6.946 Mercury Project

Dennis V. Perepelitsa

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Gerald Jay Sussman, Jack Wisdom

Exercise 2.21: Rotation of Mercury

(a) We derive the relation in the book. Kepler's Third Law can be stated as, roughly, *"Equal time, equal area"*. That is, the change in area covered by the arc of the satellite motion $\frac{dA}{dt}$ is constant. The area differential can be written as

$$dA = \frac{1}{2}\left[r^2\right] d\theta$$

Here, $r = R(t)$, the distance to the satellite, and $d\theta = Df(t)dt$, the arc differential. Furthermore, for an ellipse with semimajor and semiminor axes of length a and b, respectively, the total area is given by πab, and the period of a revolution is $\frac{2\pi}{n}$, where n is the orbital frequency.

$$\frac{dA}{dt} = \frac{\pi ab}{\frac{2\pi}{n}}$$

$$\frac{1}{2}R(t)^2 Df(t) = \frac{nab}{2}$$

$$Df(t) = nb\frac{a}{R(t)^2}$$

We can rewrite b in terms of the eccentricity $e = \sqrt{1 - \frac{a^2}{b^2}}$. Specifically, $b = a\sqrt{1-e^2}$.

$$Df(t) = n\sqrt{1-e^2}\left(\frac{a}{R(t)}\right)^2$$

We'd also like a definition for $\frac{a}{R(t)}$ that depends only on $f(t)$. A common way to do this is[1]

$$R(t) = \frac{a(1-e^2)}{1 + e\ \cos(f(t))}$$

$$\frac{a}{R(t)} = \frac{1 + e\ \cos(f(t))}{1-e^2}$$

We change coordinates to $\tau = tn$, such that $(Df)\cdot\tau = Df\cdot D\tau = nDf(\tau)$,

$$nDf(\tau) = n\sqrt{1-e^2}\left(\frac{a}{R(\tau)}\right)^2\frac{1}{n^2}$$

$$Df(\tau) = \sqrt{1-e^2}\left(\frac{a}{R(\tau)}\right)^2$$

[1] http://mathworld.wolfram.com/Ellipse.html, (50)

$$\frac{a}{R(\tau)} = \frac{1 + e\ \cos(f(\tau))}{1 - e^2}$$

The Lagrange Equation for $\theta(t)$, from (2.80), is

$$CD^2\theta(t) = -\frac{n^2\epsilon^2 C}{2}\left(\frac{a}{R(t)}\right)^3 \sin(2\theta(t) - 2f(t))$$

We change coordinates, with $D^2(\theta(\tau)) = D(nD\theta(\tau)) = n^2 D^2\theta(\tau)$,

$$D^2\theta(\tau) = -\frac{\epsilon^2}{2}\left(\frac{a}{R(t)}\right)^3 \sin(2\theta(\tau) - 2f(\tau))$$

Thus, the acceleration (and velocity) procedures for our (θ, f) generalized-coordinate system are, in terms of τ,

```
(define ((theta-acc ecc epsilon) state)
  (let ((f (ref (coordinate state) 0))
        (theta (ref (coordinate state) 1)))
    (*
     -.5
     (* epsilon epsilon)
      (/
       (cube (+ 1.0 (* ecc (cos f))))
       (cube (- 1.0 (* ecc ecc)))
      )
     (sin
      (* 2
         (- theta f)))
     )
    ))

(define ((f-vel ecc) state)
  (let ((f (ref (coordinate state) 0)))
    (*
      (sqrt (- 1.0 (* ecc ecc)))
      (square (/
               (+ 1.0 (* ecc (cos f)))
               (- 1.0 (* ecc ecc))
               ))
    )))
```

We define a state derivative $\begin{pmatrix} 1 \\ \begin{pmatrix} Df(\tau) \\ D\theta(\tau) \end{pmatrix} \\ \begin{pmatrix} D^2f(\tau) \\ D^2\theta(\tau) \end{pmatrix} \end{pmatrix}$ for numerical integration. We can evolve $\theta(\tau)$ by integrating $D^2\theta(\tau)$, and we can evolve $f(\tau)$ by integrating $Df(\tau)$ explicitly (and defining $D^2f(\tau) = 0$).

```
(define ((f-state-deriv ecc epsilon) state)
  (let ((f (ref (velocity state) 0))
        (theta (ref (velocity state) 1)))
  (up 1.0
      (up
       ((f-vel ecc) state)
       theta)
      (up
       0
       ((theta-acc ecc epsilon) state)
      ))))
```

(b) We setup a window over a long time-period (and thus choose a relatively large integration step Δt), and plot the difference $\theta(\tau) - \frac{3}{2}\tau$.

```
(define plot-diff-win (frame 0 2000 -.1 .1))

(define ((monitor-diff win) state)
  (let ((theta (ref (coordinate state) 1))
    (f (ref (coordinate state) 0)))
    (plot-point win (time state)
                ((principal-value :pi) (- theta (* 1.5 (time state))))
                )))
```

Before we evolve the state, we must determine the initial conditions. Let's start Mercury out at the end of the semimajor axis, oriented outwards. That is, $f(t=0) = \theta(t=0) = 0$. We write a helper function to compute $Df_e(t=0)$, and choose $D\theta(t=0) = 1.5$ as a stable orbit. From the statement of the problem, $e = .2$ and $epsilon = .026$.

```
(define (init-orientation-change ecc)
  (*
   (sqrt (- 1 (square ecc)))
   (square (/ (+ 1 ecc) (- 1 (square ecc))))
   ))

((evolve f-state-deriv .2 .026)
  (up
   0.0
   (up 0.0 0.0)
   (up
    (init-orientation-change .2)
    1.5))
  (monitor-diff plot-diff-win)
  1
  2000
  1.e-12)
```

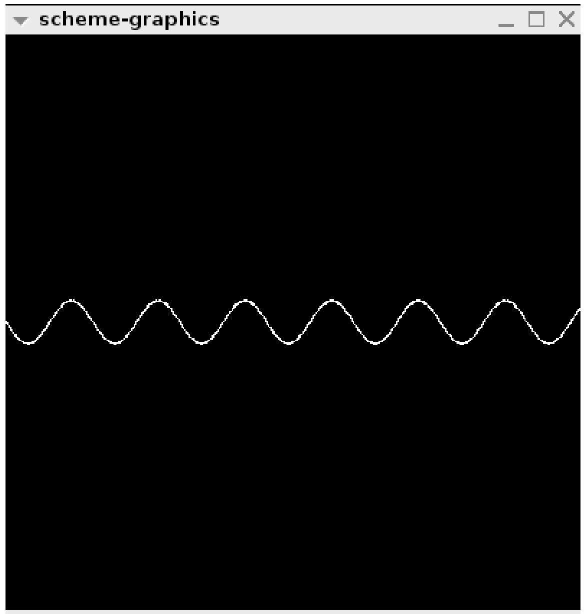

Figure 1: Plot of $\theta(\tau) - \frac{3}{2}\tau$, for $0 \leq \tau \leq 2,000$, with integration step $\Delta t = .1$. Initial angular velocity $D\theta(t = 0) = 1.5$. The extremes of the ordinate axis are $-.1$ and $.1$.

Our plot is displayed in Figure 1. We observe that the angle difference oscillates, as expected.

(c) We investigate the range of $D\theta(t = 0)$ such that Mercury exhibits resonant behavior by inspection. To eight digits, the range of initial angular velocities for which the spin-orbit resonance is stable seems to be between 1.52093252 and 1.47937647. Figures 2, 3 illustrate this graphically. Outside of this range, the angle $\theta(\tau) - \frac{3}{2}\tau$ does not oscillate around zero.

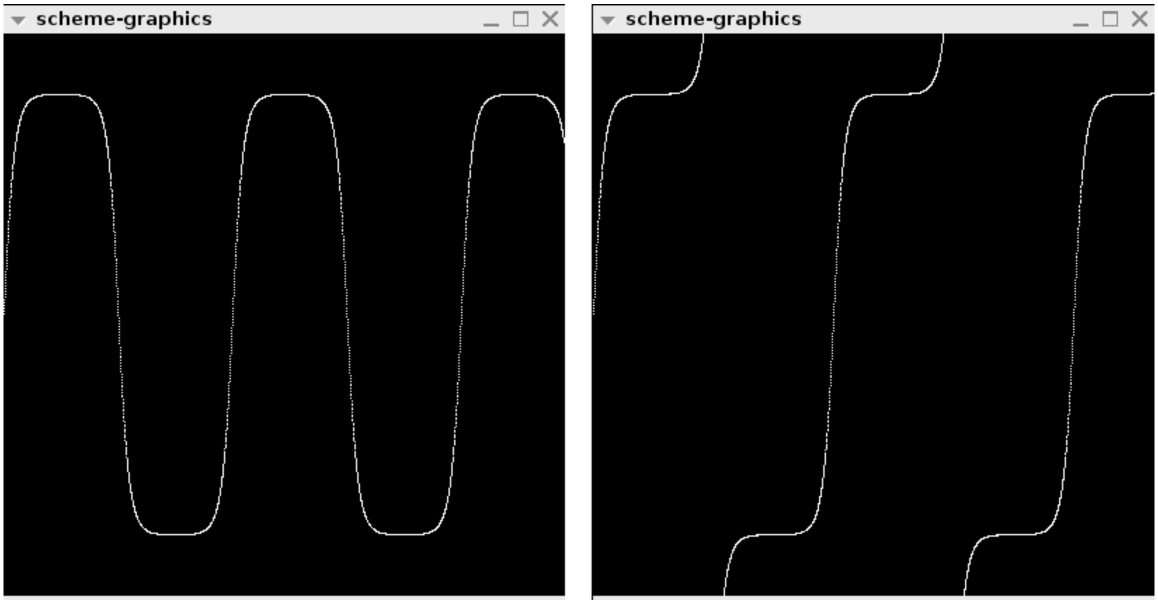

Figure 2: Plot of $\theta(\tau) - \frac{3}{2}\tau$, for $0 \leq \tau \leq 4,000$, with integration step $\Delta t = 1$. The initial angular velocities are 1.52093252 and 1.52093253 on the left and right, respectively. The extremes of the ordinate axis are -2 and 2.

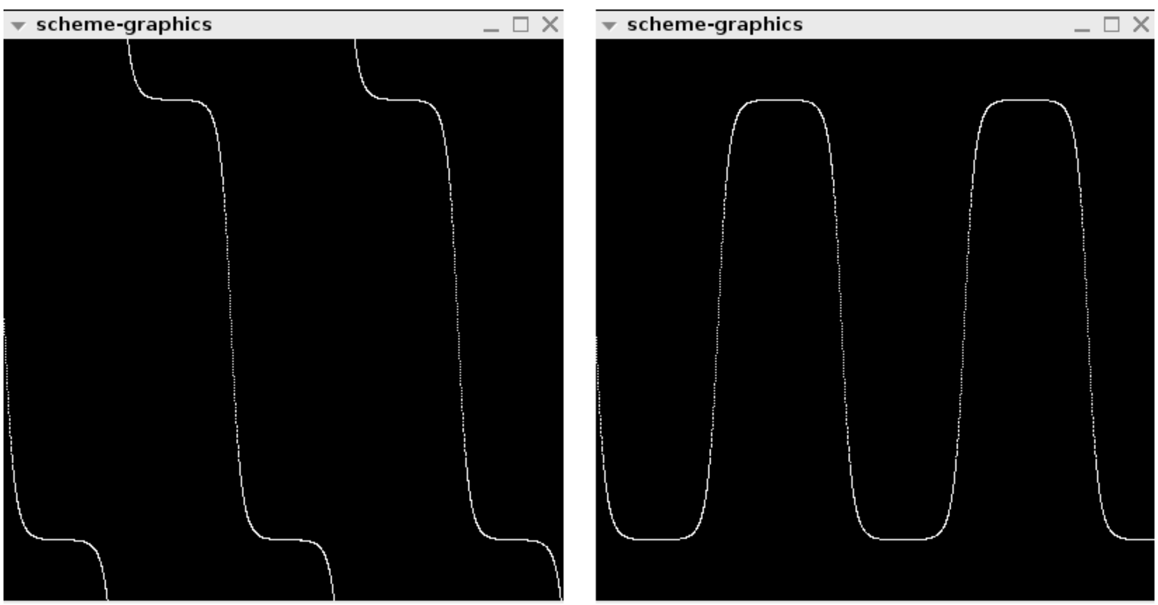

Figure 3: Plot of $\theta(\tau) - \frac{3}{2}\tau$, for $0 \leq \tau \leq 4,000$, with integration step $\Delta t = 1$. The initial angular velocities are 1.47937646 and 1.47937647 on the left and right, respectively. The extremes of the ordinate axis are -2 and 2.