6.946 Assignment 6

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Exercise 3.1: Deriving Hamilton's equations

(a) The momentum of the system is given by

$$p = \partial_2 L(t, \theta, \dot{\theta}) = ml^2\dot{\theta}$$

Thus, we have $\dot{\theta}(p) = \frac{p}{ml^2}$. The Hamiltonian, in terms of p is:

$$H(t, \theta, p) = p\dot{\theta}(p) - L(t, \theta, \dot{\theta}(p)) = \frac{p^2}{ml^2} - L(t, \theta, \frac{p}{ml^2})$$

$$= \frac{p^2}{ml^2} - \frac{1}{2}ml^2\left(\frac{p}{ml^2}\right)^2 - mgl\cos(\theta) = \frac{1}{2}\frac{p^2}{ml^2} - mgl\cos(\theta)$$

The Hamiltonian equations are given by $D\theta = \partial_2 H$ and $Dp = -\partial_1 H$.

$$D\theta = \partial_2 H(t, q, p) = \frac{p}{ml^2}$$

$$Dp = -\partial_1 H(t, q, p) = -mgl\sin(\theta)$$

As expected, the first of these is merely a restatement of the relationship between momentum and velocity.

(b) The momentum of the system is given by

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \partial_2 L(t; x, y; \dot{x}, \dot{y}) = \begin{bmatrix} m\dot{x} \\ m\dot{y} \end{bmatrix}$$

Thus, we have $\dot{x} = \frac{p_x}{m}$ and $\dot{y} = \frac{p_y}{m}$. The Hamiltonian, in terms of p is:

$$H(t; x, y; p) = p \cdot \begin{bmatrix} \dot{x}(p_x) \\ \dot{y}(p_y) \end{bmatrix} - L(t; x, y; \dot{x}(p_x), \dot{y}(p_y))$$

$$= (p_x\frac{p_x}{m} + p_y\frac{p_y}{m}) - \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2\right) + V(x, y)$$

$$= \frac{1}{2m}(p_x^2 + p_y^2) + V(x, y)$$

The Hamiltonian equations are given by $Dq = \partial_2 H$ and $Dp = -\partial_1 H$.

$$Dq = \begin{bmatrix} Dx \\ Dy \end{bmatrix} = \partial_2 H = \begin{bmatrix} \frac{p_x}{m} \\ \frac{p_y}{m} \end{bmatrix}$$

$$Dp = \begin{bmatrix} Dp_x \\ Dp_y \end{bmatrix} = -\partial_1 H = \begin{bmatrix} -\frac{d}{dx}V(x,y) \\ -\frac{d}{dy}V(x,y) \end{bmatrix} = \begin{bmatrix} -x - 2xy \\ -y - x^2 + y^2 \end{bmatrix}$$

The first pair of these are a restatement of the relationship between momentum and velocity. The second pair relates the rate of change of momentum with the gradient of the potential energy.

(c) The momentum of the system is given by

$$p = \begin{bmatrix} p_\theta \\ p_\phi \end{bmatrix} = \partial_2 L(t;\theta,\phi;\dot{\theta},\dot{\phi}) = \begin{bmatrix} mR^2\dot{\theta} \\ mR^2\dot{\phi}\sin^2\theta \end{bmatrix}$$

Thus, we have $\dot{\theta} = \frac{p_\theta}{mR^2}$ and $\dot{\phi} = \frac{p_\phi}{mR^2 \sin^2\theta}$. The Hamiltonian, in terms of p is:

$$H(t;\theta,\phi;p) = p \cdot \begin{bmatrix} \dot{\theta}(p_\theta) \\ \dot{\phi}(p_\phi) \end{bmatrix} - L(t;\theta,\phi;\dot{\theta}(p_\theta),\dot{\phi}(p_\phi))$$

$$= (p_\theta \frac{p_\theta}{mR^2} + p_\phi \frac{p_\phi}{mR^2\sin^2\theta}) - \frac{1}{2}mR^2\left(\left(\frac{p_\theta}{mR^2}\right)^2 + \left(\frac{p_\phi}{mR^2\sin^2\theta}\right)^2 \sin^2\theta\right)$$

$$= \frac{1}{2mR^2}(p_\theta^2 + p_\phi^2 \frac{1}{\sin^2\theta})$$

The Hamiltonian equations are given by $Dq = \partial_2 H$ and $Dp = -\partial_1 H$.

$$Dq = \begin{bmatrix} D\theta \\ D\phi \end{bmatrix} = \partial_2 H = \begin{bmatrix} \frac{p_\theta}{mR^2} \\ \frac{p_\phi}{mR^2\sin^2\theta} \end{bmatrix}$$

$$Dp = \begin{bmatrix} Dp_\theta \\ Dp_\phi \end{bmatrix} = -\partial_1 H = \begin{bmatrix} \frac{1}{mR^2} p_\phi^2 \frac{\cos\theta}{\sin^3\theta} \\ 0 \end{bmatrix}$$

The first pair of these are a restatement of the relationship between momentum and velocity.

Exercise 3.3: Computing Hamilton's equations

(a) We define H and compute Hamilton's equations.

```
(define ((H-part-a m l g) H-state)
  (let
      ((theta (coordinate H-state))
       (p (momentum H-state)))
    (- (/ (square p) (* 2 m (square l))) (* m g l (cos theta)))))

(show-expression
 (((Hamilton-equations
    (H-part-a 'm 'l 'g))
   (literal-function 'theta) (literal-function 'p)) 't))
```

$$\left(\begin{array}{c} 0 \\ D\theta(t) - \frac{p(t)}{l^2 m} \\ glm\sin(\theta(t)) + Dp(t) \end{array}\right)$$

These are the same as in **(3.1)**.

(b) We define H and $V(x, y)$ and compute Hamilton's equations.

```
(define ((H-part-b m V) H-state)
  (let
      ((q (coordinate H-state))
       (p (momentum H-state)))
    (+ (/ (square p) (* 2 m)) (V (ref q 0) (ref q 1)))))

(define (V x y)
  (+ (/ (+ (square x) (square y)) 2) (* (square x) y) (/ (cube y) -3)))

(show-expression
 (((Hamilton-equations
    (H-part-b 'm V))
   (up (literal-function 'x) (literal-function 'y))
   (down (literal-function 'p_x) (literal-function 'p_x))) 't))
```

$$\left(\begin{array}{c} 0 \\ \left(\begin{array}{c} Dx(t) - \frac{p_x(t)}{m} \\ Dy(t) - \frac{p_x(t)}{m} \end{array}\right) \\ \left[\begin{array}{c} 2y(t)\,x(t) + Dp_x(t) + x(t) \\ -(y(t))^2 + (x(t))^2 + Dp_x(t) + y(t) \end{array}\right] \end{array}\right)$$

These are the same as in **(3.1)**.

(c) We define H and compute Hamilton's equations.

```
(define ((H-part-c m R) H-state)
  (let ((theta (ref (coordinate H-state) 0))
        (p_theta (ref (momentum H-state) 0))
        (p_phi (ref (momentum H-state) 1)))
    (- (+
        (/ (square p_theta) (* m R R))
        (/ (square p_phi) (* m (square (* R (sin theta))))))
```

```
      (* (/ (* m R R) 2)
         (+
          (square (/ p_theta (* m R R)))
          (square (/ p_phi (* m R R (sin theta))))
          )))))

(show-expression
 (((Hamilton-equations
    (H-part-c 'm 'R))
   (up (literal-function 'theta) (literal-function 'phi))
   (down (literal-function 'p_theta) (literal-function 'p_phi))) 't))
```

$$\begin{pmatrix} 0 \\ \begin{pmatrix} D\theta(t) - \dfrac{p_\theta(t)}{R^2 m} \\ D\phi(t) - \dfrac{p_\phi(t)}{R^2 m (\sin(\theta(t)))^2} \end{pmatrix} \\ \begin{bmatrix} Dp_\theta(t) - \dfrac{(p_\phi(t))^2 \cos(\theta(t))}{R^2 m (\sin(\theta(t)))^3} \\ Dp_\phi(t) \end{bmatrix} \end{pmatrix}$$

These are the same as in **(3.1)**.

Exercise 3.4: Simple Legendre transforms

(a) We are looking for a function $G(y)$ with the following properties:

$$y = \partial_0 F(x), \quad xy = F(x) + G(y), \quad x = \partial_0 G(y)$$

The first of these gives us $y = a + 2bx$. In terms of x, this is $x = \frac{1}{2b}(y - a)$. The second of these gives us

$$xy = ax + bx^2 + G(y)$$

$$G(y) = xy - ax - bx^2 = xy - xy + bx^2 = bx^2$$

$$G(y) = b\left(\frac{y-a}{2b}\right)^2 = \frac{1}{4b}(y-a)^2$$

We double-check that the third relation is correct: $\partial_0 G(y) = \frac{1}{2b}(y-a) = x$. So $G(y) = \frac{1}{4b}(y-a)^2$ is the function related to $F(x)$ through a Legendre transform.

(c) Let $w = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$ be the active argument in $G(x, y; w_x, w_y)$. Consider the relationships between F and G given in **3.48**. The first one gives us:

$$w = \begin{bmatrix} w_x \\ w_y \end{bmatrix} = \partial_1 F = \begin{bmatrix} 2x\dot{x} + 3\dot{y} \\ 3\dot{x} + 2y\dot{y} \end{bmatrix}$$

We can rewrite the active arguments of G in terms of the active arguments of F.

$$2yw_x - 3w_y = 4xy\dot{x} - 9\dot{x} = (4xy-9)\dot{x}$$

$$2xw_y - 3w_x = 4xy\dot{y} - 9\dot{y} = (4xy-9)\dot{y}$$

The third relationship gives us

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \partial_1 G = \begin{bmatrix} dG/dw_x \\ dG/dw_y \end{bmatrix}$$

Solving both of these differential equations gives us:

$$G = \frac{yw_x^2}{4xy-9} - 3\frac{w_x w_y}{4xy-9} + f(w_y)$$

$$G = \frac{xw_y^2}{4xy-9} - 3\frac{w_x w_y}{4xy-9} + f(w_x)$$

Combining these, we have an expression for G:

$$G(t;x,y;w_x,w_y) = \frac{yw_x^2 + xw_y^2}{4xy-9} - 3\frac{w_x w_y}{4xy-9}$$

Let's double-check the other expressions, starting with the second one:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix} = G + F$$

The left side of the equality is

$$\dot{x}w_x + \dot{y}w_y = 2x\dot{x}^2 + 3\dot{y}\dot{x} + 3\dot{x}\dot{y} + 2y\dot{y}^2 = 2F$$

We must have $G = F$ for the two sides to be equal. Let's expand $(4xy-9)G$:

$$(4xy-9)G = yw_x^2 + xw_y^2 - 3w_x w_y =$$

$$y(4x^2\dot{x}^2 + 12x\dot{x}\dot{y} + 9\dot{y}^2) + x(4y^2\dot{y}^2 + 12y\dot{x}\dot{y} + 9\dot{x}^2) - 3(6x\dot{x}^2 + 9\dot{x}\dot{y} + 6y\dot{y}^2 + 4xy\dot{x}\dot{y})$$
$$= x\dot{x}^2(4xy + 9 - 18) + y\dot{y}^2(4xy + 9 - 18) + \dot{x}\dot{y}(12xy + 12xy - 27 - 12xy)$$
$$= (4xy-9)F$$

Which is what we wanted to show. Now consider the last equality. We must show

$$0 = \partial_0 F + \partial_0 G = \begin{bmatrix} \dot{x}^2 \\ \dot{y}^2 \end{bmatrix} + \begin{bmatrix} dG/dx \\ dG/dy \end{bmatrix}$$

Consider dG/dx:

$$\frac{d}{dx}G = \frac{d}{dx}\left(\frac{yw_x^2 + xw_y^2}{4xy-9}\right) - 3\frac{d}{dx}\left(\frac{w_x w_y}{4xy-9}\right)$$

$$\frac{d}{dx}G = \frac{(4xy-9)(w_y^2) - (yw_x^2 + xw_y^2)(4y)}{(4xy-9)^2} + 3\left(\frac{w_x w_y (4y)}{(4xy-9)^2}\right)$$

$$(4xy-9)^2 \frac{dG}{dx} = (4xy-9)(w_y^2) - (yw_x^2 + xw_y^2)(4y) + 3w_x w_y (4y)$$

$$(4xy-9)^2 \frac{dG}{dx} = 4xyw_y^2 - 9w_y^2 - 4y^2 w_x^2 - 4yxw_y^2 + 12yw_x w_y$$

$$(4xy-9)^2 \frac{dG}{dx} = -9w_y^2 - 4y^2 w_x^2 + 12yw_x w_y$$

$$(4xy-9)^2 \frac{dG}{dx} = -(2yw_x - 3w_y)^2$$

$$\frac{dG}{dx} = -\frac{(2yw_x - 3w_y)^2}{(4xy-9)^2}$$

$$\frac{dG}{dx} = -\dot{x}^2$$

And therefore $dG/dx + \dot{x}^2 = 0$. Similarly, the bottom of the equality pair holds as well. All four relations hold with our choice of G.

Thus, $G(x, y; w_x, w_y) = \frac{yw_x^2 + xw_y^2}{4xy-9} - 3\frac{w_x w_y}{4xy-9}$ is the function related to $F(x, y; \dot{x}, \dot{y})$ through a Legendre transform.

Exercise 3.5:

The total time derivative of a conserved quantity is zero, so we must have $DH(t, q(t), p(t)) = 0$.

$$DH(t, q(t), p(t)) = \partial_0 H + \partial_1 H \cdot Dq(t) + \partial_2 H Dp(t)$$

Hamilton's equations hold along a solution path:

$$\partial_0 H + \partial_1 H \cdot Dq(t) + \partial_2 H \cdot Dp(t) = \partial_0 H + (-Dp(t))Dq(t) + (Dq(t))Dp(t) = \partial_0 H$$

If $H(t, q(t), p(t))$ has no explicit time-dependence, then $\partial_0 H = 0$, $DH = 0$, and H is thus a conserved quantity, which is what we wanted to show.