6.946 Assignment 7

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Exercise 3.8: Sleeping top

Consider the potential energy $U_p(\theta)$. It is given that $\theta = 0$ is a local minimum or maximum, depending on the value of p. If the second derivative of U_p is positive, it is a local minimum, and a local maximum when the second derivative of U_p is negative. We compute this second derivative at $\theta = 0$:

$$U_p(\theta) = \frac{p^2}{2C} + \frac{p^2}{2A}\tan^2\frac{\theta}{2} + gMR\cos(\theta) = \frac{p^2}{2C} + \frac{p^2}{2A}\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} + gMR\cos(\theta)$$

$$\frac{dU_p(\theta)}{d\theta} = \frac{p^2}{2A}\frac{\cos^2(\theta/2)2\ \sin(\theta/2)\cos(\theta/2)\frac{1}{2} - \sin^2(\theta/2)2\ \cos(\theta/2)(-\sin(\theta/2))\frac{1}{2}}{\cos^4(\theta/2)} - gMR\sin(\theta)$$

$$= \frac{p^2}{2A}\frac{\cos(\theta/2)\sin(\theta/2)}{\cos^4(\theta/2)} - gMR\sin(\theta) = \frac{p^2}{2A}\frac{\sin(\theta/2)}{\cos^3(\theta/2)} - gMR\sin(\theta)$$

$$\frac{d^2U_p(\theta)}{d\theta^2} = \frac{p^2}{2A}\frac{\cos^3(\theta/2)\cos(\theta/2)\frac{1}{2} - \sin(\theta/2)3\ \cos^2(\theta/2)(-\sin(\theta/2))\frac{1}{2}}{\cos^6(\theta/2)} - gMR\cos(\theta)$$

$$= \frac{p^2}{4A}\frac{\cos^2(\theta/2) + 3\ \sin^2(\theta/2)}{\cos^4(\theta/2)} - gMR\cos(\theta)$$

$$\left.\frac{d^2U_p(\theta)}{d\theta^2}\right|_{\theta=0} = \frac{p^2}{4A}\frac{\cos^2(0) + 3\ \sin^2(0)}{\cos^4(0)} - gMR\cos(0) = \frac{p^2}{4A} - gMR$$

This is positive (corresponding to a local *minimum*) when

$$\frac{p^2}{4A} - gMR > 0 \quad \Rightarrow \quad p > \sqrt{4gMRA}$$

And negative (corresponding to a local *maximum*) when $p < \sqrt{4gMRA}$. Thus, the critical angular velocity $\omega_c = \frac{p_c}{C}$ above which an axisymmetric top can sleep is given by

$$w_c = \frac{\sqrt{4gMRA}}{C}$$

Exercise 3.10: Fun with phase portraits

I decided to investigate a pendulum on a rotating pivot with displacement A and period ω. In Cartesian coordinates, the location and velocity of the pivot point is given by

$$x_0(t) = A\sin(\omega t), \quad y_0(t) = A\cos(\omega t)$$

$$\dot{x}_0(t) = A\omega\cos(\omega t), \quad \dot{y}_0(t) = -A\omega\sin(\omega t)$$

If the pendulum is a bob of mass m, hanging on a massless string of length l and making an angle θ with the gravity normal, we can describe its position and velocity with

$$x(t,\theta) = x_0(t) + l\,\sin(\theta), \quad y(\theta,t) = y_0(t) + l\,\cos(\theta)$$

$$\dot{x}(t,\theta,\dot{\theta}) = \dot{x}_0(t) + l\,\cos(\theta)\dot{\theta}, \quad \dot{y}(t,\theta,\dot{\theta}) = \dot{y}_0(t) - l\,\sin(\theta)\dot{\theta}$$

The gravitational potential of the pendulum is given by $V_g = -mgy(t)$. We note that we can completely describe the pivot and pendulum system with only a single parameter θ and its derivative. A valid Lagrangian for the system is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy(t)$$

We write a procedure that represents this Lagrangian, and then convert it into a Hamiltonian, and then to a state derivative ready for integration.

```
(define ((L-rot-pend m l g A omega) state)
  (let ((theta (coordinate state))
        (thetadot (velocity state))
        (t (time state)))
    (let ((x (+ (* A (sin (* omega t))) (* l (sin theta))))
          (y (+ (* A (cos (* omega t))) (* l (cos theta))))
          (xdot (+ (* A omega (cos (* omega t))) (* l (cos theta) thetadot)))
          (ydot (+ (* -1 A omega (sin (* omega t))) (* -1 l (sin theta) thetadot)))
          )
      (+
       (* .5 m (+ (square xdot) (square ydot)))
       (* m g y)))))

(define (H-rot-pend-sysder m l g A omega)
  (Hamiltonian->state-derivative
   (Lagrangian->Hamiltonian
    (L-rot-pend m l g A omega))))
```

We setup some plotting procedures:

```
(define ((monitor-p-theta win) state)
  (let ((q ((principal-value :pi) (coordinate state)))
        (p (momentum state)))
    (plot-point win q p)))

(define (rot-pendulum-map m l g A omega)
  (let ((advance (state-advancer H-rot-pend-sysder m l g A omega))
        (map-period (/ :2pi omega)))
    (lambda (theta ptheta return fail)
      (let ((ns (advance
                 (up 0 theta ptheta)
```

```
                 map-period)))
         (return ((principal-value :pi) (coordinate ns))
                 (momentum ns))))))

(define win (frame :-pi :pi -10 30))
```

I chose to investigate the surface of section for the following initial conditions $m = 1$ kg, $l = 1$ m, $g = 9.8$ m/s^2, $A = .04$ m and $\omega = 5.2\ \omega_0$, where $\omega_0 = \sqrt{\frac{g}{l}}$ is the natural frequency of the pendulum.

```
(let ((m 1.)
      (l 1.)
      (g 9.8)
      (A .04))
  (let ((omega0 (sqrt (/ g l))))
    (let ((omega (* 5.2 omega0)))
      (explore-map
       win
       (rot-pendulum-map m l g A omega)
       200))))
```

Some trajectories are plotted in Figures 1 and 2. There is a large chaotic region in the middle range of momenta, with major and minor islands of stability.

Exercise 3.13: Fun with Henon's quadratic map

(a) Consider the Jacobian determinant of the map.

$$\left|\frac{\partial(x',y')}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\alpha + 2x\sin\alpha & -\sin\alpha \\ \sin\alpha - 2x\cos\alpha & \cos\alpha \end{vmatrix} =$$

$$(\cos^2\alpha + 2x\cos\alpha\sin\alpha) - (-\sin^2\alpha + 2x\sin\alpha\cos\alpha) = \cos^2\alpha + \sin^2\alpha = 1$$

Thus the map preserves area.

(b) We implement that map as a procedure that returns the failure condition when the orbit escapes from the area of interest $-1 \le x \le 1$, $-1 \le y \le 1$.

```
(define ((quad-map alpha) x y return fail)
  (let ((xprime (- (* x (cos alpha)) (* (- y (square x)) (sin alpha))))
        (yprime (+ (* x (sin alpha)) (* (- y (square x)) (cos alpha)))))
    (if (or (or (> xprime 1) (< xprime -1)) (or (> yprime 1) (< yprime -1)))
        (fail)
    (return xprime yprime))))
```

(c) We create a frame, and explore the map for different values of α.

```
(define window (frame -1. 1. -1. 1.))
(explore-map window (quad-map 1.0) 2000)
```

The Figures below show the map for values of $\alpha = 1.2, 1.23, 1.26, 1.29, 1.32, 1.35$. There is a marked evolution of the map around this parameter region of α. As α evolves, islands of stability form and move outwards until they disappear entirely or escape the main region of stability.

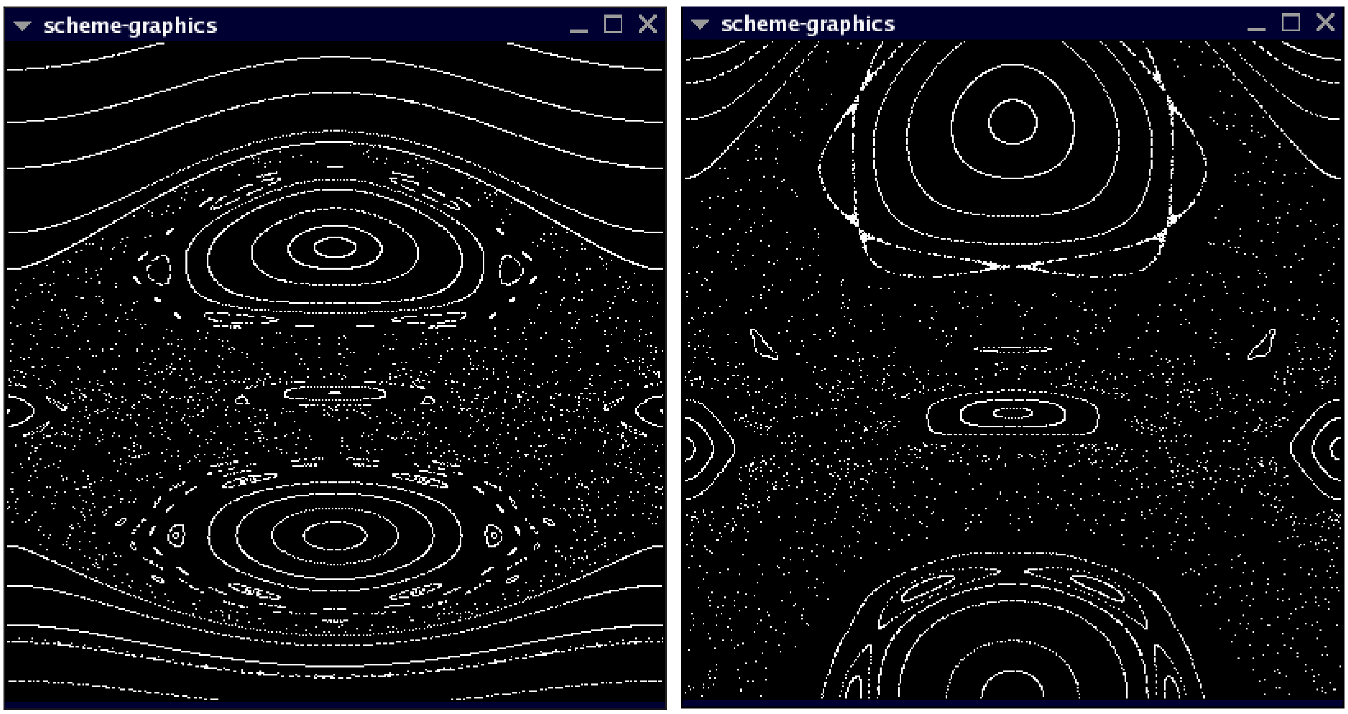

Figure 1: Both plots show several trajectories. On the left, the coordinates range from $-10 \leq p_\theta \leq 30$ and $-\pi \leq \theta \leq \pi$. On the right, the coordinates range from $0 \leq p_\theta \leq 20$ and $-\pi \leq \theta \leq \pi$.

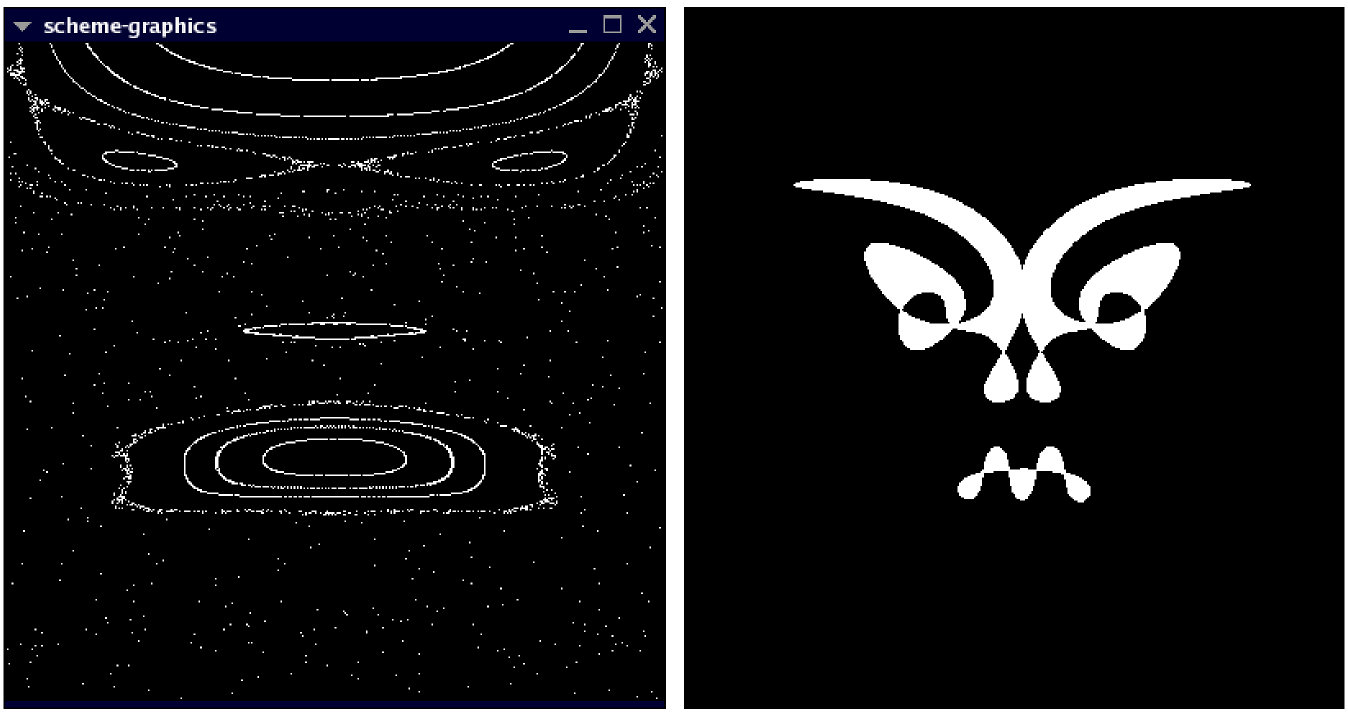

Figure 2: An eerie plot of several trajectories. On the left, the coordinates range from $5 \leq p_\theta \leq 15$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. On the right is Krotus, the Dark Lord of Fifth East, for comparison.

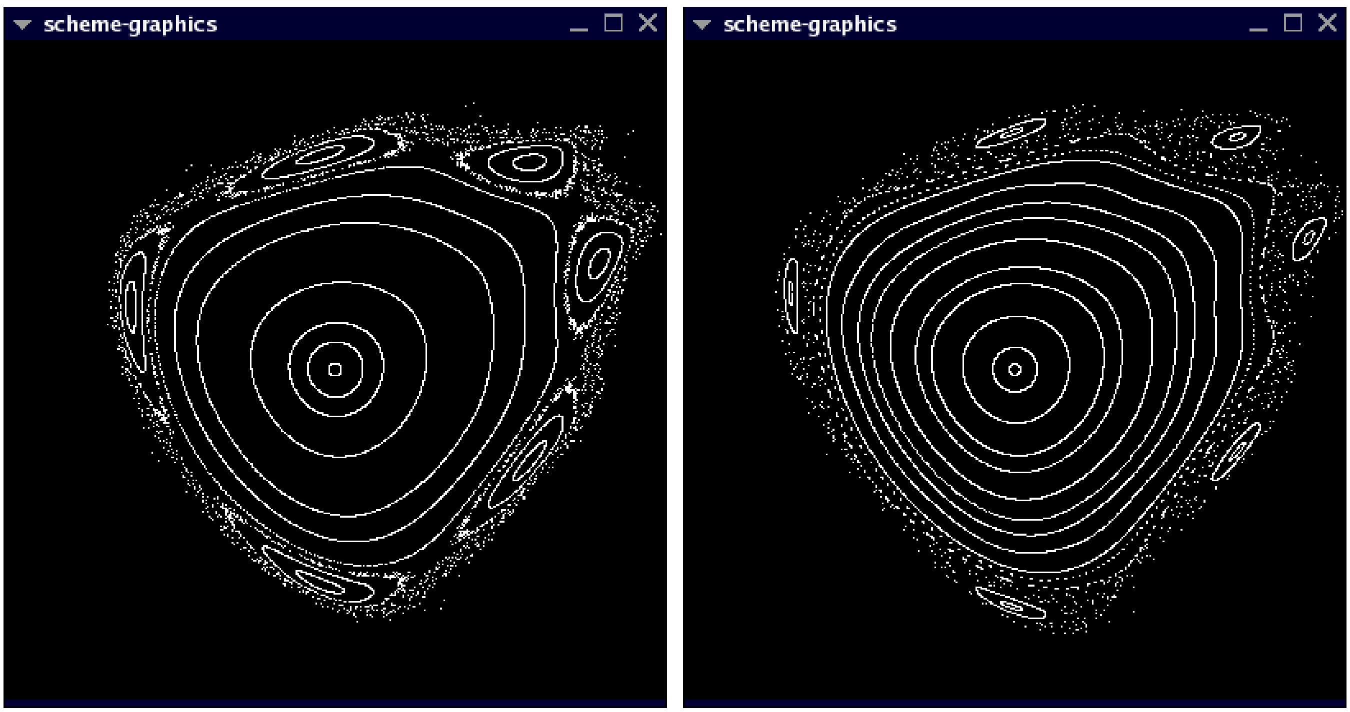

Figure 3: Both plots show several trajectories. $\alpha = 1.2$ and 1.23 on the left and right, respectively.

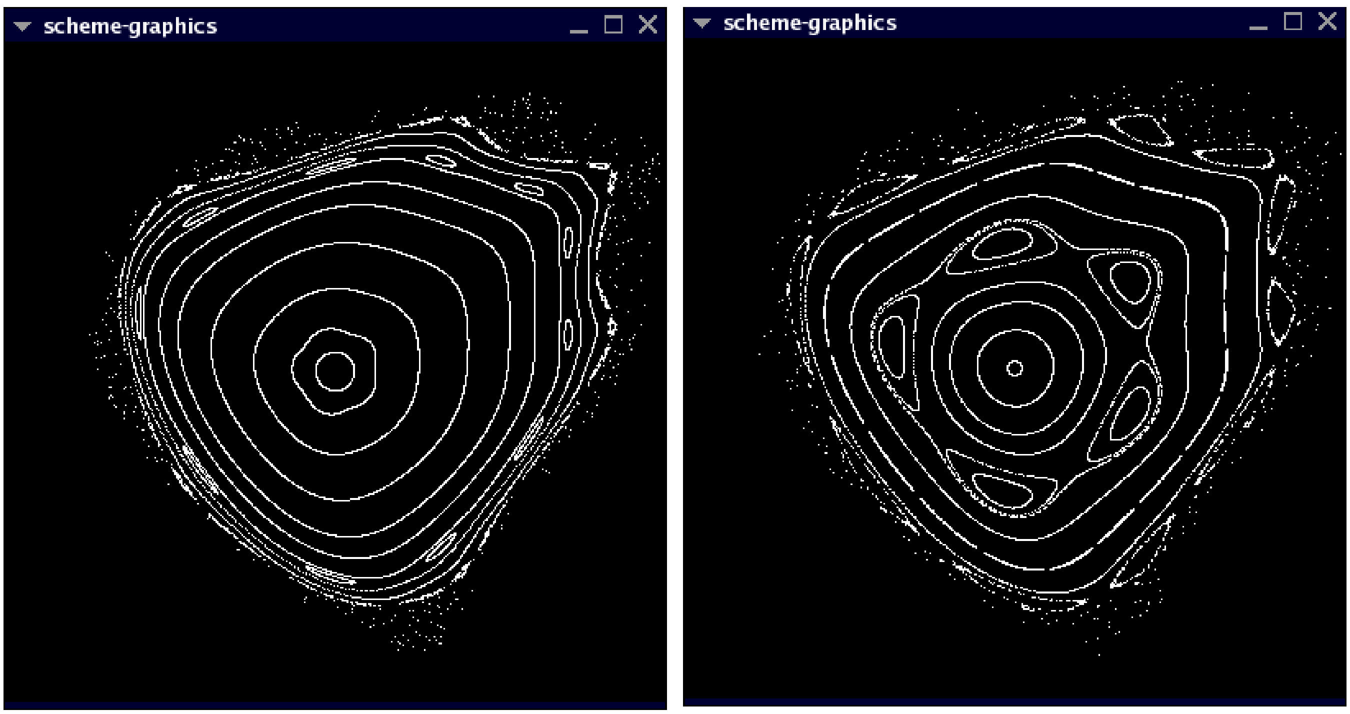

Figure 4: Both plots show several trajectories. $\alpha = 1.26$ and 1.29 on the left and right, respectively.

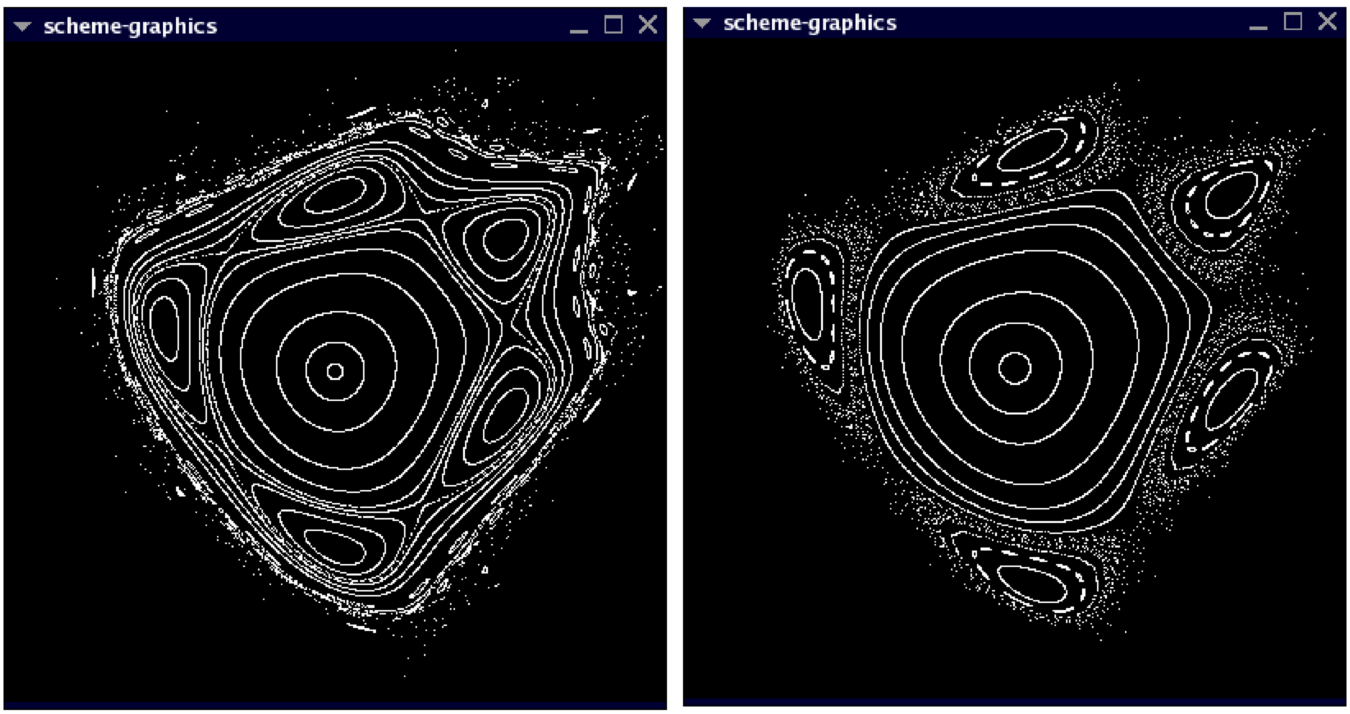

Figure 5: Both plots show several trajectories. $\alpha = 1.32$ and 1.35 on the left and right, respectively.