Exercise 2.21: Rotation of Mercury

(a) We derive the relation in the book. Kepler’s Third Law can be stated as, roughly, “Equal time, equal area”. That is, the change in area covered by the arc of the satellite motion $\frac{dA}{dt}$ is constant. The area differential can be written as

$$dA = \frac{1}{2} [r^2] d\theta$$

Here, $r = R(t)$, the distance to the satellite, and $d\theta = Df(t)dt$, the arc differential. Furthermore, for an ellipse with semimajor and semiminor axes of length $a$ and $b$, respectively, the total area is given by $\pi ab$, and the period of a revolution is $\frac{2\pi}{n}$, where $n$ is the orbital frequency.

$$\frac{dA}{dt} = \frac{\pi ab}{2\pi} = \frac{nab}{2}$$

$$\frac{1}{2}R(t)^2 Df(t) = \frac{nab}{2}$$

$$Df(t) = nb \frac{a}{R(t)^2}$$

We can rewrite $b$ in terms of the eccentricity $e = \sqrt{1 - \frac{a^2}{b^2}}$. Specifically, $b = a\sqrt{1 - e^2}$.

$$Df(t) = n\sqrt{1 - e^2} \left( \frac{a}{R(t)} \right)^2$$

We’d also like a definition for $\frac{a}{R(t)}$ that depends only on $f(t)$. A common way to do this is\(^1\)

$$R(t) = \frac{a(1 - e^2)}{1 + e \cos(f(t))}$$

$$\frac{a}{R(t)} = \frac{1 + e \cos(f(t))}{1 - e^2}$$

We change coordinates to $\tau = \tau n$, such that $(Df) \cdot \tau = Df \cdot D\tau = n Df(\tau)$,

$$nDf(\tau) = n\sqrt{1 - e^2} \left( \frac{a}{R(\tau)} \right)^2 \frac{1}{n^2}$$

$$Df(\tau) = \sqrt{1 - e^2} \left( \frac{a}{R(\tau)} \right)^2$$

\(^1\)http://mathworld.wolfram.com/Ellipse.html, (50)
\[
\frac{a}{R(\tau)} = \frac{1 + e \cos(f(\tau))}{1 - e^2}
\]

The Lagrange Equation for \( \theta(t) \), from (2.80), is

\[
CD^2\theta(t) = -\frac{n^2e^2C}{2} \left( \frac{a}{R(t)} \right)^3 \sin(2\theta(t) - 2f(t))
\]

We change coordinates, with \( D^2(\theta(\tau)) = D(nD\theta(\tau)) = n^2D^2\theta(\tau) \),

\[
D^2\theta(\tau) = -\frac{e^2}{2} \left( \frac{a}{R(t)} \right)^3 \sin(2\theta(\tau) - 2f(\tau))
\]

Thus, the acceleration (and velocity) procedures for our \((\theta, f)\) generalized-coordinate system are, in terms of \( \tau \),

\begin{verbatim}
(define ((theta-acc ecc epsilon) state)
  (let ((f (ref (coordinate state) 0))
        (theta (ref (coordinate state) 1)))
    (* -0.5
       (* epsilon epsilon)
       (/ (cube (+ 1.0 (* ecc (cos f))))
           (cube (- 1.0 (* ecc ecc))))
       (sin
        (* 2
          (- theta f)))))
  )
)

(define ((f-vel ecc) state)
  (let ((f (ref (coordinate state) 0)))
    (* (sqrt (- 1.0 (* ecc ecc)))
       (square (/ (+ 1.0 (* ecc (cos f)))
                 (- 1.0 (* ecc ecc)))))
  )
)
\end{verbatim}

We define a state derivative for numerical integration. We can evolve \( \theta(\tau) \) by integrating \( D^2\theta(\tau) \), and we can evolve \( f(\tau) \) by integrating \( Df(\tau) \) explicitly (and defining \( D^2f(\tau) = 0 \)).
(define ((f-state-deriv ecc epsilon) state)
  (let ((f (ref (velocity state) 0))
        (theta (ref (velocity state) 1)))
    (up 1.0
        (up
         ((f-vel ecc) state)
         theta)
        (up
         0
         ((theta-acc ecc epsilon) state)
         ))))

(b) We setup a window over a long time-period (and thus choose a relatively large integration step \( \Delta t \)), and plot the difference \( \theta(\tau) - \frac{3}{2}\tau \).

(define plot-diff-win (frame 0 2000 -.1 .1))

(define ((monitor-diff win) state)
  (let ((theta (ref (coordinate state) 1))
        (f (ref (coordinate state) 0)))
    (plot-point win (time state)
                ((principal-value :pi) (- theta (* 1.5 (time state))))
                )))

Before we evolve the state, we must determine the initial conditions. Let’s start Mercury out at the end of the semimajor axis, oriented outwards. That is, \( f(t = 0) = \theta(t = 0) = 0 \). We write a helper function to compute \( Df_e(t = 0) \), and choose \( D\theta(t = 0) = 1.5 \) as a stable orbit. From the statement of the problem, \( e = .2 \) and \( \epsilon = .026 \).

(define (init-orientation-change ecc)
  (*
   (sqrt (- 1 (square ecc)))
   (square (/ (+ 1 ecc) (- 1 (square ecc))))
   ))

((evolve f-state-deriv .2 .026)
  (up
   0.0
   (up 0.0 0.0)
   (up
    (init-orientation-change .2)
    1.5)
   (monitor-diff plot-diff-win)
   1
   2000
   1.e-12))
Figure 1: Plot of $\theta(\tau) - \frac{3}{2} \tau$, for $0 \leq \tau \leq 2,000$, with integration step $\Delta t = .1$. Initial angular velocity $D\theta(t = 0) = 1.5$. The extremes of the ordinate axis are $-1$ and $1$.

Our plot is displayed in Figure 1. We observe that the angle difference oscillates, as expected.

(c) We investigate the range of $D\theta(t = 0)$ such that Mercury exhibits resonant behavior by inspection. To eight digits, the range of initial angular velocities for which the spin-orbit resonance is stable seems to be between 1.52093252 and 1.47937647. Figures 2, 3 illustrate this graphically. Outside of this range, the angle $\theta(\tau) - \frac{3}{2} \tau$ does not oscillate around zero.
Figure 2: Plot of $\theta(\tau) - \frac{3}{2}\tau$, for $0 \leq \tau \leq 4,000$, with integration step $\Delta t = 1$. The initial angular velocities are 1.52093252 and 1.52093253 on the left and right, respectively. The extremes of the ordinate axis are $-2$ and 2.

Figure 3: Plot of $\theta(\tau) - \frac{3}{2}\tau$, for $0 \leq \tau \leq 4,000$, with integration step $\Delta t = 1$. The initial angular velocities are 1.47937646 and 1.47937647 on the left and right, respectively. The extremes of the ordinate axis are $-2$ and 2.