

## 6.946 Double Pendulum Project

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### Exercise 1.39: Double pendulum behavior

(a) We can describe the position and velocity of the two pendulum bobs in Cartesian coordinates. Let  $(x_1, y_1)$  be the position of the  $m_1$  (first) pendulum bob:

$$(x_1, y_1) = (l_1 \sin(\theta_1), l_1 \cos(\theta_1))$$

$$\frac{d}{dt}(x_1, y_1) = (l_1 \cos(\theta_1) \dot{\theta}_1, -l_1 \sin(\theta_1) \dot{\theta}_1)$$

Let  $(x_2, y_2)$  be the position of the  $m_2$  (second) pendulum bob. We can express it as the sum of  $(x_1, y_1)$  and the relative offset of the second bob from the first:

$$(x_2, y_2) = (x_1 + l_2 \sin(\theta_1 + \theta_2), y_1 + l_2 \cos(\theta_1 + \theta_2))$$

$$\frac{d}{dt}(x_2, y_2) = \left( \frac{d}{dt}x_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2), \frac{d}{dt}y_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \right)$$

We can define the kinetic and potential energies  $K$  and  $V$ .

$$V = -gm_1y_1 - gm_2y_2$$

$$K = \frac{m_1}{2} \left( \left( \frac{dx_1}{dt} \right)^2 + \left( \frac{dy_1}{dt} \right)^2 \right) + \frac{m_2}{2} \left( \left( \frac{dx_2}{dt} \right)^2 + \left( \frac{dy_2}{dt} \right)^2 \right)$$

We define them in terms of a state tuple  $(t, \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix})$  for later ease of use, starting with the potential energy:

```
(define ((V-dp g m_1 m_2 l_1 l_2) local)
  (let ((theta_1 (ref (coordinate local) 0))
        (theta_2 (ref (coordinate local) 1))))
    (let ((y_1 (* l_1 (cos theta_1)))
          (y_2 (+ (* l_1 (cos theta_1))
                  (* l_2 (cos (+ theta_1 theta_2))))))
      (+ (* -1 g m_1 y_1)
         (* -1 m_2 g y_2))
    )))
```

And the kinetic energy:

```
(define ((K-dp m_1 m_2 l_1 l_2) local)
  (let ((theta_1 (ref (coordinate local) 0))
        (theta_2 (ref (coordinate local) 1))))
```

```

(alpha_1 (ref (velocity local) 0))
(alpha_2 (ref (velocity local) 1)))
(let ((v1x (* l_1 alpha_1 (cos theta_1)))
      (v1y (* -1 l_1 alpha_1 (sin theta_1)))
      (v2x (+
             (* l_1 alpha_1 (cos theta_1))
             (* l_2 (+ alpha_1 alpha_2) (cos (+ theta_1 theta_2)))))
      (v2y (+
             (* -1 l_1 alpha_1 (sin theta_1))
             (* -1 l_2 (+ alpha_1 alpha_2) (sin (+ theta_1 theta_2))))))
      (+ (* 1/2 m_1 (+ (square v1x) (square v1y)))
         (* 1/2 m_2 (+ (square v2x) (square v2y)))
        )))

```

We proceed to formulate a Lagrangian  $L = V - K$ :

```

(define ((L-double-pendulum g m_1 m_2 l_1 l_2) local)
  (- ((V-dp g m_1 m_2 l_1 l_2) local) ((K-dp m_1 m_2 l_1 l_2) local)))

```

To put this into a form fit for numerical integration, we turn the Lagrangian into a state derivative, providing the numerical values of the constants:

```

(define (double-pendulum->state-derivative)
  (Lagrangian->state-derivative
   (L-double-pendulum 9.8 1 3 1 .9)))

```

(b) We create a frame in which to plot points as we evolve the state of the system:

```

(define ((monitor-theta win) state)
  (let ((theta ((principal-value :pi) (ref (coordinate state) 0))))
    (plot-point win (time state) theta)))

(define plot-win (frame 0 50 (- 0 :pi) :pi))

((evolve double-pendulum->state-derivative)
 (up 0 (up (/ :pi 2) :pi) (up 0 0))
 (monitor-theta plot-win) .005 50 1.0e-12)

```

Annoyingly, the trajectories of the pendulum bobs look different depending on what our integration step size is and, believe it or not, which machines we run the simulations on<sup>1</sup>. Every plot in this project was generated by running our code on [planets.csail.mit.edu](http://planets.csail.mit.edu). In this case, we use step sizes  $\Delta t = .005$  and  $\Delta t = .001$  for comparison. The plots of the trajectories of the  $m_1$  and  $m_2$ , respectively, bob are shown in Figure 1 and Figure 2, respectively.

We now write a procedure to save points which are spaced  $\Delta t = .01$  apart<sup>2</sup>:

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<sup>1</sup>Could this be a floating point issue?

<sup>2</sup>I realize the project description called for points spaced  $\Delta t = .125$  apart. However, I have found that the plots of the squared differences looked much better with a smaller granularity, and so have decided to use this smaller  $\Delta t = .01$ . I hope this is acceptable.

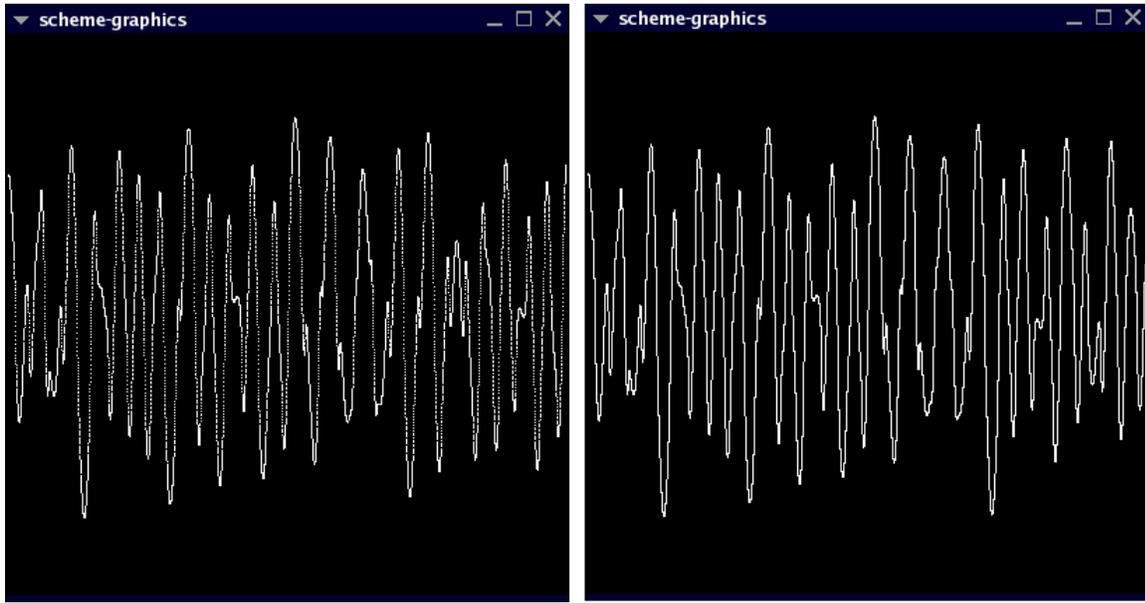


Figure 1: Trajectory of the  $m_1$  pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ . Notice the divergence towards the end.

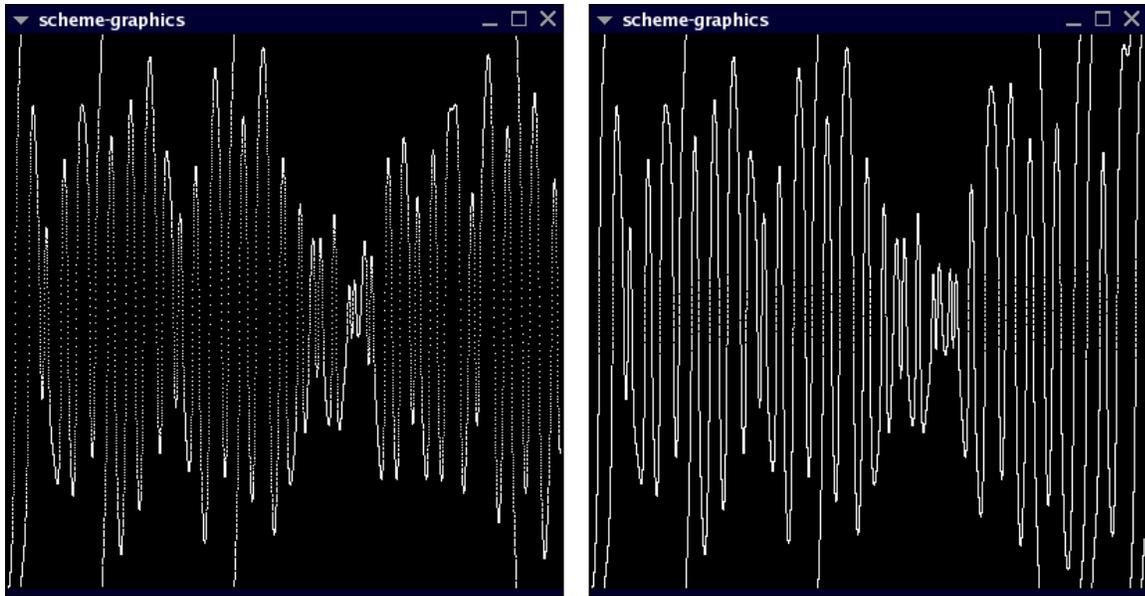


Figure 2: Trajectory of the  $m_2$  pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ . Again, notice the divergence towards the end.

```

(define ((save-points) state)
  (if (or
      (< (abs (- (/ (time state) .01) (floor (/ (time state) .01)))) .0001)
      (< (abs (- (/ (time state) .01) (ceiling (/ (time state) .01)))) .0001))
    (set! set-of-state-points
      (append
        set-of-state-points
        (list
          (list (time state)
                (list (ref (coordinate state) 0) (ref (coordinate state) 1))
                (list (ref (velocity state) 0) (ref (velocity state) 1))
                ))))))

```

We run the simulation again, saving the state points.

```

(define set-of-state-points (list))

((evolve double-pendulum->state-derivative)
 (up 0 (up (/ :pi 2) :pi) (up 0 0))
 (save-points) .001 50 1.0e-12)

(define set-of-state-points-1 set-of-state-points)

```

We will use these in part **(d)**.

**(c)** We define a procedure to extract the total energy  $V + K$ .

```

(define (get-energy state g m_1 m_2 l_1 l_2)
  (+ ((V-dp g m_1 m_2 l_1 l_2) state) ((K-dp m_1 m_2 l_1 l_2) state)))

```

We plot the total energy as the state evolves.

```

(define ((monitor-energy win g m_1 m_2 l_1 l_2) state)
  (let ((e (get-energy state g m_1 m_2 l_1 l_2)))
    (plot-point win (time state) e)))

(define plot-energy-win (frame 0 50 -1e-10 1e-10))

((evolve double-pendulum->state-derivative)
 (up 0 (up (/ :pi 2) :pi) (up 0 0))
 (monitor-energy plot-energy-win 9.8 1 3 1 .9) .005 50 1.0e-12)

```

The results for step sizes  $\Delta = .005$  and  $\Delta = .001$  are shown in Figure 3. Conservation of energy is nearly obeyed - the absolute error is less than  $10^{-10}$  Joules.

**(d)** We repeat the experiment of part **(b)**, but this time move the second bob  $10^{-10}$  higher than before. This corresponds to a new  $\theta'_2 = \theta_2 - \Delta\theta_2$ , with  $\Delta\theta_2 = \sin^{-1}\left(\frac{10^{-10}}{l_2}\right) = \frac{10^{-10}}{.9}$ . The new initial conditions from which we evolve the state are:



Figure 3: Total energy of the double-pendulum system from  $t = 0$  to  $t = 50$ , with step size of  $\Delta t = .005$ . The y-axis goes from  $-10^{-10}$  to  $10^{-10}$ .

```
((evolve double-pendulum->state-derivative)
 (up 0 (up (/ :pi 2) (- :pi (/ 1.e-10 .9))) (up 0 0))
 (monitor-theta plot-win) .005 50 1.0e-12)
```

Plots of the angles of the  $m_1$  and  $m_2$  pendulum bob, respectively, integrated with step sizes  $\Delta t = .005$  and  $\Delta t = .001$  are shown in Figure 4 and Figure 5, respectively.

We again record the state points spaced  $\Delta t = .01$  apart:

```
(define set-of-state-points (list))

((evolve double-pendulum->state-derivative)
 (up 0 (up (/ :pi 2) (- :pi (/ 1.e-10 .9))) (up 0 0))
 (save-points) .001 50 1.0e-12)
```

```
(define set-of-state-points-2 set-of-state-points)
```

We now intend to plot the squared differences of the distances between the positions of the  $m_2$  bobs in the two experiments. We define some helper procedures:

```
(define (x2 thetas)
  (let ((theta_1 (ref thetas 0))
        (theta_2 (ref thetas 1)))
    (* 1.0 (cos theta_1))))

(define (y2 thetas)
  (let ((theta_1 (ref thetas 0))
```

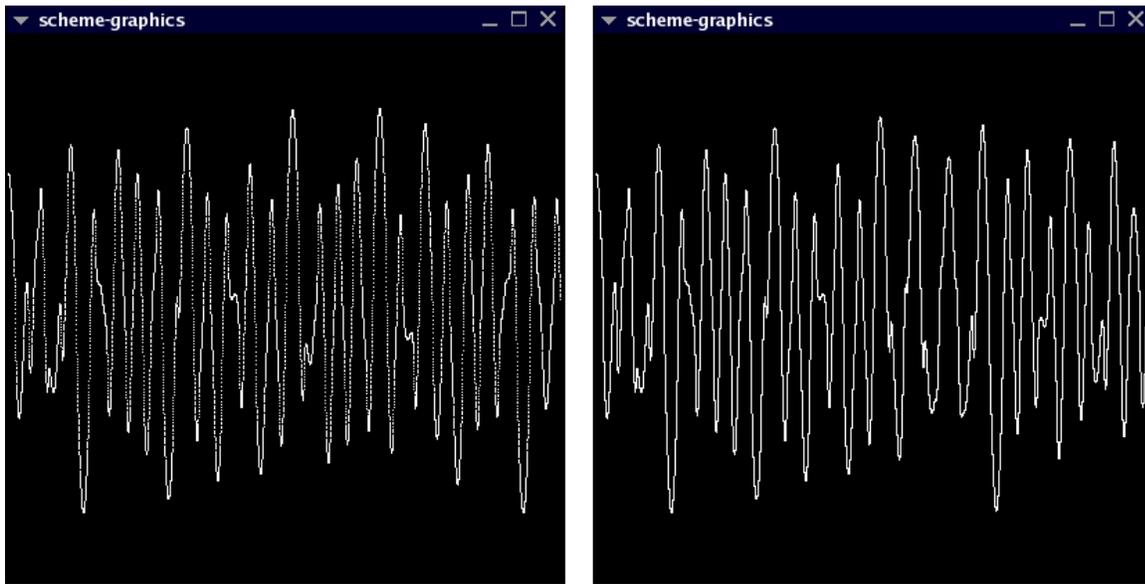


Figure 4: Trajectory of the first pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ .

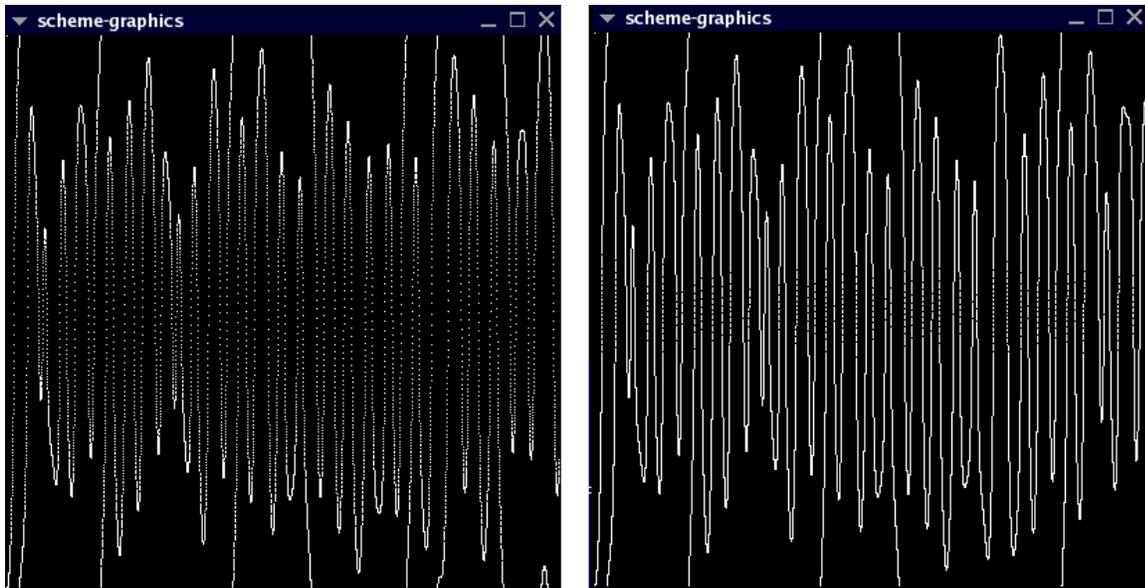


Figure 5: Trajectory of the second pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ .

```

    (theta_2 (ref thetas 1)))
  (+ (x2 thetas)
    (* .9 (sin (+ theta_1 theta_2))))
))

```

```

(define (diff thetas1 thetas2)
  (+ (square (- (x2 thetas1) (x2 thetas2)))
    (square (- (y2 thetas1) (y2 thetas2)))
  ))

```

We extract the list of squared differences:

```

(define squared-differences
  (map diff
    (map cadr set-of-state-points-1)
    (map cadr set-of-state-points-2)))

```

We setup up plotting procedures and plot the differences.

```

(define plot-diffs-win (frame 0 50 0 10))

(define (recursive-plot y t)
  (let ()
    (plot-point plot-diffs-win t (car y))
    (if (> (length y) 1)
      (recursive-plot (cdr y) (+ t .01))
      )))

```

```

(recursive-plot squared-differences 0.0)

```

We do the same for the log of the differences:

```

(define log-squared-differences
  (map log squared-differences))

(define plot-diffs-win (frame 0 50 -50 2))

(recursive-plot log-squared-differences 0.0)

```

The results are displayed in Figure 6. The bobs are very close together for about half the time, then diverge in an unpredictable manner. Furthermore, since the plot on the right looks almost linear, this divergence seems to be exponential.

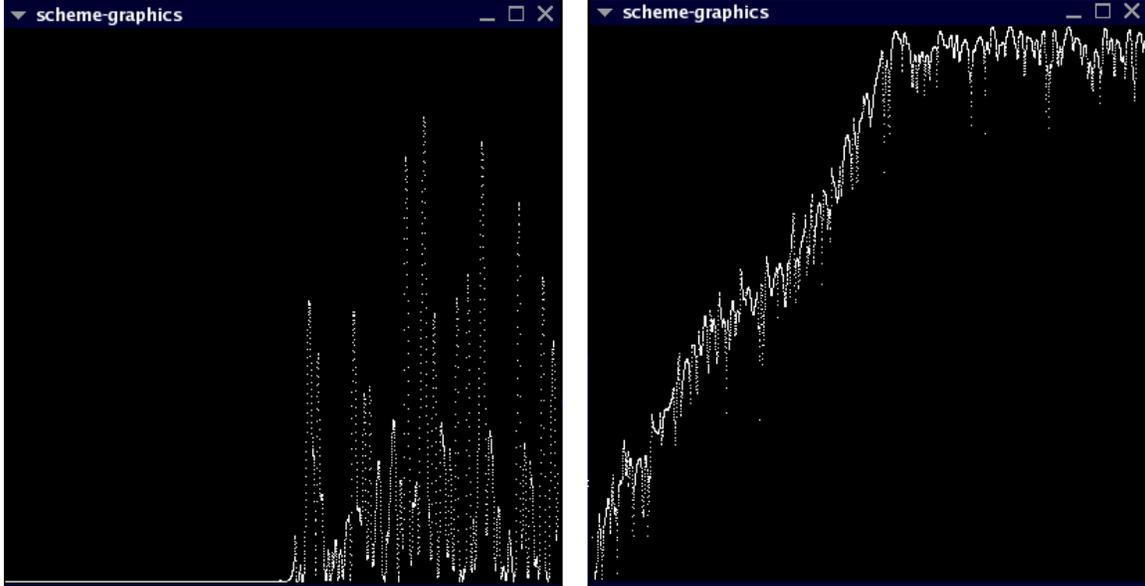


Figure 6: Plots of squared differences of the distances between the  $m_2$  bobs, from  $t = 0$  to  $t = 50$ . The plot on the left is on a linear scale from  $0m^2$  to  $10m^2$ . The plot on the right is logarithmic, and the scale is from  $-50$  to  $2$ .

(e) We repeat the above steps, with a different setup. As above, our plots of the trajectories were taken with step sizes  $\Delta t = .005$  and  $\Delta t = .001$ .<sup>3</sup>

Plots of the trajectory of a system with initial conditions  $\theta_1(0) = \frac{\pi}{2}$ ,  $\theta_2(0) = 0$ ,  $\dot{\theta}_1(0) = 0$ ,  $\dot{\theta}_2(0) = 0$  are shown in Figures 7 and 8. We also plot the total energy of the system, in Figure 9. Energy seems to be mostly conserved, to within  $10^{-10}$ , particularly with the small step size.

We now move the  $m_2$  bob slightly higher. Plots of the trajectory of a system with initial conditions  $\theta_1(0) = \frac{\pi}{2}$ ,  $\theta_2(0) = \frac{10^{-10}}{9}$ ,  $\dot{\theta}_1(0) = 0$ ,  $\dot{\theta}_2(0) = 0$  are shown in Figures 10 and 11.

There seems to be little, if any, noticeable deviation between the two sets of initial conditions. We plot the logarithm of the squared differences in the distances between the  $m_2$  bobs in Figure 12.

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<sup>3</sup>However, in this case it seems to make little difference, except in the plot of the energy. I have included them anyway, for the sake of completeness.

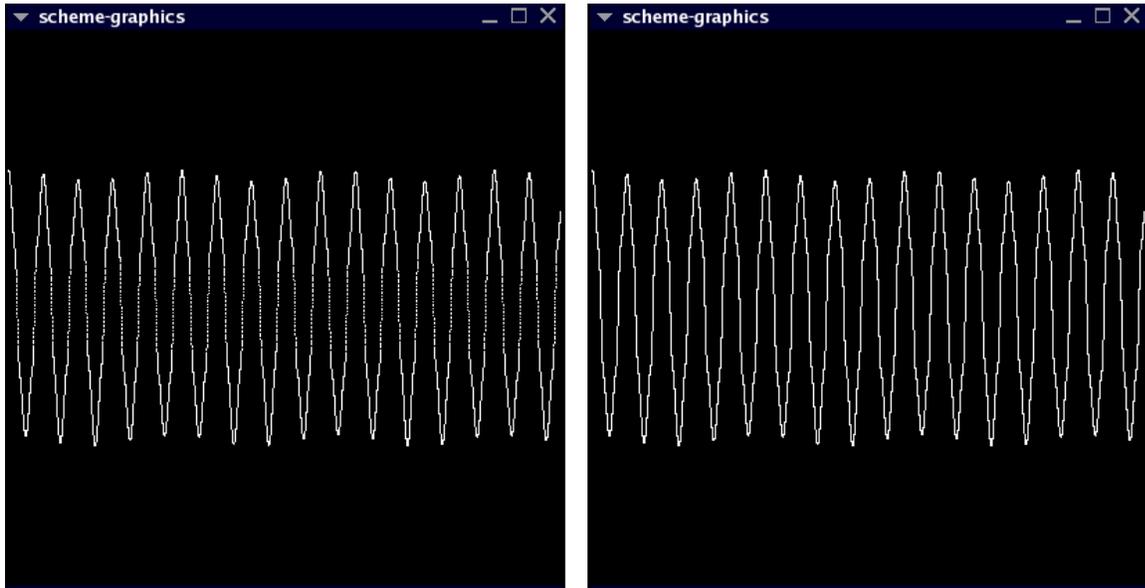


Figure 7: Trajectory of the  $m_1$  pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ .

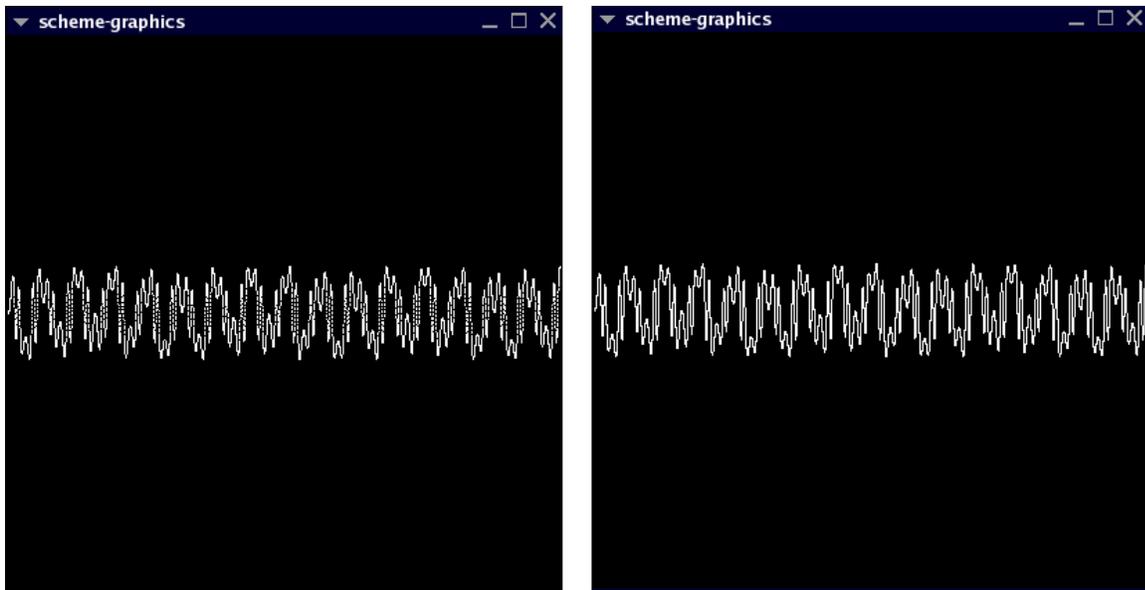


Figure 8: Trajectory of the  $m_2$  pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ .

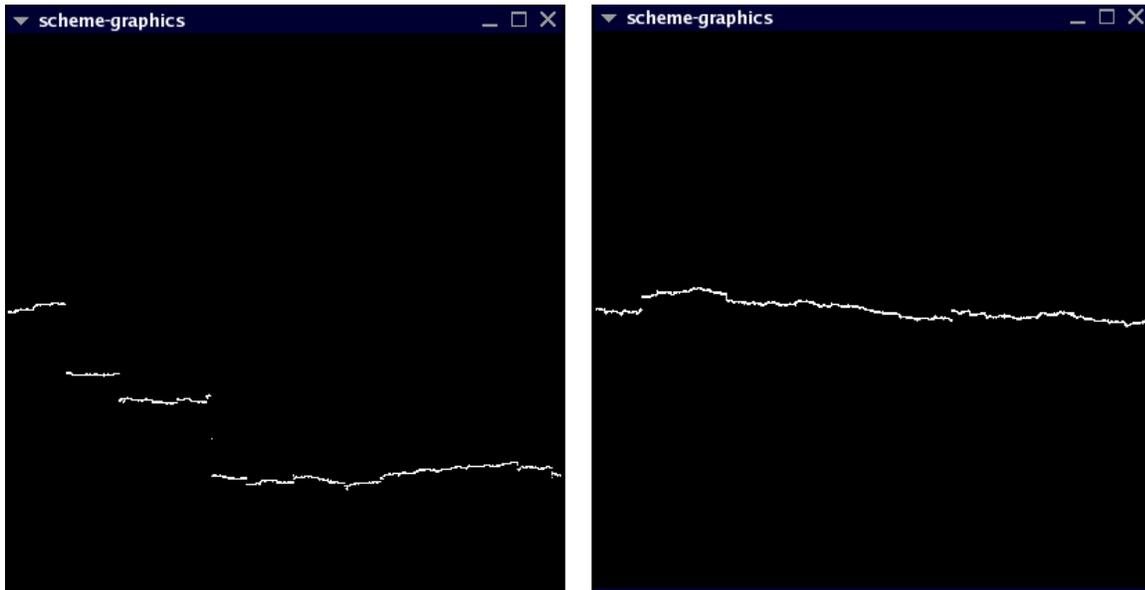


Figure 9: Total energy of the double-pendulum system from  $t = 0$  to  $t = 50$ , with step sizes of  $\Delta t = .005$  on the left and  $\Delta t = .001$  on the right. The y-axis goes from  $-10^{-10}$  to  $10^{-10}$ .

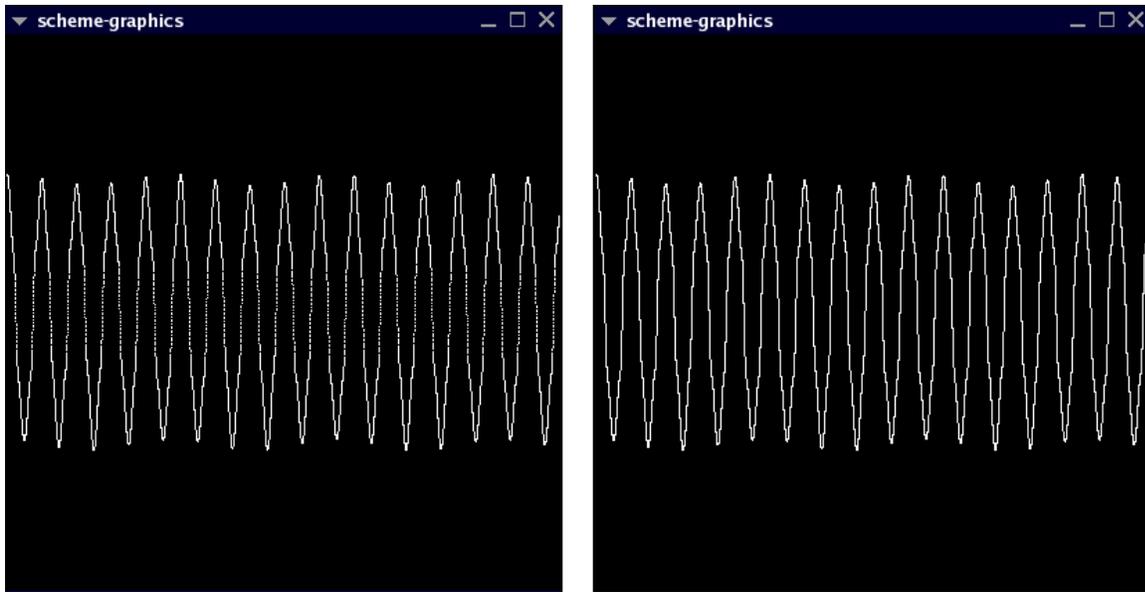


Figure 10: Trajectory of the  $m_1$  pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ .

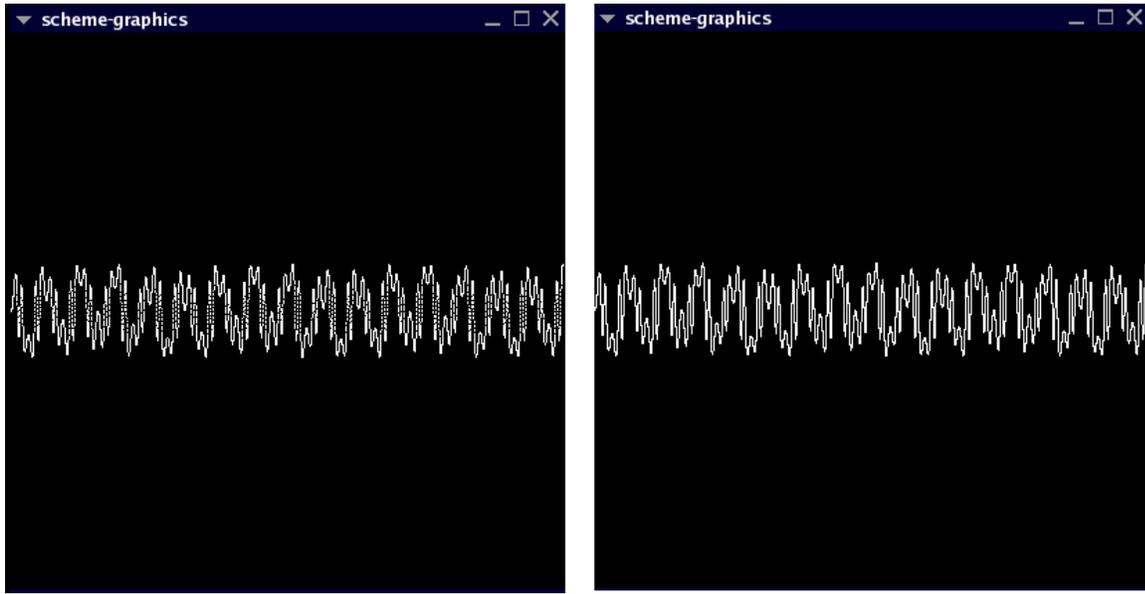


Figure 11: Trajectory of the  $m_2$  pendulum bob from  $t = 0$  to  $t = 50$ . The figure on the left was integrated with a step size of  $\Delta t = .005$ , and on the right with  $\Delta t = .001$ .

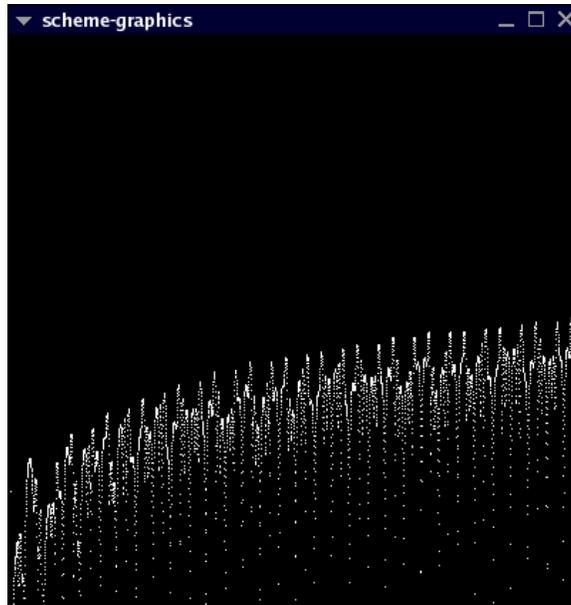


Figure 12: Plot of the logarithm of the squared differences of the distances between the  $m_2$  bobs, from  $t = 0$  to  $t = 50$ . The scale is from  $-50$  to  $-30$ .