# 6.946 Assignment 4[1]

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3 October 2006
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## Exercise 2.2: Steiner's Theorem

The moment of inertia with respect to the first line is given by $I = \sum_\alpha m_\alpha (\xi_\alpha^\perp)^2$, where $m_\alpha$ is the mass of the individual particle, and $\xi_\alpha$ is the vector from the reference point to the individual point. Since the axis of rotation goes through the center of mass, the first line goes through the reference point.

Let $\vec{R}$ be a perpendicular vector from the first line to the second, and let $\xi_\alpha^{\perp\prime} = \xi_\alpha^\perp + \vec{R}$ be the reference vector from the second line to the individual particle. The new moment of inertia $I'$ is given by:

$$I' = \sum_\alpha m_\alpha (\xi_\alpha^{\perp\prime})^2 = \sum_\alpha m_\alpha (\xi_\alpha^\perp + \vec{R})^2 = \sum_\alpha m_\alpha \left( (\xi_\alpha^\perp)^2 + 2\xi_\alpha^\perp \vec{R} + R^2 \right) =$$

$$= \sum_\alpha m_\alpha (\xi_\alpha^\perp)^2 + \sum_\alpha m_\alpha 2\xi_\alpha^\perp \vec{R} + R^2 \sum_\alpha m_\alpha = I + \sum_\alpha m_\alpha 2\xi_\alpha^\perp \vec{R} + MR^2$$

By the definition of the center of mass, $\sum_\alpha m_\alpha \xi_\alpha^\perp = 0$, and the middle term drops out. So

$$I' = I + MR^2$$

## Exercise 2.3: Some useful moments of inertia

In the continuous case, we can replace the equation for the moment of inertia with

$$I = \int_M (r_\perp)^2 dm,$$

where $r_\perp$ is the perpendicular distance from the axis of rotation to the position of $dm$, and we integrate over the entire mass.

**(a)** We introduce a density $\delta = \frac{dm}{dV} = \frac{M}{\frac{4}{3}\pi R^3}$. Consider a spherical coordinate system, and let the axis of rotation be the z-axis.

$$I = \int_M r_\perp^2 dm = \int_V r_\perp^2 \frac{dm}{dV} dV = \delta \int_{\rho=0}^{\rho=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} r_\perp^2 (\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho)$$

Since the axis of rotation is the z-axis, the perpendicular distance is just $r_\perp = \rho \sin\phi$, and we continue the calculation[2]:

$$I = \delta \int_{\rho=0}^{\rho=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (\rho \sin\phi)^2 \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho = \delta 2\pi \int_{\rho=0}^{\rho=R} \int_{\phi=0}^{\phi=\pi} \rho^4 \sin^3\phi \, d\phi \, d\rho$$

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[1] This only contains the first four of six problems. I'm turning the other two problems in on Tuesday.

[2] We use the easily verifiable fact that $\int \sin^3(x)dx = \frac{2}{3} sin^2(x) cos(x) + \frac{1}{3} cos(x)$.

$$= \delta 2\pi \frac{R^5}{5} \int_{\phi=0}^{\phi=\pi} \sin^3 \phi \, d\phi = \delta 2\pi \frac{R^5}{5} \left[ \frac{1}{3} \sin^2 \phi \cos \phi + \frac{2}{3} \cos \phi \right]_{\phi=0}^{\phi=\pi}$$

$$= \delta 2\pi \frac{R^5}{5} \frac{4}{3} = \frac{M}{\frac{4}{3}\pi R^3} 2\pi \frac{R^5}{5} \frac{4}{3} = \frac{24\pi M R^5}{60\pi R^3} = \frac{2}{5} M R^2$$

**(b)** We introduce a mass-per-surface-area constant $\delta = \frac{dm}{dS} = \frac{M}{4\pi R^2}$. Consider a spherical coordinate system, and let the axis of rotation be the z-axis.

$$I = \int_M r_\perp^2 dm = \int_S r_\perp^2 \frac{dm}{dS} dS = \delta \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} r_\perp^2 (R^2 \sin \phi \, d\phi \, d\theta)$$

Since the axis of rotation is the z-axis, the perpendicular distance is just $r_\perp = R \sin \phi$, and we continue the calculation:

$$I = \delta R^2 \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (R \sin \phi)^2 \sin \phi \, d\phi \, d\theta \;= \delta R^4 2\pi \int_{\phi=0}^{\phi=\pi} \sin^3 \phi \, d\phi$$

$$= \delta R^4 2\pi \left[ \frac{1}{3} \sin^2 \phi \cos \phi + \frac{2}{3} \cos \phi \right]_{\phi=0}^{\phi=\pi} = \delta R^4 2\pi \frac{4}{3} = \frac{M}{4\pi R^2} R^4 2\pi \frac{4}{3} = \frac{8\pi M R^4}{12\pi R^2} = \frac{2}{3} M R^2$$

## Exercise 2.4: Jupiter

**(a)** We can write the rotational inertia of both a sphere with equal density and a planet as $\int_V \frac{dm}{dV} r^2 dV$. In the case of the planet, $\frac{dm}{dV}$ is a constant density, but in the case of the planet, this term is large when $r$ is small and small when $r$ is large. Thus, those mass infinitesimals contribute less to the total value of the integral, since they are integrated at a lower value of $r$ than in the case of the sphere. Thus, the moment of inertia is smaller for a planet than for a sphere of uniform density of the same mass and radius.

**(b)** We use a spherical coordinate system, and align the z-axis with the axis of rotation. The variable density is given by $\frac{dm}{dV} = \frac{M}{R^3} \frac{\sin(\pi r/R)}{4r/R}$.

$$I = \int_M r_\perp^2 dm = \int_V r_\perp^2 \frac{dm}{dV} dV = \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (r_\perp^2) \left( \frac{M}{R^3} \frac{\sin(\pi r/R)}{4r/R} \right) (r^2 \sin \phi \, d\phi \, d\theta \, dr)$$

$$= \frac{M}{4R^2} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (r_\perp^2) \; r \sin(\pi r/R) \sin \phi \, d\phi \, d\theta \, dr$$

$$= 2\pi \frac{M}{4R^2} \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=\pi} (r_\perp^2) \; r \sin(\pi r/R) \sin \phi \, d\phi \, dr$$

Since the axis of rotation is the z-axis, the perpendicular distance is just $r_\perp = r \sin \phi$, and we continue the calculation:

$$= 2\pi \frac{M}{4R^2} \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=\pi} (r^2 \sin^2 \phi) \; r \sin(\pi r/R) \sin \phi \, d\phi \, dr$$

$$= 2\pi \frac{M}{4R^2} \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=\pi} \sin^2 \phi \; r^3 \sin(\pi r/R) \sin \phi \, d\phi \, dr$$

$$= 2\pi \frac{M}{4R^2} \int_{r=0}^{r=R} r^3 \sin(\pi r/R) \; dr \int_{\phi=0}^{\phi=\pi} \sin^3 \phi \; d\phi$$

We know from **2.3 (a)** that the integral on the right is $\frac{4}{3}$. We look up the integral on the left on the internet[3].

$$= 2\pi \frac{M}{4R^2} \left[ \frac{3(\left(\frac{\pi}{R}\right)^2 r^2 - 2)\sin(\pi r/R)}{\left(\frac{\pi}{R}\right)^4} - \frac{r(\left(\frac{\pi}{R}\right)^2 r^2 - 6)\cos(\pi r/R)}{\left(\frac{\pi}{R}\right)^3} \right]_{r=0}^{r=R} \left[\frac{4}{3}\right]$$

$$= 2\pi \frac{M}{3R^2} \left[ (0 - \frac{R(\left(\frac{\pi}{R}\right)^2 R^2 - 6)\cos(\pi)}{\left(\frac{\pi}{R}\right)^3}) - (0 - 0) \right] = \frac{2M\pi}{3R^2} \left[ R^4 \frac{\pi^2 - 6}{\pi^3} \right]$$

$$= \frac{2MR^2}{3} \frac{\pi^2 - 6}{\pi^2} = \frac{2}{3} \left( 1 - \frac{6}{\pi^2} \right) MR^2$$

Which is less than the moment of inertia of a sphere with equal, albeit evenly distributed, mass - $\frac{2}{3}MR^2$.

### Exercise 2.5: A constraint on the moments of inertia

Let there be some arrangement of particles $\{\alpha\}$ with mass $\{m_\alpha\}$ and position $r_\alpha = r_{\alpha,1}\hat{e}_1 + r_{\alpha,2}\hat{e}_2 + r_{\alpha,3}\hat{e}_3$ with respect to some set of orthonormal axes $\{\hat{e}_i\}$. Consider the sum of any two moments of inertia, say $I_{11}$ and $I_{22}$, and compare it with $I_{33}$.

$$I_{11} + I_{22} = \sum_\alpha m_\alpha \left( (\hat{e}_1 \times r_\alpha)^2 + (\hat{e}_2 \times r_\alpha)^2 \right) = \sum_\alpha m_\alpha \left( (r_{\alpha,2}^2 + r_{\alpha,3}^2) + (r_{\alpha,1}^2 + r_{\alpha,3}^2) \right)$$

$$I_{33} = \sum_\alpha m_\alpha \left( \hat{e}_3 \times r_\alpha \right)^2 = \sum_\alpha m_\alpha \left( r_{\alpha,1}^2 + r_{\alpha,2}^2 \right)$$

So $I_{11} + I_{22} \geq I_{33}$. (In particular, there is only equality if $\sum_\alpha r_{\alpha,3}^2 = 0$.) By symmetry, this is true for any choice of orthonormal axes. The sum of any two moments of inertia are greater than or equal to the third.

### Exercise 2.6: Principal moments of inertia

**(a)** We choose a set of axes, and show that they are the principal axes. Let $(0, 0, 0)$ be the origin and center of mass, and let $r_1 = (\frac{1}{\sqrt{3}}, 0, -\frac{1}{4}\sqrt{\frac{2}{3}})$, $r_2, r_3 = (-\frac{1}{2\sqrt{3}}, \pm\frac{1}{2}, -\frac{1}{4}\sqrt{\frac{2}{3}})$, and $p_4 = (0, 0, \frac{3}{4}\sqrt{\frac{2}{3}})$ be the four points. Thus, the distance between any two points is 1. The orthogonal axis we choose is $(\hat{x}, \hat{y}, \hat{z})$. We intend to calculate the components of the moments of inertia for every combination $(i, j)$:

$$I_{ij} = \sum_\alpha m_\alpha (e_i \times r_\alpha) \cdot (e_j \times r_\alpha)$$

Before we do this, however, we determine the cross product of the axes with the different $\{r_i\}$:

[3] http://integrals.wolfram.com/index.jsp

$$\hat{x} \times r_1 = \frac{1}{4}\sqrt{\frac{2}{3}}\hat{y},\ \hat{y} \times r_1 = -\frac{1}{4}\sqrt{\frac{2}{3}}\hat{x} - \frac{1}{\sqrt{3}}\hat{z},\ \hat{z} \times r_1 = \frac{1}{\sqrt{3}}\hat{y}$$

$$\hat{x} \times r_2 = \frac{1}{4}\sqrt{\frac{2}{3}}\hat{y} + \frac{1}{2}\hat{z},\ \hat{y} \times r_2 = -\frac{1}{4}\sqrt{\frac{2}{3}}\hat{x} + \frac{1}{2\sqrt{3}}\hat{z},\ \hat{z} \times r_2 = -\frac{1}{2}\hat{x} - \frac{1}{2\sqrt{3}}\hat{y}$$

$$\hat{x} \times r_3 = \frac{1}{4}\sqrt{\frac{2}{3}}\hat{y} - \frac{1}{2}\hat{z},\ \hat{y} \times r_3 = -\frac{1}{4}\sqrt{\frac{2}{3}}\hat{x} + \frac{1}{2\sqrt{3}}\hat{z},\ \hat{z} \times r_3 = \frac{1}{2}\hat{x} - \frac{1}{2\sqrt{3}}\hat{y}$$

$$\hat{x} \times r_4 = -\frac{3}{4}\sqrt{\frac{2}{3}}\hat{y},\ \hat{y} \times r_4 = \frac{3}{4}\sqrt{\frac{2}{3}}\hat{x},\ \hat{z} \times r_4 = 0$$

We calculate the off-diagonal components $I_{xy}$, $I_{yz}$ and $I_{xz}$, and let the mass term be a constant.

$$I_{xy} = m\sum_{\alpha}(\hat{x} \times r_\alpha) \cdot (\hat{y} \times r_\alpha) = m(0 + \frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} + 0) = 0$$

$$I_{yz} = m\sum_{\alpha}(\hat{y} \times r_\alpha) \cdot (\hat{z} \times r_\alpha) = m(0 + \frac{1}{8}\sqrt{\frac{2}{3}} - \frac{1}{8}\sqrt{\frac{2}{3}} + 0) = 0$$

$$I_{xz} = m\sum_{\alpha}(\hat{x} \times r_\alpha) \cdot (\hat{z} \times r_\alpha) = m(\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{24} - \frac{\sqrt{2}}{24} + 0) = 0$$

By symmetry, $I_{yx} = I_{zy} = I_{zx} = 0$, and the off-diagonal elements are zero. Thus, our chosen axes are the principal axes. We now find the principal moments of inertia.

$$I_{xx} = m\sum_{\alpha}(\hat{x} \times r_\alpha) \cdot (\hat{x} \times r_\alpha) = m\frac{1}{16}\frac{2}{3} + 2m\left(\frac{1}{16}\frac{2}{3} + \frac{1}{4}\right) + m\frac{9}{16}\frac{2}{3} = m$$

$$I_{yy} = m\sum_{\alpha}(\hat{y} \times r_\alpha) \cdot (\hat{y} \times r_\alpha) = m\left(\frac{1}{16}\frac{2}{3} + \frac{1}{3}\right) + 2m\left(\frac{1}{16}\frac{2}{3} + \frac{1}{12}\right) + m\frac{9}{16}\frac{2}{3} = m$$

$$I_{zz} = m\sum_{\alpha}(\hat{z} \times r_\alpha) \cdot (\hat{z} \times r_\alpha) = m\frac{1}{3} + 2m\left(\frac{1}{4} + \frac{1}{12}\right) + 0 = m$$

Let $R$ be the distance between the points, or, the scale of the arrangement. We chose $R$ to be 1 earlier, but we can extend this calculation to a general tetrahedral arrangement by multiplying the principal moments of inertia by $R^2$. Thus, the principal axes are $\hat{x}, \hat{y}, \hat{z}$, and the corresponding principal moments of inertia are each $mR^2$, which is what we wanted to show.

**(b)** Let the cube with side length $R$ centered at $(0,0,0)$ be described by $-\frac{R}{2} \leq x, y, z \leq \frac{R}{2}$. We will show that $(\hat{x}, \hat{y}, \hat{z})$ are the principal axes. Let the cube have a uniform density $\delta = \frac{M}{R^3}$.

First, we compute the off-diagonal inertial moments. Let $r(x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}$ be the vector from the origin to the point at $(x,y,z)$.

$$I_{xy} = \int_V \delta(\hat{x} \times r) \cdot (\hat{y} \times r) dV = \delta \int_V (-z\hat{y} + y\hat{z}) \cdot (z\hat{x} - x\hat{z}) dV = \delta \int_V -xy dV$$

But by symmetry, this is zero, and so are the rest of the off-diagonal components. Thus, the axes we have chosen are the principal axes, and we go to calculate the principal moments.

$$I_{xx} = \int_V \delta(\hat{x} \times r) \cdot (\hat{x} \times r) dV = \delta \int_V (-z\hat{y} + y\hat{z}) \cdot (-z\hat{y} + y\hat{z}) dV = \delta \int_V z^2 + y^2 dV$$

$$= \delta \int_V z^2 + y^2 dV = \delta R \int_{-R/2}^{R/2} \int_{-R/2}^{R/2} z^2 + y^2 \ dz dy = 2\delta R^2 \int_{-R/2}^{R/2} z^2 dz = 2\delta R^2 \frac{1}{12} R^3 = \frac{1}{6} M R^2$$

Similarly, $I_{yy} = I_{zz} = \frac{1}{6} M R^2$.

**(c)** The center of mass of this arrangement is $\left(\frac{-1+1+1+0+0}{5}, \frac{0+0+1+0+0}{5}, \frac{0+0+0+0+1}{5}\right) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. Our plan of attack is to choose any set of orthogonal axes, and to calculate the tensor matrix $I$. The eigenvalues of the matrix are the principal moments of inertia, and the eigenvectors are the principal axes. Since it contributes only a multiplicative term, we let the mass $m_\alpha = 1$ for all $\alpha$.

We choose $(\hat{x}, \hat{y}, \hat{z})$ as our axes, and calculate the cross product of each of these with the relative displacement vectors for the points. First we calculate these new vectors:

$$r_1 = \left(\frac{4}{5}, -\frac{1}{5}, -\frac{1}{5}\right), \ r_2 = \left(-\frac{6}{5}, -\frac{1}{5}, -\frac{1}{5}\right), \ r_3 = \left(\frac{4}{5}, \frac{4}{5}, -\frac{1}{5}\right)$$

$$r_4 = \left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}\right), \ r_5 = \left(-\frac{1}{5}, -\frac{1}{5}, \frac{4}{5}\right)$$

Then the crossproducts.

$$\hat{x} \times r_1 = \frac{1}{5}\hat{y} - \frac{1}{5}\hat{z}, \ \hat{y} \times r_1 = -\frac{1}{5}\hat{x} - \frac{4}{5}\hat{z}, \ \hat{z} \times r_1 = \frac{1}{5}\hat{x} + \frac{4}{5}\hat{y}$$

$$\hat{x} \times r_2 = \frac{1}{5}\hat{y} - \frac{1}{5}\hat{z}, \ \hat{y} \times r_2 = -\frac{1}{5}\hat{x} + \frac{6}{5}\hat{z}, \ \hat{z} \times r_2 = \frac{1}{5}\hat{x} - \frac{6}{5}\hat{y}$$

$$\hat{x} \times r_3 = \frac{1}{5}\hat{y} + \frac{4}{5}\hat{z}, \ \hat{y} \times r_3 = -\frac{1}{5}\hat{x} - \frac{4}{5}\hat{z}, \ \hat{z} \times r_3 = -\frac{4}{5}\hat{x} + \frac{4}{5}\hat{y}$$

$$\hat{x} \times r_4 = \frac{1}{5}\hat{y} - \frac{1}{5}\hat{z}, \ \hat{y} \times r_4 = -\frac{1}{5}\hat{x} + \frac{1}{5}\hat{z}, \ \hat{z} \times r_4 = \frac{1}{5}\hat{x} - \frac{1}{5}\hat{y}$$

$$\hat{x} \times r_5 = -\frac{4}{5}\hat{y} - \frac{1}{5}\hat{z}, \ \hat{y} \times r_5 = \frac{4}{5}\hat{x} + \frac{1}{5}\hat{z}, \ \hat{z} \times r_5 = \frac{1}{5}\hat{x} - \frac{1}{5}\hat{y}$$

Now we calculate entries in the matrix $I$, using $I_{ij} = \sum_\alpha (\hat{e}_i \times r_\alpha) \cdot (\hat{e}_i \times r_\alpha)$.

$$I_{xx} = \frac{2}{25} + \frac{2}{25} + \frac{17}{25} + \frac{2}{25} + \frac{17}{25} = \frac{8}{5}$$

$$I_{yy} = \frac{17}{25} + \frac{37}{25} + \frac{17}{25} + \frac{2}{25} + \frac{17}{25} = \frac{18}{5}$$

$$I_{zz} = \frac{17}{25} + \frac{37}{25} + \frac{32}{25} + \frac{2}{25} + \frac{2}{25} = \frac{18}{5}$$

And the off-diagonal moments.

$$I_{xy} = \frac{4}{25} - \frac{6}{25} - \frac{16}{25} - \frac{1}{25} - \frac{1}{25} = \frac{4}{5}$$

$$I_{yz} = -\frac{1}{25} - \frac{1}{25} + \frac{4}{25} - \frac{1}{25} + \frac{4}{25} = \frac{1}{5}$$

$$I_{xz} = \frac{4}{25} - \frac{6}{25} + \frac{4}{25} - \frac{1}{25} + \frac{4}{25} = \frac{1}{5}$$

Since the $I^T = I$, we have

$$I = \frac{1}{5}\begin{pmatrix} 8 & 4 & 1 \\ 4 & 18 & 1 \\ 1 & 1 & 18 \end{pmatrix}$$

I used Scientific Python (SciPy) to solve for the eigenvalues and eigenvectors of this matrix.

```
>>> import scipy; import scipy.linalg

>>> scipy.linalg.eig(1/5. * scipy.array([[8,4,1],[4,18,1],[1,1,18]]))
(array([ 1.31273832+0.j,  4.03274534+0.j,  3.45451633+0.j]),
 array([[-0.94401077,  0.30599901, -0.12332182],
       [ 0.3254508 ,  0.80245031, -0.50015525],
       [ 0.05408738,  0.51228712,  0.85710936]]))
```

Thus, the principal axes are, to three decimal places, $-.944\hat{x} + .305\hat{y} + -.123\hat{z}$, $.325\hat{x} + .802\hat{y} + -.500\hat{z}$, and $.054\hat{x} + .512\hat{y} + .857\hat{x}$, with associated principal moments of inertia 1.313m, 4.033m and 3.455m.