}
\]

\[
I_{xz} = \frac{4}{25} - \frac{6}{25} + \frac{4}{25} - \frac{1}{25} + \frac{4}{25} = \frac{1}{5}
\]

Since the \( I^T = I \), we have

\[
I = \frac{1}{5} \begin{pmatrix} 8 & 4 & 1 \\ 4 & 18 & 1 \\ 1 & 1 & 18 \end{pmatrix}
\]

I used Scientific Python (SciPy) to solve for the eigenvalues and eigenvectors of this matrix.

```python
>>> import scipy; import scipy.linalg

>>> scipy.linalg.eig(1/5. * scipy.array([[8,4,1],[4,18,1],[1,1,18]]))
(array([ 1.31273832+0.j, 4.03274534+0.j, 3.45451633+0.j]),
 array([-0.94401077, 0.30599901, -0.12332182],
 [ 0.32545081, 0.80245031, -0.50015525],
 [ 0.05408738, 0.51228712, 0.85710936])))
```

Thus, the principal axes are, to three decimal places, 
\(-.944\hat{x} + .305\hat{y} + -.123\hat{z}, .325\hat{x} + .802\hat{y} + -.500\hat{z}\), and 
\(.054\hat{x} + .512\hat{y} + .857\hat{z}\), with associated principal moments of inertia \(1.313m\), \(4.033m\) and \(3.455m\).