Exercise 3.8: Sleeping top

Consider the potential energy $U_p(\theta)$. It is given that $\theta = 0$ is a local minimum or maximum, depending on the value of $p$. If the second derivative of $U_p$ is positive, it is a local minimum, and a local maximum when the second derivative of $U_p$ is negative. We compute this second derivative at $\theta = 0$:

$$U_p(\theta) = \frac{p^2}{2C} + \frac{p^2}{2A} \tan^2 \frac{\theta}{2} + gM \cos(\theta) = \frac{p^2}{2C} + \frac{p^2}{2A} \sin^2(\theta/2) + gM \cos(\theta)$$

$$\frac{dU_p(\theta)}{d\theta} = \frac{p^2 \cos^2(\theta/2) \sin(\theta/2) \cos(\theta/2) \frac{1}{2} - \sin^2(\theta/2) \cos(\theta/2) \frac{1}{2} - gM \sin(\theta)}{\cos^4(\theta/2)}$$

$$\frac{d^2U_p(\theta)}{d\theta^2} = \frac{p^2 \cos^3(\theta/2) \cos(\theta/2) \frac{1}{2} - \sin(\theta/2) \cos^2(\theta/2) \frac{1}{2} - gM \cos(\theta)}{\cos^6(\theta/2)}$$

$$\left. \frac{d^2U_p(\theta)}{d\theta^2} \right|_{\theta=0} = \frac{p^2 \cos^2(0) + 3 \sin^2(0)}{4A} - gM \cos(0) - gM = \frac{p^2}{4A} - gM$$

This is positive (corresponding to a local minimum) when

$$\frac{p^2}{4A} - gM > 0 \Rightarrow p > \sqrt{4gMRA}$$

And negative (corresponding to a local maximum) when $p < \sqrt{4gMRA}$. Thus, the critical angular velocity $\omega_c = \frac{p}{C}$ above which an axisymmetric top can sleep is given by

$$w_c = \frac{\sqrt{4gMRA}}{C}$$

Exercise 3.10: Fun with phase portraits

I decided to investigate a pendulum on a rotating pivot with displacement $A$ and period $\omega$. In Cartesian coordinates, the location and velocity of the pivot point is given by

$$x_0(t) = A \sin(\omega t), \quad y_0(t) = A \cos(\omega t)$$

$$\dot{x}_0(t) = A \omega \cos(\omega t), \quad \dot{y}_0(t) = -A \omega \sin(\omega t)$$
If the pendulum is a bob of mass $m$, hanging on a massless string of length $l$ and making an angle $\theta$ with the gravity normal, we can describe its position and velocity with

$$x(t, \theta) = x_0(t) + l \sin(\theta), \quad y(\theta, t) = y_0(t) + l \cos(\theta)$$

$$\dot{x}(t, \theta, \dot{\theta}) = \dot{x}_0(t) + l \cos(\theta) \dot{\theta}, \quad \dot{y}(t, \theta, \dot{\theta}) = \dot{y}_0(t) - l \sin(\theta) \dot{\theta}$$

The gravitational potential of the pendulum is given by $V_g = -mgy(t)$. We note that we can completely describe the pivot and pendulum system with only a single parameter $\theta$ and its derivative. A valid Lagrangian for the system is

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy(t)$$

We write a procedure that represents this Lagrangian, and then convert it into a Hamiltonian, and then to a state derivative ready for integration.

(define ((L-rot-pend m l g A omega) state)
  (let ((theta (coordinate state))
        (thetadot (velocity state))
        (t (time state)))
    (let ((x (+ (* A (sin (* omega t))) (* l (sin theta))))
           (y (+ (* A (cos (* omega t))) (* l (cos theta))))
           (xdot (+ (* A omega (cos (* omega t))) (* l (cos theta) thetadot)))
           (ydot (+ (* -1 A omega (sin (* omega t))) (* -1 l (sin theta) thetadot))))
      (+
       (* .5 m (+ (square xdot) (square ydot))))
       (* m g y)))))

(define (H-rot-pend-sysder m l g A omega)
  (Hamiltonian->state-derivative
   (Lagrangian->Hamiltonian
    (L-rot-pend m l g A omega))))

We setup some plotting procedures:

(define ((monitor-p-theta win) state)
  (let ((q ((principal-value :pi) (coordinate state)))
         (p (momentum state)))
    (plot-point win q p)))

(define (rot-pendulum-map m l g A omega)
  (let ((advance (state-advancer H-rot-pend-sysder m l g A omega))
         (map-period (/ :2pi omega)))
    (lambda (theta ptheta return fail)
     (let ((ns (advance
              (up 0 theta ptheta))
           (up 0 theta ptheta))
           (return fail))))

I chose to investigate the surface of section for the following initial conditions \( m = 1 \text{ kg}, \ l = 1 \text{ m}, \ g = 9.8 \text{ m/s}^2, \ A = .04 \text{ m} \text{ and } \omega = 5.2 \omega_0, \) where \( \omega_0 = \sqrt{\frac{g}{l}} \) is the natural frequency of the pendulum.

\[
\begin{align*}
\text{let } & \quad (m 1.) \\
& \quad (l 1.) \\
& \quad (g 9.8) \\
& \quad (A .04)) \\
\text{let } & \quad ((\text{omega0} (\text{sqrt} (/ \ g 1)))) \\
\text{let } & \quad ((\text{omega} (* 5.2 \text{ omega0}))) \\
& \quad \text{(explore-map win}} \\
& \quad \text{(rot- pendulum-map m l g A omega))} \\
\end{align*}
\]

Some trajectories are plotted in Figures 1 and 2. There is a large chaotic region in the middle range of momenta, with major and minor islands of stability.

**Exercise 3.13: Fun with Henon’s quadratic map**

(a) Consider the Jacobian determinant of the map.

\[
\begin{vmatrix}
\frac{\partial (x', y')}{\partial (x, y)} &=& \left| \begin{array}{cc}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y}
\end{array} \right|
&=& \begin{vmatrix}
\cos \alpha + 2x \sin \alpha & -\sin \alpha \\
\sin \alpha - 2x \cos \alpha & \cos \alpha
\end{vmatrix} = \cos^2 \alpha + 2x \cos \alpha \sin \alpha - (\sin^2 \alpha + 2x \sin \alpha \cos \alpha) = \cos^2 \alpha + \sin^2 \alpha = 1
\end{vmatrix}
\]

Thus the map preserves area.

(b) We implement that map as a procedure that returns the failure condition when the orbit escapes from the area of interest \(-1 \leq x \leq 1, -1 \leq y \leq 1\).

\[
\begin{align*}
\text{define } & \quad ((\text{quad-map-alpha} \text{ alpha} \ x \ y \ return \ fail)) \\
\text{let } & \quad ((\text{xprime} (- (* \ x \ (\cos \alpha))) (* (- \ y \ (\text{square} \ x)) \ (\sin \alpha)))) \\
& \quad ((\text{yprime} (+ (* \ x \ (\sin \alpha))) (* (- \ y \ (\text{square} \ x)) \ (\cos \alpha)))) \\
\text{if } & \quad \text{ (or (or (> xprime 1) (< xprime -1)) (or (> yprime 1) (< yprime -1))))} \\
& \quad \text{ (fail)} \\
& \quad \text{ (return xprime yprime))})
\end{align*}
\]

(c) We create a frame, and explore the map for different values of \( \alpha \).

\[
\text{define window (frame -1. 1. -1. 1.))} \\
\text{(explore-map window (quad-map 1.0) 2000)}
\]

The Figures below show the map for values of \( \alpha = 1.2, 1.23, 1.26, 1.29, 1.32, 1.35 \). There is a marked evolution of the map around this parameter region of \( \alpha \). As \( \alpha \) evolves, islands of stability form and move outwards until they disappear entirely or escape the main region of stability.
Figure 1: Both plots show several trajectories. On the left, the coordinates range from $-10 \leq p_{\theta} \leq 30$ and $-\pi \leq \theta \leq \pi$. On the right, the coordinates range from $0 \leq p_{\theta} \leq 20$ and $-\pi \leq \theta \leq \pi$. 
Figure 2: An eerie plot of several trajectories. On the left, the coordinates range from $5 \leq p_\theta \leq 15$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. On the right is Krotus, the Dark Lord of Fifth East, for comparison.
Figure 3: Both plots show several trajectories. $\alpha = 1.2$ and 1.23 on the left and right, respectively.
Figure 4: Both plots show several trajectories. $\alpha = 1.26$ and $1.29$ on the left and right, respectively.
Figure 5: Both plots show several trajectories. $\alpha = 1.32$ and 1.35 on the left and right, respectively.