6.946 Assignment 9

Dennis V. Perepelitsa

8 December 2006
Gerald Jay Sussman, Jack Wisdom

Exercise 4.5: Homoclinic paradox

The lobes of the stable and unstable manifolds, while tracing an area of finite, nonzero size, have infinitesimally small width. Thus, the lobes of a (stable or unstable) manifold that sweep out an area as they approach the fixed point can then turn around, and sweep the same area, just inside the previous lobes. This phenomenon is pictured in the computed stable and unstable manifolds in Figure 1.

In the figure, lobes of the stable and unstable manifolds pack behind one another without crossing. This way, an infinite number of copies of the area enclosed by crossings of the stable and unstable manifolds can fit inside a finite area.

Figure 1: A close-up of the crossings of the computed stable and unstable manifolds around the $(I = 0, \theta = 0)$ fixed point of the $K = 1.5$ standard map. The axes are $-1 \leq \theta \leq 1$ and $-1 \leq I \leq 1$.

Exercise 4.6: Computing homoclinic tangles

(a) We have a way of computing the unstable manifold by iterating linearized eigensolutions around the fixed points of the standard map forward in time. The first order of business is to devise a way to compute the *stable* manifolds. We can do this by iterating the *inverse* of the standard map forward in time, remembering to use the appropriate eigenvectors.

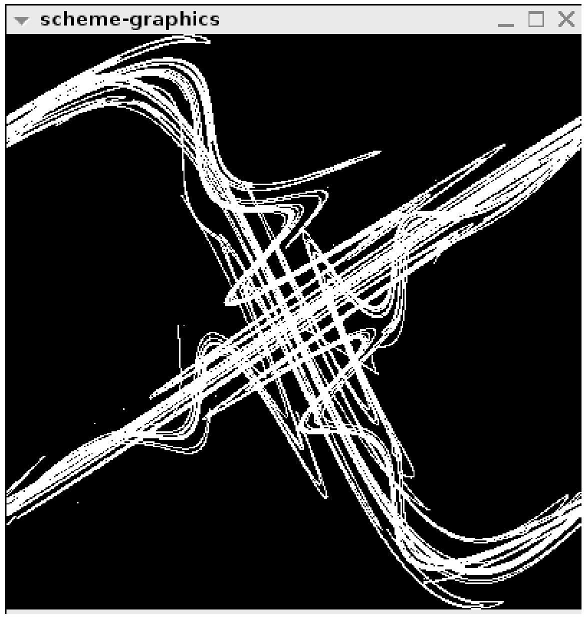

Figure 2: Computed stable and unstable manifolds around the ($I = 0,\ \theta = 0$) fixed point of the $K = 1.5$ standard map. The axes are $-\pi \le \theta \le \pi$ and $-\pi \le I \le \pi$.

```
(define ((inverse-standard-map K) theta I return failure)
  (let ((nI (+ I (* K (sin theta)))))
    (return ((principal-value :pi) (- theta I))
            ((principal-value :pi) (- I (* K (sin (- theta I))))))))
```

Let's examine the homoclinic tangle around the fixed point ($I = 0,\ \theta = 0$) for coupling constant $K = 1.5$. As shown in Exercise 4.3, the stability matrix M for this point, and associated characteristic multiplier of the positive eigenvalue e^{λ} and linearized eigenvector x_{λ} are

$$M = \begin{pmatrix} 1 & K \\ 1 & 1+K \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ 1 & 2.5 \end{pmatrix},\ e^{\lambda} = 10.80\ldots,\ x_{\lambda} = \begin{pmatrix} 0.810\ldots \\ 0.502\ldots \end{pmatrix}$$

On the other hand, the stability matrix for the inverse map, characteristic multiplier and linearized eigenvector are[1]

$$M = \begin{pmatrix} -1 & 1 \\ 1+K & -K \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2.5 & -1.5 \end{pmatrix},\ e^{\lambda} = 7.389\ldots,\ x_{\lambda} = \begin{pmatrix} 0.447 \\ 0.949 \end{pmatrix}$$

We choose a large interval $[a, b]$ over which to plot each of the manifolds using the `plot-parametric-fill` procedure, manually stopping computation when a reasonable part of the manifold in question has been plotted.

```
(define mywin (frame :-pi :pi :-pi :pi))
```

[1] We have changed the order of the basis here to (θ, I).

Figure 3: Computed unstable manifold around the ($I = 0,\ \theta = 0$) fixed point of the $K = 1.5$ standard map. The "linearization" scale ranges is, from left to right, .01, .001 and .0001, respectively.

```
(plot-parametric-fill mywin
  (unstable-manifold (standard-map 1.5)
    0 0 0.810 0.502 10.80 .1) .1 1e13 (cylinder-near? .01))

(plot-parametric-fill mywin
  (unstable-manifold (inverse-standard-map 1.5)
    0 0 0.949 .447 7.389 .1) .1 1e13 (cylinder-near? .01))
```

The results are shown in Figure 2.[2]

(b) The homoclinic tangle looks as we would expect. The stable and unstable manifolds are asymptotic to eigensolutions around the fixed point and sweep out successively larger lobes as they approach it. The stable and unstable manifolds cross each other in the center of the picture a very large number of times. However, despite all this activity, neither manifold ever seems to cross itself.

(c) The major estimation in the process is the choice of the `eps` parameter - the scale on which the linearized map is a good enough approximation to the standard map. If the error involved in the calculation is what is causing the apparent behavior of the unstable manifold, then it should change radically over different choices of the parameter. To see if this is the case, we calculate the unstable manifold for `eps` = .01, .001, .0001, as pictured in Figure 3.

Since it takes a long time to calculate a decent portion of the manifold for `eps` = .0001, it is less detailed than the other two. However, we see that under a regime of scales that differ by a magnitude of 100, the unstable manifold looks the same - sweeping side to side with an increasingly larger range as it approaches the fixed point. This gives us faith in its reality.

[2]We are using these coordinates instead of $0 \leq \theta \leq 2\pi$, $0 \leq I \leq 2\pi$, so as not to cut the homoclinic tangle into four parts on our screen.

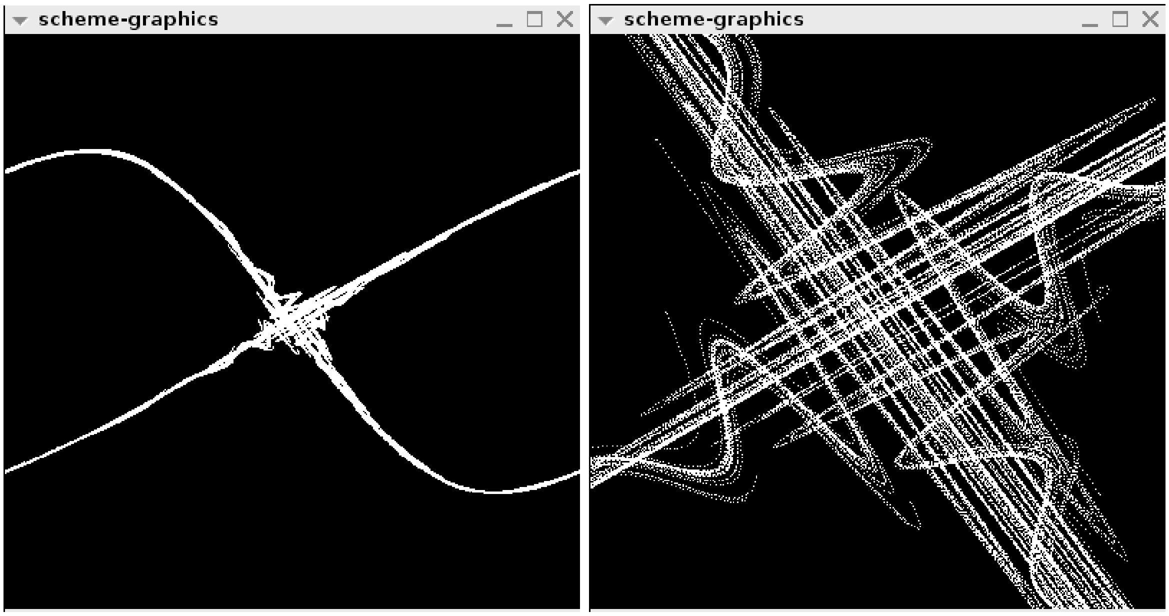

Figure 4: Computed homoclinic tangle (and a close-up) around the ($I = 0,\ \theta = 0$) fixed point of the $K = .8$ standard map. The coordinates are $-\pi \leq \theta, I \leq \pi$ on the left plot and $-.5 \leq \theta, I \leq .5$ on the right plot.

(d) We determine the characterstic multiplier and linearized eigenvectors for $K = .8$ as above, and compute the homoclinic tangle, as pictured in Figure 4. A surface of section created with the `explore-map` procedure shows the chaotic region for this map, pictured in Figure 5. The homoclinic tangle and the chaotic region fill approximately the same relatively small area[3]. This is evidence of the fact that the homoclinic tangle dictates the chaotic behavior of trajectories around this unstable fixed point.

[3]Curiously, the $K = .8$ homoclinic tangle fills a smaller area than the $K = 1.5$ tangle. This is coincident with the fact that the $K = 1.5$ map has a larger chaotic region, and reinforces our beliefs about the relationship between relative sizes of the homoclinic tangle and a chaotic region

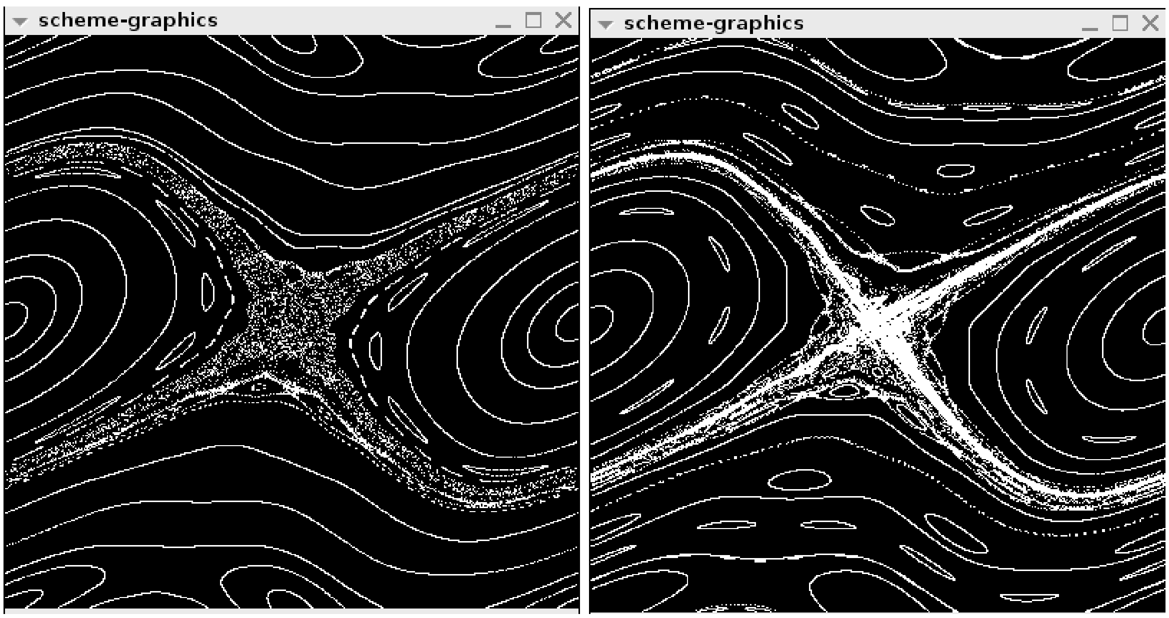

Figure 5: Surface of section for the $K = .8$ standard map. The coordinates are $-\pi \leq \theta, I \leq \pi$ for both plots. The plot on the right is a surface of section with superimposed homoclinic tangle.