QUOTE OF THE WEEK

By 1970, I had become convinced not only that black holes are an inevitable consequence of general relativity theory and that they are likely to exist in profusion in the universe, but also that their existence implies the mutability of physical law. If time can end in a black hole, if space can be crumpled to nothingness at its center, if the number of particles within a black hole has no meaning, then why should we believe that there is anything special, anything unique, about the laws of physics that we discover and apply? These laws must have come into existence with the Big Bang as surely as space and time did.

> --- John Archibald Wheeler with Ken Ford
> *Geons, Black Holes and Quantum Foam*, page 351

ASSIGNMENT: WEEK 4

For the week 2/23 through 3/1

INTRODUCTION

Chapter 2 of EBH presented the first Big Idea of the text: the *metric* that relates wristwatch time between two events to the incremental change in coordinates between those two events. Chapter 3 employs the second Big Idea: the *Principle of Maximal Aging*. Remember the twin paradox? The twin who relaxes at home ages more than the twin who streaks to a distant star and returns. Nature commands the free stone*: Follow the path of maximal aging.* In an inertial frame the path of maximal aging is a straight worldline. And it is true in general relativity. Why? Because in general relativity you can always find a local inertial frame in which the free stone is currently moving. And in that local inertial frame special relativity applies. So the special relativity command "Follow a straight worldline in an inertial frame." becomes the general relativity command "Follow a straight worldline in a *local* inertial frame." One big job of general relativity is to patch together local inertial frames to describe curved spacetime. In the patched-together spacetime of general relativity, the Principle of Maximal Aging summarizes Nature's command to the stone.

This week we apply the two Big Ideas to predict the motion of a stone as it plunges radially toward a black hole. The main result is a new expression for the total energy of the stone. We use this expression for energy to predict the changing values of velocity of our plunging stone as measured in different reference frames. One striking prediction: The far-away bookkeeper concludes that the stone *slows down* as it approaches the horizon, coasting to rest at the horizon over an unlimited far-away time. In contrast, the plunging observer riding on the stone zips through the horizon, streaks ever downward inside the horizon, and is tidally

squeezed and pulled into spaghetti near the crunch-point at the center of the black hole. Weird? Welcome to general relativity!

Thorne fills in the human and historical background of our tale: The almost irrational resistance of Einstein and others to the idea of a black hole; the prediction by a nineteen-year-old Indian Subrahmanyan Chandrasekhar (for whom the current orbiting Chandra X-ray observatory is named) that a white dwarf cannot sustain itself against the pull of gravity if its mass is greater than about 1.4 times the mass of our Sun; the public humiliation of Chandrasekhar by the eminent astrophysicist Sir Arthur Eddington, resulting in Chandrasekhar's retreat from the study of black holes for three decades, during which others took up the task.

READINGS

EBH Chapter 3, Plunging
AND
Thorne Chapter 3, Black Holes Discovered and Rejected
AND
Thorne Chapter 4, The Mystery of the White Dwarfs
AND
Notes: Sections 1 and 2 of "How Gravitational Forces Arise from Curvature"
OPTIONAL
The homework exercises refer to pages B-12 thru B-14 of Project B, Inside the Black Hole in EBH. You may want to "read around" these pages to master the context.

EVENING SEMINAR

Dr. Fred Baganoff will talk about the giant black hole in the center of our galaxy, Sagittarius A*, on Wednesday evening, February 23, from 7 to 8:30pm in Room 37-181. Feel free to come early for cookies and conversation!

RECITATION

At the usual hour 2:30 to 4pm in Room 2-131
At least one of the following questions will appear on the recitation quiz this week.

1. What "physical impossibilities" did Einstein invoke to disprove the possible existence of a black hole? Were Einstein's predictions about these physical impossibilities correct? What was wrong with his arguments?

2. An observer riding inside a freely-falling small capsule approaches the horizon of a spherical black hole of mass one million times the mass of our Sun. Which of the following statements are true as she crosses the horizon? EXPLAIN your choice.
(a) She will experience a sudden jolt. (b) The numerical value of the speed of light she measures inside her capsule will differ from the c measured far from the black hole. (c) She will be crushed at the center simultaneously with crossing the horizon as

recorded on her clock. (d) She will not be able to tell, from experiments inside the capsule, when she crosses the horizon.

3. The observer described in question 3 is now inside the horizon. Which of the following statements are true? EXPLAIN your choice. (e) Looking out through the transparent capsule, she will not be able to see any distant stars. (f) She will not be able to receive email from her friends outside the horizon. (g) The plunger's friends, looking inward from a great distance, will see her being crushed at the center of the black hole. (h) The plunger will NOT be able to see the crunch point ahead of her as she approaches it.

PROBLEM SET
Due before 4 pm Tuesday, March 1 in Room 4-339B.

PROBLEM 1. PLUNGING FROM REST AT INFINITY
This is a modification of Exercise 1, page 3-28 of EBH.
Black hole Alpha has a mass M = 8 kilometers and a horizon radius of $2M$ = 16 kilometers. A stone starting from rest at a great distance falls radially into black hole Alpha. In the following consider all speeds to be positive and express these speeds as a decimal fraction of the speed of light.

A What is the speed of the stone measured by the shell observer at r = 50 kilometers as the stone passes the shell observer?

B What is the bookkeeper speed of the stone as it passes r = 50 kilometers?

C What is the speed of the stone measured by the shell observer at r = 32 kilometers as the stone passes the shell observer?

D What is the bookkeeper speed of the stone as it passes r = 32 kilometers?

E In a two or three sentences, explain why the speed change in parts A and C reckoned by the bookkeeper as the stone moves inward is qualitatively different from the speed change in parts B and D measured by shell observers at smaller radii.

PROBLEM 2. ENERGY CONVERSION USING A BLACK HOLE
This is a modification of Exercise 6, page 3-29 of EBH.
PLEASE READ that exercise before attempting this one.

NOTE that a shell observer can use special relativity to compute the kinetic energy of a particle of mass m that passes his shell with velocity that he measures to be v_{shell} .

Black hole Beta has a mass M = 10 kilometers and a horizon at $2M$ = 20 kilometers. A bag of garbage of mass m starts from rest at a power station that is a great distance from the black hole. This bag of garbage falls radially onto a shell of radius (reduced circumference) r = 30 kilometers. This shell has a machine that converts all of the kinetic energy of the incoming garbage into a light flash.

A What is the energy, measured by the shell observer at r = 30 kilometers, of the photons produced by this machine as a fraction of the rest energy m of the garbage?

B The machine now directs the resulting flash of light radially outward. What is the energy of this flash as it arrives back at the power station, energy expressed as a fraction of the original rest energy m of the garbage?

C Now the garbage (which was brought to rest at r = 30 kilometers when its kinetic energy was converted to photons) is released from the shell at radius r and falls into the black hole. What is the increase in mass of the black hole, expressed as a fraction of the original rest energy m of the garbage?

D What is the fractional efficiency of this energy converter, that is (energy output)/(energy input)?

Advice: Check your answers by looking for possible conservation relations among your numerical results in this exercise.

PROBLEM 3. HITTING A NEUTRON STAR

This exercise is a *major* revision of exercise 3, page 3-28 in Chapter 3 of EBH.

A particular nonrotating neutron star has a mass M = 1.35 times the mass of Sun and a radius of 10 kilometers. A stone starting from rest at a great distance falls onto the surface of this neutron star. Express all speeds as decimal fractions of the speed of light, and consider all speeds to be positive.

A If this neutron star were a black hole with the same mass, what would be the r-value of its horizon in kilometers?

B With what speed does the stone hit the surface of the neutron star as measured by someone standing on the surface?

C With what speed does the stone hit the surface of the neutron star according to the far-away bookkeeper?

Several years ago, it was thought that astronomical gamma-ray bursts might be caused by stones (asteroids) impacting neutron stars. Carry out a preliminary analysis of this hypothesis by assuming that the stone is made of iron. The observer standing on the surface

of the neutron star (as the shell observer, Section 6, page 3-17 of EBH) can use special relativity to calculate the kinetic energy of impact. The impact kinetic energy is very much greater than the binding energy of iron atoms in the stone, greater than the energy needed to completely remove all 26 electrons from each iron atom, and greater even than the energy needed to shatter the iron nucleus into its component 26 protons and 30 neutrons. So we neglect all these binding energies in our estimate. The result is a vaporized gas of 26 electrons and 56 nucleons (protons and neutrons). We want to find the average energy of photons (gamma rays) emitted by this gas.

D Explain in a few brief sentences why, just after impact, the electrons have very much less kinetic energy than the nucleons. So in what follows we neglect the initial kinetic energy of the electron gas just after impact.

E Very quickly the nucleons share their kinetic energy with the electrons. Assume that both the proton and the neutron have the rest energy (mass) 1 GeV. Estimate the temperature in MeV (= 2/3 times the average kinetic energy per particle in MeV) of the electron-nucleon gas ("plasma").

F The hot gas emits thermal radiation with characteristic photon energy approximately equal to the temperature. What is the characteristic energy of photons reaching a distant observer, in MeV?

NOTE: It is now known that astronomical gamma-ray bursts release much more energy than an asteroid falling onto a neutron star. Gamma ray bursts are now thought to arise from the birth of new black holes in distant galaxies.

PROBLEM 4. THE PLUNGER

An observer falls from rest starting a great distance from a black hole. Call this observer the **plunger**. The plunger falls past two shells an incremental reduced circumference dr apart at a radius r.

A How far apart dr_{shell} are these two shells as measured by a shell observer?

B The shell observer and the plunger can use special relativity to transform observed distances and times between them. How far apart dr_{plunge} the shells are as measured by the plunger?

C How long dt_{shell} as measured by the shell observer does it take the plunger to fall between these two shells?

D How long dt_{plunge} as measured by the plunger does it take for the two shells to pass her?

E What is the speed dr_{plunge}/dt_{plunge} at which the plunger measures the shells to be passing her? Compare this speed with the speed dr_{shell}/dt_{shell} of the passing plunger as measured by the shell observer. Account for the similarity of or difference between these two expressions for speed.

PROBLEM 5. CAN THE BOOKKEEPER BE A REAL OBSERVER?

NOTE ON UNITS: The supplementary notes by Bertschinger set c = 1 but carry the constant G along. G does not appear in the metrics in EBH, for reasons explained in Section 6 of Chapter 2. To make things simple, just set G = 1 in Bertschinger's notes for purposes of solving the exercises in this assignment.

Bertschinger's notes stress that bookkeeper coordinates, those that appear on the RIGHT side of the metric, can be entirely arbitrary, are often chosen for convenience of calculation, and need not represent distances or times recorded by any observer. In contrast, distance and time measured by an observer appear on the LEFT side of the metric: proper time $d\tau$ and proper distance ds (called $d\sigma$ in EBH) between two nearby events.

Nevertheless, it is fair to ask for any set of bookkeeper coordinates: Is there an observer whose clock reads bookkeeper time? The answer is "sometimes." In this exercise you will answer this question for two different cases: Schwarzschild time and rain time. In both cases use the truncated metric for motion in a plane ($\theta = \pi/2$).

A. Schwarzschild bookkeeper time. Is there an observer whose clock reads Schwarzschild bookkeeper time? Answer this question by setting $d\tau = dt$ in the Schwarzschild metric. Is there a location of a stationary clock ($dr = d\phi = 0$) such that the resulting equation is valid? How does this result square with the analysis of bookkeeper time in Chapter 2 of EBH?

B. Bookkeeper rain time. (References: Section 5 of the second set of Notes, "Gravity, Metrics and Coordinates" and EBH pages B-12 to B-14) Is there an observer whose clock reads rain time? Answer this question by setting $d\tau = dt$ in the rain metric (equation 21 of the Notes). Is there a location of a *stationary* clock ($dr = d\phi = 0$) such that the resulting equation is valid? Is there a radial velocity dr/dt_{rain} such that the equation is valid? There is a velocity equation in Chapter 3 that has the same form. What is the physical relation between these two expressions for velocity? What is the relation between the rain observer and the plunger in Exercise 4 above?

PROBLEM 6. ONE WAY MOTION INSIDE THE HORIZON

Answer the questions in QUERY 9 on page B-14 of EBH Project B Inside the Black Hole. In particular (part E) make a decisive argument showing that ANY object launched in the radially outward direction from a raindrop inside the horizon nevertheless moves with decreasing radius r.

PROBLEM 7. LAGRANGIAN MECHANICS

Classical (nonrelativistic) mechanics uses the Euler-Lagrange equation with integrand $f(x, \dot{x}; t) = T - U$ where T is the kinetic energy and U is the potential energy of a particle with position x and speed $\dot{x}$. In mechanics, the function $T - U$ is called the *Lagrangian*.

A. Using $U = m\,\Phi_N(x,y,z)$ and the appropriate expression for T in Cartesian coordinates (x,y,z), show that the Euler-Lagrange equation is identical to the usual form of Newton's laws. Assume three dimensions of space.

B. Use the metric of flat spacetime in spherical coordinates to write v^2 in terms of dr/dt and $d\phi/dt$ for a particle moving in the two-dimensional equatorial plane $\theta = \pi/2$.

C. Rewriting T in spherical coordinates and assuming that $\Phi_N = \Phi_N(r)$ depends only on r, use the Euler-Lagrange equations to obtain differential equations for $r(t)$ and $\phi(t)$. Identify the orbital angular momentum and show that it does not change with time.