PROJECT H. Cosmology

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1   Current Cosmology

We live in a golden age of astrophysics and cosmology. Observational satellites above Earth's atmosphere view the electromagnetic spectrum from microwave through gamma rays. These observations couple with those from ground-based observations in the visible and radio portions of the spectrum to yield a flood of images and data that arouse public interest and stimulate advances in theory. For the first time in human history data and testable models inform our view of the Universe most of the way back to its beginning.

Briefly, the current view is that some 13.7 billion years ago the Universe underwent a rapid expansion from a hot **Big Bang** which created a variety of particles. Most of the particles decayed or annihilated with antiparticles, leaving behind copious amounts of photons and neutrinos, a small density of **dark matter** particles of uncertain form, and finally trace amounts of the familiar electrons, protons and free neutrons (the latter decay with a half-life 11 minutes). During the first three minutes, collisions and cooling allowed some neutrons to bind to two protons to create stable helium nuclei and a tiny amount of stable lithium (three protons plus one neutron). About 380,000 years after the Big Bang the temperature dropped to the point that electrons could bind to nuclei, creating atoms and freeing photons to stream uninhibited through the Universe. The **cosmic microwave background radiation** we see in all directions brings us photons from that moment, but vastly redshifted due to the subsequent expansion of the Universe. The small variations in this background radiation (a few parts in 100,000) tell us about the **spatial curvature** of the Universe and the constituents of the Universe (described in this project). From these fluctuations grew the galaxies and clusters of galaxies that we now observe.

Gravitational compression into stars ignited nuclear burning which cooked lighter elements into heavier ones up to the most stable element, iron. Explosions of some dying stars spewed these elements into space, in the process creating nuclei heavier than iron. The initial expansion rate of the Universe slowed down due to the preponderant influence of matter attracting other matter, both visible matter made of atoms and the dark matter whose presence is inferred form the stable rotation of galaxies and motion within galaxy

clusters. Furthermore, the present Universe appears to be increasing its rate of expansion due to a so-far mysterious **dark energy**. If the most popular current cosmological models are correct, this accelerating expansion could continue indefinitely. This apparently crazy prediction will be described and justified more fully in this project.

In addition to nailing down more accurate numbers given by current observations, the goal of future observations is to explore the nature of dark matter which evidently accounts for about 23 percent of the matter in the Universe and dark energy which makes up about 73 percent. Baryonic matter, made of protons and neutrons in the form of atoms and their associated electrons, accounts for only 4 percent of the contents of the Universe.

2 Robertson-Walker Metric

The simplest models of the Universe assume that the Universe is **homogeneous** (the same everywhere) and **isotropic** (the same as viewed in all directions). As far as we can tell from our Earth-bound perspective, these assumptions appear to be correct when averages are taken over large enough regions of the Universe. In the words of cosmologist Martin Rees, "A box 200 million light-years on a side . . . can accommodate the largest aggregates. Such a box, wherever it was placed in the universe, would contain roughly the same number of galaxies, grouped in a statistically similar way, into clusters, filamentary structures, and so on."

Assuming symmetry, the homogeneity and isotropy of space, uniform spacetime curvature, and the possibility of expansion, what form can the spacetime metric take? In the 1930s Howard Robertson and Arthur Walker independently found the the metric that satisfies these conditions, a result that we call the **Robertson-Walker metric**. In a two-dimensional spatial plane the Robertson-Walker metric has the form:

$$d\tau^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2 d\phi^2\right) . \tag{1}$$

Here, $R(t)$ is a unitless **scale factor** that describes the expansion (or contraction) of the Universe. The scale factor $R(t)$ is *defined* to have the value unity at the current age of the Universe, t_0:

$$R(t_0) \equiv 1 \qquad t_0 \text{ is the present time .} \tag{2}$$

The constant K in equation (1) is called the **curvature parameter**. As the name implies, K describes the overall curvature of the Universe. (It describes only the *spatial* curvature, i.e. curvature with $dt = 0$. *Spacetime* can be curved regardless of the value of K.) Different values of K lead to three qualitatively different descriptions of this geometry, as
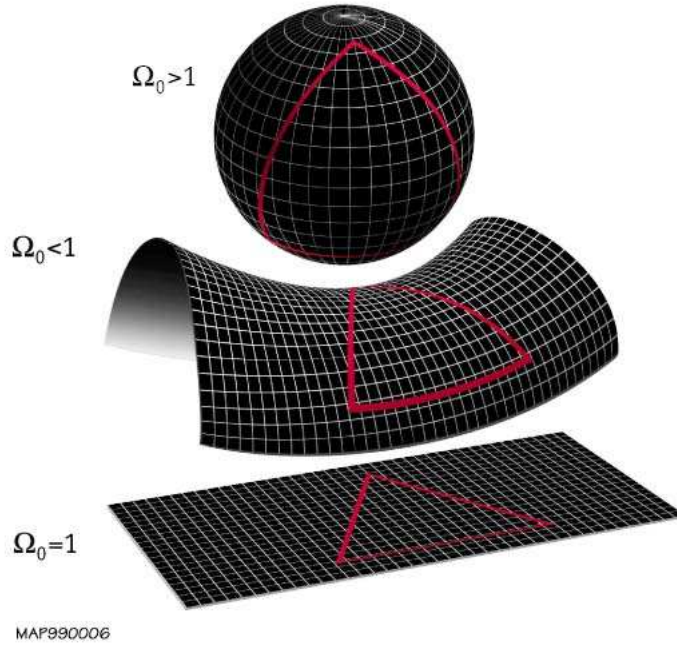
Figure 1: Spatial geometry of the Universe. Shown is the equatorial "plane" $\theta = \pi/2$ used in the metric of equation (1). The sign of the curvature parameter K is the same as the sign of $\Omega_0 - 1$, where Ω_0 is defined later in the text and called $\Omega_{\text{tot},0}$. Imprinted on each surface is a triangle made up of geodesics. (Image courtesy of the WMAP Science Team, from the WMAP website.)

shown in Figure 1. For $K = 0$ the Universe is **spatially flat**, having zero spatial curvature. Current observations support this value. For a positive value of K the Universe is **closed**, having positive curvature like a sphere. For a negative value of K the Universe is **open**, having negative curvature like a saddle. From equation (1) it is apparent that K has the units of inverse length squared. From it we can form a length $|K|^{-1/2}$ called the **curvature length**.

Alternative curvatures of the Universe

Write the spacelike form of the Robertson-Walker metric (1), that is with $d\sigma^2 = -d\tau^2$ on the left side. Consider the resulting metric at the present age of the Universe, so that $R(t) = R(t_0) = 1$. Choose an arbitrary point as the center $(r = 0)$ of your coordinate system. All of the following questions concern two events near to one another in space that occur simultaneously $(dt = 0)$ in the frame described by (1).

A. Show that very close to the origin $(r = 0)$, the proper distance between these two events is described by Euclidean geometry, no matter what the value of K.

B. For $K = 0$ (zero curvature) and r remote from the origin, is the proper distance between these two events greater than, equal to, or less than the distance reckoned using Euclidean geometry? What is the largest possible value of r?

C. For $K > 0$ (positive curvature) and r remote from the origin, is the proper distance between these two events greater than, equal to, or less than the distance reckoned using Euclidean geometry? What is the largest possible value of r?

D. For $K < 0$ (negative curvature) and r remote from the origin, is the proper distance between these two events greater than, equal to, or less than the distance reckoned using Euclidean geometry? What is the largest possible value of r?

The scale factor $R(t)$ helps us to deal with the confusion of lengths that change in an expanding Universe. It implies a set of **comoving coordinates** along with the physical (expanding) coordinates. Imagine drawing a grid onto a sheet of rubber and trying to describe the positions of dots marked onto the sheet as you stretch the rubber as it lies over a fixed flat floor. The physical positions of the dots (for example, their positions relative to a fixed point on the floor) will change in a complicated way as you pull on the sheet. However, relative to the grid you drew on the sheet (which is also being stretched) the dots are stationary and their positions are trivial to describe: constant. The bookkeeper coordinates on the grid which stretches with the rubber are analogous to comoving coordinates in the Universe.

The scale factor $R(t)$ tells us how to convert physical length (i.e., measured in light-years) into comoving length (i.e., measured using bookkeeper coordinates on your rubber grid) at some time t:

$$\ell_{\text{physical}}(t) = R(t) \, \ell_{\text{comoving}} \, . \tag{3}$$

The square of this relation is coded on the right side of equation (1) in the parenthesis with its coefficient $R^2(t)$. The quantities dr and $r d\phi$ are the comoving bookkeeper distances. Two stones with constant coordinate separation dr and $r d\phi$ will increase their meter-stick-measured physical separation as the scale factor $R(t)$ increases.

QUERY 2

Measuring Distance

Start with the spacelike form of the Robertson Walker metric (1) and assume that the Universe is flat (for which there is current evidence). At a time equal to half the current age of the Universe $(t = t_0/2)$ two stones have the incremental *coordinate* separation dr $(d\phi = 0)$.

A. What will be the *measured* separation between the stones at that time $(t = t_0/2)$?

B. Assume that the stones maintain the same bookkeeper coordinate separation until the present age of the Universe $(t = t_0)$. What will be the *measured* separation between the stones now?

C. How can a very long-lived observer (or observing machine) verify that the coordinate separation dr remains constant from $t = t_0/2$ to $t = t_0$?

The time t in equation (1) is the *local* time measured on clocks that ride on dust particles assumed to be locally at rest (with unchanging bookkeeper coordinates) as the scale factor changes. As usual, the proper time increment $d\tau$ is the time on the wristwatch of a particle moving from one event to another nearby event separated by the coordinates $dr, d\phi, dt$.

3 Friedmann-Robertson-Walker (FRW) Model of the Universe

The Robertson-Walker metric (1) does not provide a cosmological model, because it says nothing about how the scale factor $R(t)$ develops with time. In 1922 Alexander Alexandrovich Friedmann combined the Robertson-Walker metric with Einstein's field equations to

obtain what we now call the **Friedmann equation**, which relates the time rate of change of the scale factor to the total mass-energy density ρ_{tot}, assumed to be uniform throughout the Universe. The mass-energy density is however a function of time, $\rho_{\text{tot}}(t)$. The resulting model of the Universe is called the **Friedmann-Robertson-Walker model** or simply the **FRW cosmology**. The Friedmann equation is:

$$H^2(t) \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\rho_{\text{tot}}(t) - \frac{K}{R^2} \qquad \text{Friedmann equation .} \tag{4}$$

In this equation $\dot{R}$ is the first time derivative of R (local time, remember). The Friedmann equation defines the **Hubble parameter** H to be equal to $\dot{R}/R$ (note that it has the dimensions of inverse time), and will therefore change value as the scale factor $R(t)$ evolves with time. *So remember whenever you see H that it is really $H(t)$.* In this project we almost always use the value of H at the present time t_0 and give it the symbol H_0.

$$H_0 \equiv H(t_0) \qquad \text{Hubble parameter NOW .} \tag{5}$$

AN ASIDE ON UNITS: In the Friedmann equation (4), R is dimensionless, K has the units of inverse distance squared, and distance, time, and mass are all measured in the unit of length. Therefore density has the units of length/length^3 = length^{-2}. If you want to measure mass in conventional units, such as kilograms, then (see Section 6, Chapter 2 of EBH) the Friedmann equation becomes.

$$H^2(t) \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_{\text{tot}}(t) - \frac{Kc^2}{R^2} \qquad \text{Friedmann equation, conventional units .} \tag{6}$$

For simplicity we will use equation (4) in the following analysis.

Equation (4) can be written in a form that shows how expansion (H, stretching and flattening spacetime) fights with density (ρ_{tot}, whose gravitational attraction curves spacetime) to determine the overall curvature K. Multiply through by R^2 and rearrange:

$$K = R^2 \left(\frac{8\pi}{3}\rho_{\text{tot}} - H^2\right) . \tag{7}$$

A large density in (7) tends to increase the value of K, giving the Universe positive curvature. In contrast, a large expansion rate H tends to lower the value of K. In all cases, $\rho_{\text{tot}}(t)$ and $R(t)$ vary together so as to make K independent of time. This remarkable fact reflects the conservation of energy: $(HR)^2$ is proportional to the kinetic energy of an object in an expanding universe while the other term in equation (7) is proportional to minus the gravitational energy. The Einstein field equations link geometry and energy.

We need a benchmark value for the density ρ_{tot}, something with which to compare observed values. A useful reference density is the **critical density** $\rho_{\text{crit}}(t)$, which is the total density for which spacetime is flat, a condition described by the value $K = 0$. For densities greater than the critical density ($\rho_{\text{tot}} > \rho_{\text{crit}}$) the Universe has a closed geometry ($K > 0$). For densities less than the critical density ($\rho_{\text{tot}} < \rho_{\text{crit}}$) the Universe has an open geometry ($K < 0$). The Friedmann equation (4) shows that the Hubble parameter H is a function of time. Therefore the critical density also changes with time. We define the **critical density** $\rho_{\text{crit}, 0}$ as the value NOW, determined by the the Hubble constant H_0, the present value of the Hubble parameter. Substitute this value and $K = 0$ into the Friedman equation (4) to obtain:

$$\rho_{\text{crit}, 0} \equiv \frac{3H_0^2}{8\pi} \qquad \text{critical density NOW .} \tag{8}$$

The ratio of density today to critical density is a parameter used widely in cosmology. This parameter is given the Greek symbol capital omega, Ω:

$$\Omega_{\text{tot}, 0} \equiv \frac{\rho(t_0)_{\text{tot}}}{\rho_{\text{crit}, 0}} . \tag{9}$$

We retain the subscript zero as a reminder that we mean the density measured NOW relative to the critical value NOW. Combining equations (7) and (9) NOW (when $R \equiv 1$) gives a simple relation between the curvature parameter K and density parameter $\Omega_{\text{tot}, 0}$:

$$K = H_0^2(\Omega_{\text{tot}, 0} - 1) . \tag{10}$$

QUERY 3

Value of the critical density NOW

A. Estimate the numerical value of the critical density in equation (8) in units of (meters of mass)/meter^3 = meter^{-2}. For the value of H_0 see equation (30) and the following equations later in this project.

B. Express your estimate of the value of the critical density as a fraction of the density of water, one gram per cubic centimeter.

C. Express your estimate of the value of the critical density in units of hydrogen atoms (effectively, protons) per cubic meter.

The Friedmann equation (4) relates the rate of change of the scale factor $R(t)$ to the contents of the Universe. Before we can solve this equation for $R(t)$, we need to list the contributions to the total density and determine the time dependence of each. We now examine some of the strange properties of the components of the total density ρ_{tot} inferred from recent observations.

4 Contents of the Universe: How Density Components Vary with the Scale Factor $R(t)$

The Friedman-Robertson-Walker model of the Universe has been widely accepted for 40 years but observations during that time have significantly modified our picture of the contents of the Universe. Our knowledge has grown significantly over even just the last 5 years. Such is the excitement of being at the research edge of so large a subject.

We group the contents of the Universe into three broad categories: matter, radiation, and dark energy. Each category is chosen because of the way its contribution to the total density changes with time. We describe these changes in terms of the scale factor $R(t)$, leaving until later (Section 6) the derivation of how this scale factor changes with time.

4.1 Matter

The first category we refer to as **matter**. By matter we mean particles or nonrelativistic objects with rest mass much greater than the mass-equivalent of their kinetic energy. Objects in this category are:

- **STARS**, including white dwarfs, neutron stars, and black holes.

- **GAS**, mostly hydrogen, with a smattering of other elements and dust.

- **NEUTRINOS**, very light (but recently determined to be massive) particles produced, among other ways, by the decay of free neutrons.

- **DARK MATTER**, the non-luminous stuff, as yet unidentified, that makes up most of the matter in the Universe.

Stars, interstellar gas and dust are made of atoms. Cosmologists sometimes call atomic matter **baryonic** matter because most of the mass is made of baryons (essentially protons and neutrons). The mass of an electron is negligible compared to the mass of atomic nucleus, so even though electrons are not technically baryons the distinction is unimportant here.

Current observations lead to the estimate that **luminous matter**, the stars we can see, make up about one percent of the density of the Universe, with stars and gas together totaling four percent. It seems incredible that all the stars, individually and in galaxies and groups of galaxies, that we observe from Earth's surface and from satellites, taken together, have only a minor influence on the development of the Universe. Yet this is what we observe.

Cosmic background neutrinos have never been directly detected but their presence is inferred from our understanding of nuclear physics in the early universe. They contribute at most a small fraction of one percent to all the mass in the Universe.

Dark matter is currently estimated to account for approximately 23 percent of the mass-energy of the Universe. What is dark matter? And how do we know that it contributes so large a fraction? We do not know what dark matter is, but from observations we infer its density and some of its properties. From the **rotation curves** of galaxies (the tangential velocities of gas as a function of distance from the center) we can derive the magnitude of gravitational forces needed to keep the galaxies from flying apart, and, by implication, the amount and distribution of matter in them. The results show that luminous matter in a galaxy, which of course is all that we can observe directly, typically provides only a few percent of the mass required to bind the galaxy. Dark matter was originally postulated in the 1970s to complete the total needed to keep each galaxy together as it rotates. Observations on the dynamics of galaxy clusters — first made in the 1930s and greatly refined in the 1980s and 1990s — provide further evidence for the presence of dark matter.

The energy density of a gas of particles (whether particles of baryonic matter or dark matter) is the number density n times the energy per particle: nE. For nonrelativistic matter the energy per particle is well approximated by its mass m, so the matter density becomes $\rho_{\mathrm{mat}} = nm$. The mass of the particle is a constant (independent of the expansion of the Universe). However, the number density n, the number of particles per unit volume, drops as the volume increases, varying with the scale factor as R^{-3}, since volume is length cubed. By the definition in equation (2), the scale factor $R(t)$ has the value unity at the present age of the Universe t_0. Call $\rho_{\mathrm{mat},0}$ the value of the matter density at the present time. Then at any time t we have:

$$\rho_{\mathrm{mat}}(t) = \rho_{\mathrm{mat},0} R^{-3}(t) \ . \tag{11}$$

Equation (11) tells us that if we know the matter density today and the scale factor $R(t)$ as a function of time, we can determine the value of the matter density at some other time.

4.2 Radiation

The category **radiation** consists today of photons. At much earlier times, the neutrinos were relativistic particles with kinetic energy much greater than their rest mass energy. At such times, they were a significant part of the radiation component.

At the present stage of the Universe radiation is a whisper, but it used to be a shout. Shortly after the Big Bang, radiation contributed the dominant fraction of the mass-equivalent

energy density of the Universe. In the hot ionized plasma of the early Universe, radiation and matter were tightly coupled: photons were continually scattered by free electrons. About 380,000 years after the Big Bang the Universe cooled to a temperature of about 3000 K at which point electrons combine with protons to create hydrogen gas (with some helium and a trace amount of lithium). At that time the Universe became transparent to radiation, and photons were essentially decoupled from matter. The cosmic microwave background radiation we observe in all directions is a view of that early transition from opaque to transparent, with later expansion lowering the temperature to 2.73 degrees Kelvin. It is amazing to think that the low-energy photons that we detect as background radiation between the stars have been streaming freely for billions of years, not interacting with a single thing until they enter our detectors.

The number of photons emitted by all the stars in the history of the Universe is tiny compared with the number of photons created in the hot Big Bang. In the early Universe these photons were continually being emitted, absorbed, and scattered, but the *number* of photons is approximately constant as the Universe expands. Therefore the number of photons per unit volume varies inversely as the scale factor: $R^{-3}(t)$, just as matter does. But there is an additional effect for photons. The equation $E = hf = hc/\lambda$ connects the energy E of a photon to the wavelength λ of the corresponding classical wave. As this wave propagates through an expanding space, its wavelength increases in proportion to $R(t)$. (This increased wavelength is observed as the redshift of light from distant galaxies. This effect is related to, but different from, the Doppler shift of light from a receding object in the flat spacetime of special relativity.) An increasing wavelength implies a *decrease* in the energy of each photon, an energy proportional to R^{-1}. This leads to an extra (inverse) power of R compared with that for matter in equation (11) because of the drop in energy of each photon as the Universe expands. Let $\rho_{\text{rad},0}$ represent the density of radiation at t_0, the present age of the Universe. Then the radiation density for other values of the scale factor $R(t)$ obeys the equation

$$\rho_{\text{rad}}(t) = \rho_{\text{rad},0} R^{-4}(t) \ . \tag{12}$$

QUERY 4   **Mass density of radiation**

The cosmic microwave background radiation has a nearly perfect black-body spectrum with current temperature $T = 2.726$ K. The energy density (energy/length^3) of blackbody radiation in conventional units is given by the equation

$$u_{\text{rad}} = \frac{\pi^2}{15} \frac{(k_{\text{B}}T)^4}{(c\hbar)^3} \equiv a_{\text{rad}}T^4 \ . \tag{13}$$

Here k_{B} is the Boltzmann constant, c is the speed of light, and $\hbar \equiv h/2\pi$ where h is the Planck constant. The summary symbol a_{rad} is called the **radiation constant**.

A. Find the present value of the *mass* density corresponding to the cosmic background radiation, in kilograms per cubic meter.

B. Your result of part A is what fraction or multiple of the critical density, $\rho_{\text{crit},0}$?

C. Take the average energy of a photon in the gas of cosmic background radiation surrounding us to be $k_{\text{B}}T$. Estimate the number of photons per cubic meter. Compare your result with the critical mass density expressed in the number of hydrogen atoms (effectively, protons) per cubic meter.

D. At what absolute temperature T would blackbody radiation mass density be equal to the value of the critical density $\rho_{\text{crit},0}$ at the present time?

4.3   Dark Energy

After matter and radiation, the remaining contributor to the contents of the Universe is rather bizarre stuff which has been given the name **dark energy**. Dark energy is entirely unrelated to *dark matter*, the major component of *matter*. Dark energy is detected only indirectly, through its effects on cosmic expansion. Its composition is unknown. Dark energy is the component of the total density that accounts for the observed (and initially surprising) increase with time in the rate of expansion of the Universe. Observations described in

Sections 8 and 9 lead to the estimate that approximately 73 percent of the mass of the Universe is in the form of dark energy.

Dark energy is a generic term which encompasses all of the various possibilities for its composition. One possibility is the so-called **vacuum energy**. We often think of the vacuum as "nothing," but that is not the picture offered by modern physics through quantum field theory, which defines the vacuum to be the state of lowest possible energy. As the Universe expands, this lowest possible density does not drop but rather remains constant. Of what does vacuum energy consist? One way of thinking of it is **virtual particles** that are continually created and annihilated, according to quantum field theory. The existence of virtual particles is a well-known and well-tested consequence of the standard model of particle physics. For example, the presence of virtual particles in the surrounding vacuum has a small but detectable effect on the energy levels of hydrogen. Virtual particles surely have gravitational effects, but it has proved very difficult to estimate correctly the magnitude of these effects.

Cosmological effects of vacuum energy are described using the **cosmological constant** symbolized by the capital Greek lambda, Λ. In 1917, Einstein added the cosmological constant to his original field equations in order to make the Universe static, to keep it from collapsing from what he assumed must be an everlasting constant state. Einstein later removed the cosmological constant from the field equations when Hubble showed in 1929 that the Universe is expanding, but the cosmological constant continues to pop up in different theories of cosmology, as it does here as a possible form for dark energy. The presence of the cosmological constant in modern theory does *not* require a static Universe. In the 1960s, Zel'dovich and Gliner showed that vacuum energy is equivalent to the cosmological constant.

Other more complicated candidates for dark energy could have time-dependent energy density, but there is no current consensus about these possibilities. A full description of dark energy may have to await the development of a complete theory of quantum gravity, which does not yet exist. In this project we will not consider time-dependent dark energy.

IF vacuum energy accounts for dark energy, THEN as the Universe expands the density of dark energy remains constant: $\rho_{\text{dark energy}}(t) = \text{constant}$. We use the subscript Λ for dark energy to remind ourselves that we are assuming vacuum energy accounts for dark energy, and take ρ_Λ to be the the symbol for dark energy at the present time. We are assuming that this density has this same value at every time, earlier or later:

$$\rho_{\text{dark energy}}(t) = \rho_\Lambda(t) = \rho_\Lambda = \text{constant} . \tag{14}$$

The time-independent density of vacuum energy contrasts with the density of matter (proportional to R^{-3}) and the density of radiation (proportional to R^{-4}), both of which

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decrease as the Universe expands. As a result, dark energy influences the development of the Universe more and more as $R(t)$ increases.

4.4 Variation of the Total Density with the Scale Factor $R(t)$

We can now write an expression for the time dependence of total density from equations (11), (12), and (14),

$$\rho_{\text{tot}}(t) = \frac{\rho_{\text{mat},0}}{R^3(t)} + \frac{\rho_{\text{rad},0}}{R^4(t)} + \rho_\Lambda \ . \tag{15}$$

Divide through by the critical density at the present time, equation (8), to express the result as fractions of the present critical density, as in equation (9):

$$\frac{\rho_{\text{tot}}(t)}{\rho_{\text{crit},0}} = \frac{\rho_{\text{mat},0}}{\rho_{\text{crit},0}} R^{-3}(t) + \frac{\rho_{\text{rad},0}}{\rho_{\text{crit},0}} R^{-4}(t) + \frac{\rho_\Lambda}{\rho_{\text{crit},0}} \ . \tag{16}$$

We want to plot equation (16) as a function of the scale factor $R(t)$. To do this we need numerical values for the three fractional densities in that equation. These fractional densities also define contributions to the total density parameter Ω defined in equation (9).

In Sections 8 and 9 we describe current observations that yield the approximate values:

$$\Omega_{\text{mat},0} \equiv \frac{\rho_{\text{mat},0}}{\rho_{\text{crit},0}} = 0.27 \pm 0.04 \ . \tag{17}$$

$$\Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{\text{crit},0}} = 0.73 \pm 0.04 \ . \tag{18}$$

In QUERY 4 you showed that at the present time, the background radiation yields an energy density of approximately 5×10^{-5} times the critical density. Assuming that the neutrinos are relativistic would increase this by 68% implying

$$\Omega_{\text{rad},0} \equiv \frac{\rho_{\text{rad},0}}{\rho_{\text{crit},0}} = 8.4 \times 10^{-5} \ . \tag{19}$$

Actually the neutrinos are nonrelativistic today so this is not the correct value today; nonetheless, their contribution to the density today is so small that the error made by including them in equation (19) is negligible.

Figure 2 graphs equation (16) with numerical values given in equations (17), (18), and (19). Because each of the individual quantities is proportional to a power of $R(t)$, during times when one component dominates the total density, ρ versus R is a straight line on the log-log graph. Figure 2 shows that the radiation contribution has little effect at present, but was dominant at early stages because of the multiplier R^{-4} in equation (16). For a while after the radiation-dominated era, matter had the greatest influence on the evolution of the Universe. But the influence of matter is also fading by now because of the multiplier R^{-3}.
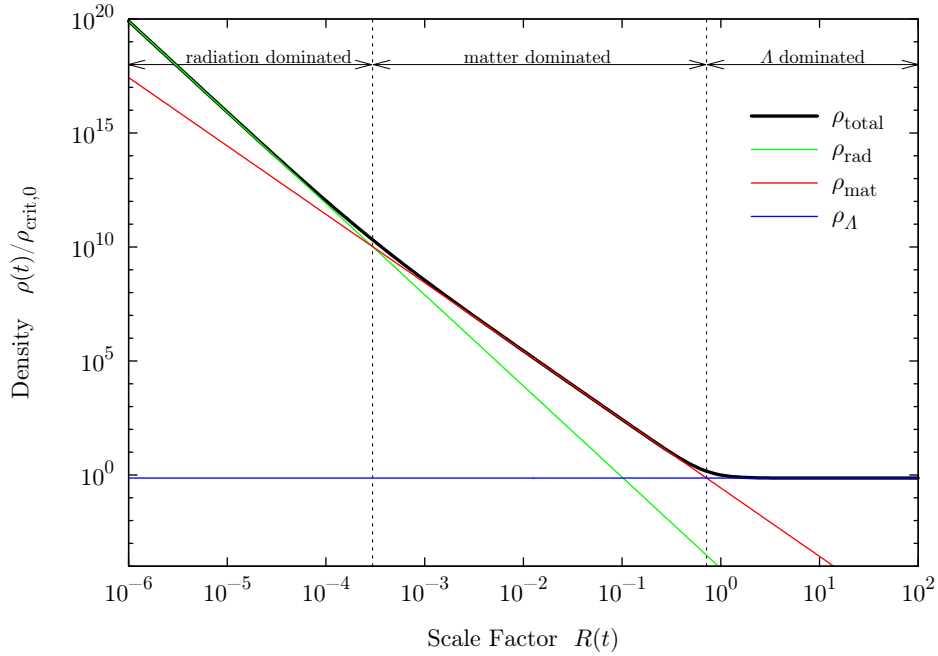
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Figure 2: Total mass-energy density of the Universe in units of the present critical value plotted versus the expansion scale factor. The vertical dashed lines denote transitions between the radiation-dominated early phase, the matter-dominated middle era, and the dark energy-dominated late stage of the Universe.

The contribution of dark energy was negligible in the distant past but has an increasing effect at the present and later stages of expansion, because its density remains constant as densities of matter and radiation decay away with the increase in $R(t)$. If the data and assumptions behind Figure 2 are correct, we are at the beginning of the era dominated by dark energy.

5 Evolution of Model Universes with Different Curvature

Using the Friedmann equation, we can analyze the development of alternative model Universes with different assumptions for the curvature K. We repeat the Friedmann equation (4), here:

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\rho_{\text{tot}} - \frac{K}{R^2} \qquad \text{Friedmann equation .} \tag{20}$$

Remember that this is a differential equation for the scale factor $R(t)$.

To put equation (20) in a more useful form, divide through by H_0^2 and substitute for the critical density from equation (8):

$$\left(\frac{H}{H_0}\right)^2 = \left(\frac{\dot{R}}{H_0 R}\right)^2 = \frac{\rho_{\text{tot}}}{\rho_{\text{crit},0}} - \frac{K}{H_0^2 R^2} \ . \tag{21}$$

Re-express equation (21) in terms of the components of Ω_{tot} defined in equations (17), (18), and (19):

$$\left(\frac{H}{H_0}\right)^2 \equiv \left(\frac{\dot{R}}{H_0 R}\right)^2 = \Omega_{\text{mat},0} R^{-3} + \Omega_{\text{rad},0} R^{-4} + \Omega_{\Lambda,0} - \frac{K}{H_0^2 R^2} \ . \tag{22}$$

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For the present time t_0, when $R(t_0) = 1$, we can write equation (22) in the very simple form:

$$1 = \Omega_{\text{mat}, 0} + \Omega_{\text{rad}, 0} + \Omega_{\Lambda, 0} - \frac{K}{H_0^2} \qquad \text{NOW} . \qquad (23)$$

This equation allows us to determine the curvature parameter K from measurements of $\Omega_{\text{mat}, 0}$, $\Omega_{\text{rad}, 0}$, and $\Omega_{\Lambda, 0}$. Current observations lead to the conclusion that, within measurement uncertainties of about 4% in Ω_0, the Universe is flat $(K = 0)$ — see equations (17), (18), and (19).

For any arbitrary time, equation (22) can be rearranged to read:

$$\dot{R}^2 - H_0^2 \left[\Omega_{\text{mat}, 0} R^{-1} + \Omega_{\text{rad}, 0} R^{-2} + \Omega_{\Lambda, 0} R^2 \right] = -K . \qquad (24)$$

Compare equation (24) with the corresponding Newtonian expression derived from the conservation of energy for a particle moving in the x-direction subject to a potential $V(x)$:

$$\dot{x}^2 + \frac{2V(x)}{m} = \frac{2E_{\text{total}}}{m} \qquad \text{Newtonian} . \qquad (25)$$

In the Newtonian case we can get a qualitative feel for the particle motion by plotting $V(x)$ as a function of position and drawing a straight line at the value of E_{total}. We use equation (24) for a similar purpose, to get a qualitative feel for the evolution of the Universe. Rewrite equation (24) as:

$$\dot{R}^2 + V_{\text{eff}}(R) = -K . \qquad (26)$$

Here the $-K$ on the right takes the place of total energy, and $V_{\text{eff}}(R)$ is an effective potential given by the equation

$$V_{\text{eff}}(R) \equiv -H_0^2 \left[\Omega_{\text{mat}, 0} R^{-1} + \Omega_{\text{rad}, 0} R^{-2} + \Omega_{\Lambda, 0} R^2 \right] . \qquad (27)$$

We summarize here the assumptions on which equations (24), (26), and (27) are based.

ASSUMPTIONS FOR THE DEPENDENCE OF $\dot{R}$ ON $R(t)$

1. The Universe is homogeneous (the same in all locations).

2. The Universe is isotropic (the same as viewed in all directions).

3. Dark energy is vacuum energy and therefore its density is constant, independent of $R(t)$.

4. *Background assumptions*: There are no other forms of mass-energy in the Universe; spacetime has four dimensions; general relativity is correct; the Standard Model of particle theory is correct, and so on.
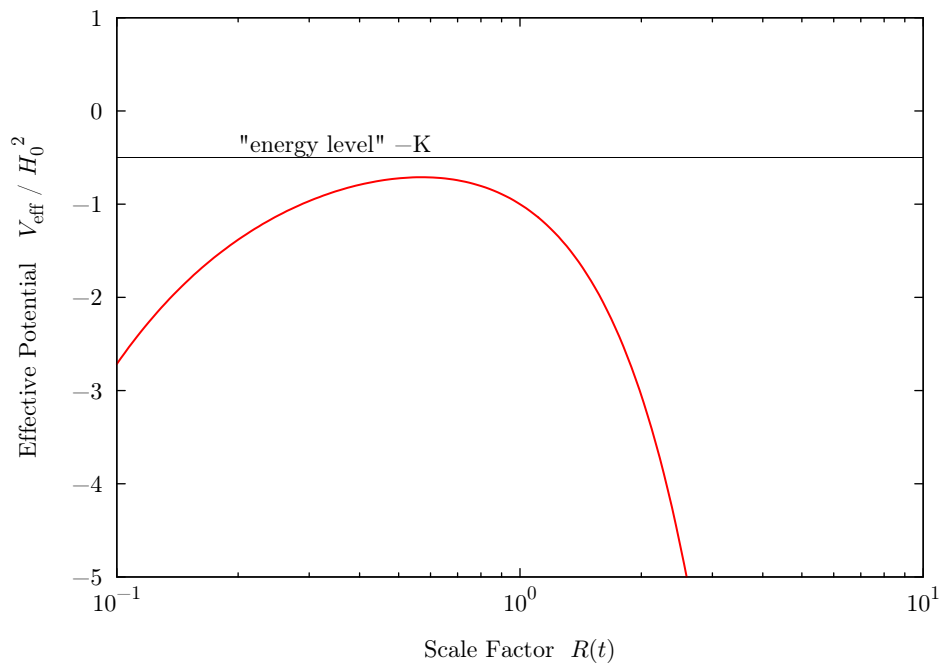
Figure 3: Effective potential governing the evolution of $R(t)$ according to equations (26) and (27). The "energy level" is set by $V_{\text{eff}}/H_0^2 = -K$. The figure shows an example of a closed universe that expands forever. Our Universe has $K = 0$ to a good approximation and will apparently expand forever.

Figure 3 plots V_{eff}/H_0^2 as a function of $R(t)$, using the values of the densities given in equations (17), (18), and (19). For the range of $R(t)$ plotted, radiation has negligible effect. Figure 3 carries a lot of information about the history and alternative futures of the Universe according to different values of K. In the Newtonian analogy, an effective potential with a *positive* slope yields a force tending to slow down positive motion along the horizontal axis, while the portion of the effective potential with a *negative* slope yields a force tending to speed up positive motion along the horizontal axis. These two conditions occur to the left and the right, respectively, of the peak at $R(t) \approx 0.57$. By analogy, then, $R(t)$ decelerates to the left of $R(t) \approx 0.57$ and accelerates to the right of $R(t) \approx 0.57$. This acceleration is due to dark energy. (CAUTION: Cosmological models described in older textbooks, written before dark energy was shown to be significant in the observed expansion of the Universe (about 1999), effectively assume that $\Omega_{\Lambda,0} = 0$ so that the expansion never accelerated.)

To check your understanding of the physical results summarized in Figure 3, answer questions in the following three QUERIES. Answers to some of these questions appear at the end of this project.

The Friedmann-Robertson-Walker Universe

The heavy line in Figure 3 represents the prediction of the Friedmann-Robertson-Walker Universe. Answer the following questions about the predictions of this model under the assumption that the Universe begins with a Big Bang. Make a reasonable assumption about the qualitative influence of radiation on $V_{\text{eff}}(R)$ for small $R(t)$.

1. True or False: The descending curve to the right of $R(t) \approx 0.57$ says that the Universe is contracting after $R(t)$ reaches this value.

2. Can the Universe be closed and expand forever?

3. Can the Universe be closed and recontract?

4. Can the Universe be open and expand forever?

5. Can the Universe be open and recontract?

6. Can the Universe be flat and expand forever?

7. Can the Universe be flat and recontract?

8. Describe qualitatively the evolution of a flat Universe ($K = 0$). Be specific about the evolution of $R(t)$ in the region to the right of the peak in the curve of V_{eff}/H_0^2.

9. What point on the graph of Figure 3 corresponds to a value of K that would lead to a static Universe? How could the Universe arrive at this configuration starting from a Big Bang? Is this static configuration stable or unstable, and what are the physical meanings of the terms *stable* and *unstable*?

6   Solution for the Scale Factor $R(t)$

Thus far we have stuffed all our ignorance about the time development of the Universe into the scale factor $R(t)$, as given in equation (15) and plotted along the horizontal axes in Figures 2 and 3. We need to determine how $R(t)$ itself develops with time. To do this we

integrate the Friedman equation (4) as modified in equation (24). Using equation (23), the latter equation can be rearranged to read:

$$\frac{dR}{dt} = H_0[\Omega_{\text{mat},0}(R^{-1} - 1) + \Omega_{\text{rad},0}(R^{-2} - 1) + \Omega_{\Lambda,0}(R^2 - 1) + 1]^{1/2} . \tag{28}$$

By eliminating the curvature we have shown that the components of Ω_{tot} completely determine the expansion of the universe — they are *important!* Now invert this equation and derive an integral with the limits from NOW ($R(t_0) = 1$) to any other arbitrary $R(t)$:

$$t - t_0 = \frac{1}{H_0} \int_1^R \frac{dR'}{[\Omega_{\text{mat},0}(R'^{-1} - 1) + \Omega_{\text{rad},0}(R'^{-2} - 1) + \Omega_{\Lambda,0}(R'^2 - 1) + 1]^{1/2}} . \tag{29}$$

Here R' is the dummy variable of integration. We can integrate equation (29) numerically from the present time t_0 to either a future time t ($R > 1$) or to an earlier time ($R < 1$). The Big Bang occurred when $R = 0$.

QUERY 7

Various kinds of Universes

Integrate equation (29) in three simplifying cases, under the assumption that spacetime is flat ($K = 0$).

A. Assume the Universe contains only matter and that $\Omega_{\text{mat},0} = 1$. Find an expression for $R(t)$ and the exact value of $H_0 t_0$.

B. Assume the Universe contains only radiation and that $\Omega_{\text{rad},0} = 1$. Find an expression for $R(t)$ and the exact value of $H_0 t_0$.

C. Assume that the Universe contains only dark energy and that $\Omega_{\Lambda,0} = 1$. Find an expression for $R(t)$.

D. *Optional.* Discuss the validity of your results for parts A, B, and C for $t < t_0$ and in particular for $t = 0$.

A major and a minor difficulty stand in the way of carrying out the integration of the full expression (29). The minor difficulty is expressing the result in units most convenient for us. Recall that the scale factor $R(t)$ is unitless and is defined to have the value unity at present time [equation (2)]. If we choose to express time in years, then the time derivative $\dot{R}(t)$ will have the units years^{-1}. Then, the current value of the Hubble constant H_0 will also be expressed in the unit of years^{-1}. Recent observations yield the following approximate value for H_0:

$$H_0 = 71 \pm 4 \; \frac{\text{kilometers/second}}{\text{Megaparsec}} . \tag{30}$$

We can convert this to the required units using the conversion factors:

$$1 \text{ parsec} \approx 3.26 \text{ light years} . \tag{31}$$

$$1 \text{ light year} \approx 9.461 \times 10^{12} \text{ kilometers} . \tag{32}$$

$$1 \text{ year} \approx 3.157 \times 10^{7} \text{ seconds} . \tag{33}$$

You can verify that the result is the value:

$$H_0 \approx 7.26 \times 10^{-11} \text{ year}^{-1} . \tag{34}$$

It is not a coincidence that the coefficient $H_0^{-1} = 1.38 \times 10^{10}$ years in equation (29) has a value approximately equal to the estimated age of the Universe: $t_0 \approx 13.7$ billion years. If $R(t)$ were a linear expansion, $R = At$ for some constant A, then because $R = R(t_0) = 1$ today, the age would be $t_0 = A^{-1}$. The Hubble constant is $H_0 \equiv \dot{R}(t_0)/R(t_0) = A$. So, for the case of linear expansion, $t_0 = H_0^{-1}$. Although the actual solution $R(t)$ is not linear in our universe, $R(t_0)/t_0$ is close to $\dot{R}(t_0) = H_0$ because the universe has recently made the transition from deceleration to acceleration. Therefore the age of the universe approximately equals the **Hubble time** H_0^{-1}.

Integrating equation (29) requires that we know the values of the components of the total density. Remember that the total density Ω_{tot} determines the curvature parameter according to equation (23). In Section 8 we report observations that tend to support the conclusion that Universe is flat ($\Omega_{\text{tot},0} = 1$) or very nearly so. Therefore (29) has been integrated numerically for several cases as shown in Figure 4. The closed model ($\Omega_{\text{tot},0} > 1$) recollapses; all other models expand forever. The model with dark energy undergoes clearly accelerated expansion at late times.

7 WHY is the Rate of Expansion of the Universe Increasing?

Figure 4 displays changes in the scale factor $R(t)$ of the Universe as a function of time. The slope of the curve at any point is the velocity $\dot{R}(t)$ of the expansion factor at that time. Changes in the slope correspond to changes in this velocity. We can call the rate of change of velocity the *acceleration* of the scale factor, $\ddot{R}(t)$. Why does the Universe change its rate of expansion? For the matter-dominated era, one can understand that the matter mutually
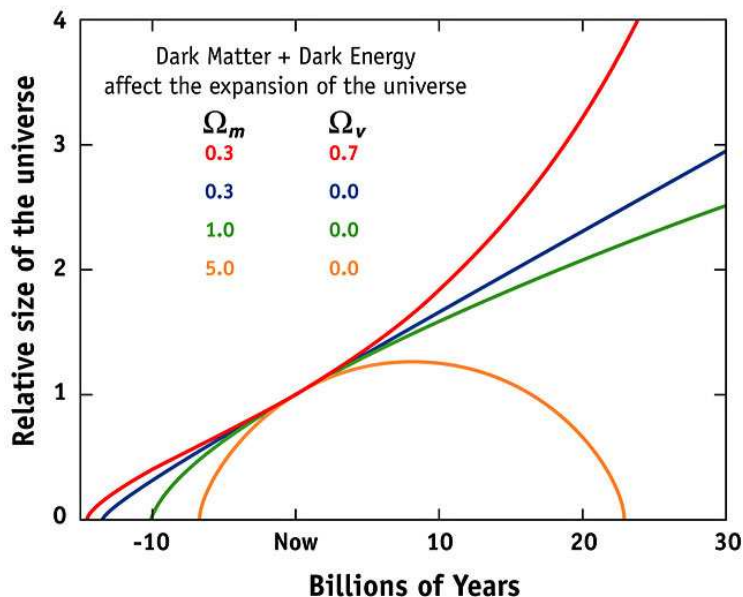
EXPANSION OF THE UNIVERSE

Figure 4: Expansion scale factor $R(t)$ versus time. The present matter density parameter is $\Omega_{\mathrm{mat},0} = \Omega_m$ while the present dark energy parameter is $\Omega_{\Lambda,0} = \Omega_v$. The Universe is flat if $\Omega_m + \Omega_v = 1$. The top curve has parameters close to the presently favored values $\Omega_{\mathrm{mat},0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$. (Image courtesy of the WMAP Science Team, from the WMAP website.)

attracts and "holds back" or "slows down" the expansion, as shown in Figure 3. But the expansion in the dark energy-dominated era clearly violates this explanation, since the rate of expansion increases there. What is the physical reason for this increased expansion rate? This question is the subject of the present section.

In the following QUERY 8 you will show that the second time derivative, the acceleration $\ddot{R}$ of the scale factor, depends not only on the density ρ_{tot} but also on pressure. Pressure, along with total density, appears in Einstein's field equations. In special relativity, pressure and energy density transform into each other under Lorentz transformations in a way analogous to (but not the same as) electric and magnetic fields. Density in one frame implies pressure in another. Since the Einstein field equations are written to be valid in any frame, pressure must make a contribution to gravity. Positive pressure has an attractive gravitational effect similar to positive density.

The gravitational effect of pressure may seem paradoxical: the *greater* the positive pres-

sure, the *more negative* the value of the acceleration. We are used to watching pressure expand things like a bicycle tire. The stretching surface of an expanding balloon is often used as an analogy to the expanding space of our Universe. These images can carry the incorrect implication that positive pressure is what makes the Universe expand. A balloon is expanded by pressure *differences*: the pressure inside the balloon is higher than the pressure outside combined with the balloon tension. Pressure differences produce mechanical forces. By contrast, we are considering a homogeneous pressure, the same everywhere – there is no "outside" of the Universe for it to expand into. There is no mechanical force of pressure in this case, only a gravitational force.

Here comes the big surprise. In QUERY 9 you show that dark energy leads to *negative* pressure. In contrast to positive pressure, negative pressure tends to *increase* the rate of expansion of the Universe. And recent observations bring evidence that we live in a Universe whose rate of expansion is increasing, not decreasing as our model would predict if only matter and radiation were present.

Now for the details.

The first law of thermodynamics says that as the volume of a box of gas increases by dV, the energy of the gas inside it decreases by an amount PdV where P is the pressure of the gas. [We use capital P to represent the pressure because lower case p is easily confused with rho (ρ), which often appears in the same equation.] The energy change PdV goes into the work done by the force due to the pressure acting on the outward-moving wall of the box. The energy of the gas is simply the volume that it occupies times its energy density. However, we measure energy in units of mass, so the energy density is just the mass density ρ_{tot}. Therefore we have

$$d(\rho_{\text{tot}} V) = -P_{\text{tot}} dV \ . \tag{35}$$

It turns out that this relation holds whenever the volume of a gas changes, regardless of the shape of the box. It even holds if there are no walls at all! It implies a general result: an expanding gas cools.

Acceleration of the Scale Factor

A. Divide the energy conservation equation (35) through by dt (in other words, consider the differential energy change in an increment of time dt) and apply it to a local volume V that has the current value V_0 and expands (or possibly contracts) with the Universe according to the equation $V = V_0 \, R^3$. Show that

$$\dot{\rho}_{\text{tot}} = -3\frac{\dot{R}}{R} \left(\rho_{\text{tot}} + P_{\text{tot}} \right) \ . \tag{36}$$

B. Rewrite the Friedmann equation (20) as

$$\dot{R}^2 = \frac{8\pi}{3} \rho_{\text{tot}} R^2 - K \ . \tag{37}$$

Take the time derivative of both sides of (37) and substitute equation (36) to obtain the equation for the acceleration of the cosmic scale factor:

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3} \left(\rho_{\text{tot}} + 3P_{\text{tot}} \right) \ . \tag{38}$$

This equation predicts that for a positive total density and positive total pressure, the scale factor will decelerate with time. The surprise comes when the pressure term is calculated for dark energy and we see that $\ddot{R}/R$ can become positive — this is what is usually meant by the phrase "the Universe is accelerating".

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Pressure from Different Sources

A. Solve equation (36) for P_{tot} and show that the result is:

$$P_{\text{tot}} = -\frac{R}{3\dot{R}}\dot{\rho}_{\text{tot}} - \rho_{\text{tot}} \ . \tag{39}$$

Equation (39) is linear in $\dot{\rho}_{\text{tot}}$ and ρ_{tot}. Therefore we can apply it separately to the different components of which ρ_{tot} and P_{tot} are composed. In parts B through D below, apply equation (39) to each component of the density to find the the individual pressures due to matter, dark energy, and radiation. (Hint: In each case express the first term on the right of (39) in terms of the corresponding fractional density.)

B. Apply equation (39) to nonrelativistic matter for which $\rho_{\text{mat}}(t) = \rho_{\text{mat},0}R^{-3}(t)$. What is the pressure $P_{\text{mat}}(t)$?

C. Apply equation (39) to dark energy for which $\rho_\Lambda(t) = \text{constant} = \rho_{\Lambda,0}$. What is the pressure $P_\Lambda(t)$? This surprising result has a large effect on the fate of the Universe.

D. Finally, apply equation (39) to a gas of photons. Though we can neglect ρ_{rad} in describing how the Universe behaves today, Figure 2 shows that in the early Universe ρ_{rad} was in fact larger than the corresponding matter term ρ_{mat} and could not be neglected. For radiation, $\rho_{\text{rad}}(t) = \rho_{\text{rad},0}R^{-4}(t)$. What is the pressure of radiation $P_{\text{rad}}(t)$?

E. Substitute your results of parts B through D into equation (38) to find an expression for $\ddot{R}$ as a function of $R(t)$:

$$\ddot{R} = -\frac{4\pi}{3}[\rho_{\text{mat},0}R^{-2} + 2\rho_{\text{rad},0}R^{-3} - 2\rho_{\Lambda,0}R] \ . \tag{40}$$

F. Assuming that $\rho_{\text{rad},0}$ is negligible, show that the condition for acceleration today $(R = 1)$ is

$$\Omega_{\Lambda,0} > \frac{1}{2}\Omega_{\text{mat},0} \ . \tag{41}$$

The result of part C of QUERY 9 tells us that the pressure of the vacuum is negative, a result unfamiliar in elementary thermodynamics. However, it is perfectly physical —

neither the density nor the pressure of the vacuum arise from physical particles. The vacuum has constant energy density produced by quantum fluctuations. Conservation of energy — represented by equation (36) — then implies that the pressure must be negative. Negative pressure — but not negative mass density — is physically allowed.

At early time the contributions of matter and radiation overwhelm the contribution of dark energy in equation (40), implying the expansion decelerated. Only for $R > 0.57$ does the dark energy term overtake the other terms leading to accelerated expansion.

Now that we have a model for the time development of the Universe, we need to validate the assumptions that went into details of applying that model, namely the values of $\Omega_{\mathrm{mat},0}$ and $\Omega_{\Lambda,0}$ given in equations (17) and (18) along with the value of $\Omega_{\mathrm{rad},0}$ given in equation (19).

8   Observations of the Contents of the Universe

8.1   Galaxy Rotation: Evidence for Dark Matter

How do we know that dark matter exists around galaxies? The most direct evidence comes from observing the orbits of stars or gas around a galaxy. Spiral galaxies are perfect for this exercise — their rotating disks contain neutral hydrogen gas that emits radiation with a rest wavelength of 21 cm. If the galaxy is edge on, then as gas orbits the galaxy it moves directly away from us on one side of the galaxy and directly towards us on the other side. Using the Doppler effect, astronomers can measure the speed of the gas as a function of its distance from the center of the galaxy. The result is a **rotation curve**.

Figure 5 shows the rotation curve of a nearby edge-on spiral galaxy. It is quite different from a graph of the orbital speeds of planets in the Solar System, which decrease with increasing distance from the sun according to Kepler's Third Law. Spiral galaxies by contrast almost always have nearly-constant rotation curves at radii outside of their dense centers.

The evidence for dark matter comes when we ask what one would *expect* the rotation curve to be if the gravitating mass were composed of only the observed stars and gas. Optical measurements of spiral galaxies show that the **surface luminosity density**, $\Sigma(R)$, is exponential to a very good approximation:

$$\Sigma(R) = \Sigma_0 \exp(-R/h) \ . \tag{42}$$

The surface density is defined as the total luminosity emitted along a column perpendicular to the galactic disk; if $\rho(r)$ is the volume density, then $\Sigma(R) = \int_{-\infty}^{\infty} \rho(\sqrt{R^2 + z^2}) \, dz$. It has
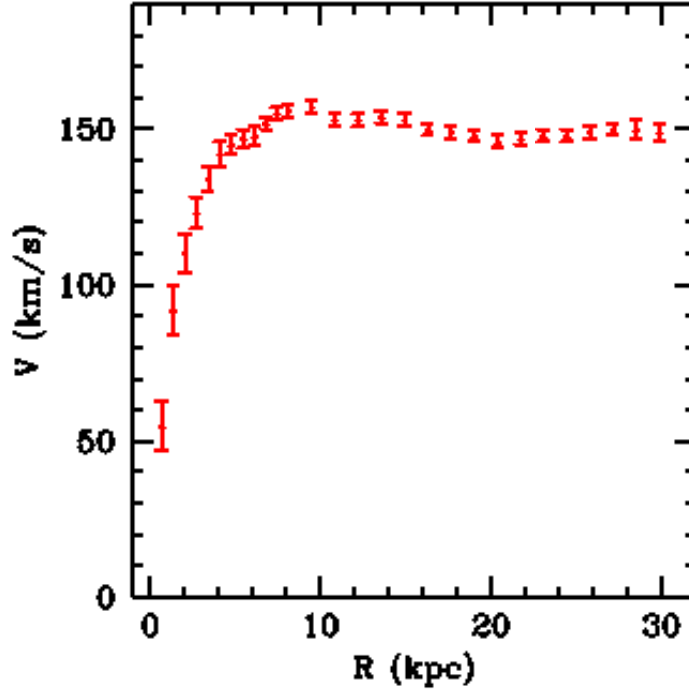
Figure 5: Rotation curve for spiral galaxy NGC 3198, from Begeman 1989, *Astronomy and Astrophysics*, 223, 47.

units of luminosity (typically Watts or solar luminosities, $L_\odot$) per unit area (typically square meters or square parsecs).

The form of equation (42) has two constants: Σ_0, the central surface luminosity density, and h, the disk scale length. For NGC 3198 the approximate values for these parameters are

$$\Sigma_0 = 100 \ L_\odot/\text{Parsec}^2 \ , \quad h = 2.7 \text{ kilparsec} . \tag{43}$$

One solar luminosity ($L_\odot$) is the amount of power emitted by the sun in optical light. To get the total luminosity emitted between radii R and $R + dR$, multiply by the area of the annulus: $dL = \Sigma(R) \, 2\pi R \, dR$. The total light emitted out to radius R follows immediately by integration.

To predict the rotation curve arising from luminous matter we need to how much *mass* there is, not how much *light*. If the luminous matter in galaxies is mainly stars like the sun, then the light in solar luminosities, $L_\odot$, equals approximately the mass in solar masses,

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$M_\odot$. In other words, if the total light emitted interior to radius R is $L(R)$, then the total luminous mass (stars and gas) is $M(R) = \Upsilon L(R)$ where $\Upsilon \equiv 1 M_\odot/L_\odot$ is a factor called the **mass-to-light ratio**. Not all stars have the same mass-to-light ratio. A reasonable range for spiral galaxies is $0.5 < \Upsilon < 5$ (remember the units of Υ are Solar masses per Solar luminosities).

Query 10 applies these ingredients so that you can show for yourself that NGC 3198 contains substantial amounts of dark matter.

Using galaxy rotation curve measurements, and a related technique in clusters of galaxies that uses X-ray emitting gas, for about 20 years astronomers have estimated $0.1 < \Omega_{\mathrm{mat},0} < 0.4$. The uncertainty was relatively large (a factor of 4!) because measurements based on luminous "tracers" (such as gas in galaxies or clusters) are insensitive to dark matter far away from galaxies. However, over the last decade increasingly sophisticated measurements of dark matter in and around galaxies have led to a consensus range $0.2 < \Omega_{\mathrm{mat},0} < 0.35$.

Dark Matter from a Rotation Curve

Figure 5 shows the rotation curve of NGC 3198. Combining it with the surface luminosity density of equation (42), one can show that the galaxy contains far more mass than can be accounted for by the stars.

Make the following assumptions:

1. Motion of stars in a galaxy can be described using Newtonian mechanics.

2. The mass density of the galaxy is spherically symmetric (but a function of radius R). (The circular speed has approximately the same value regardless of whether the mass is distributed in a thin disk or in a more spherical halo.)

3. Stars in the galaxy move in circular orbits at a speed V that is a function of radius R.

4. The surface mass density follows the same function as the surface luminosity density, implying that the mass enclosed in a sphere of radius R is

$$M(R) = \Upsilon \int_0^R \Sigma_0 e^{-r/h} \, 2\pi r \, dr \ . \tag{44}$$

A. Set up the Newtonian equation of motion and use it to find an expression for the circular speed V as a function of radius R, in terms of the enclosed mass $M(R)$.

B. Carry out the integration in equation (44) and use it to obtain a prediction for $V(R)$. Qualitatively describe the predicted $V(R)$. Does it have a maximum value? Does it approach a nonzero constant as $R \to \infty$? If not, how does it behave for $R \gg h$?

C. The observed rotation curve will exceed the predicted one if there is dark matter not accounted for by equation (44). Using Figure 5 and assuming that the luminous matter predominates for $R < 5$ kpc, what is the maximum mass-to-light ratio Υ for the luminous matter in NGC 3198?

D. Using the results of the previous parts together with Figure 5, determine the ratio of total mass to luminous mass contained within 30 kpc from the center of NGC 3198.

8.2 Type IA Supernovae: Standard Candles Revealing Dark Energy?

The rate at which the Universe expands actually increases with time. This amazing conclusion was reached in 1998 by two separate groups, one group including Adam Riess, Alexei V. Filippenko, and eighteen other people, the other group including Saul Perlmutter and thirty-one other people.

These observers use light from an exploding supernova of a particular kind, the so-called Type Ia supernova. A Type Ia supernova results when a white dwarf star gradually accretes mass from a large binary companion, finally reaching a mass at which the white dwarf becomes unstable and explodes into a supernova. The "slow fuse" on the gradual accretion process may lead to an explosion of almost the same size on each such occasion, giving us a "standard candle" of the same intrinsic brightness, provided that nuclear burning has left every white dwarf with the same nuclear composition. If so, the brightness of the explosion as seen from Earth provides a measure of the distance to the supernova. The cosmological redshift of the light tells us how fast the supernova is receding. Because supernovae (the plural of supernova) are so bright, they can be seen at a very great distance, which brings us information about the Universe much of the way back to the Big Bang.

The figures on page G-20 of EBH display the results of observations by both groups. The observed brightness is plotted on the vertical scale as a function of the redshift on the horizontal scale. Calibration of the vertical scale is in stellar magnitude, a logarithmic measure of observed brightness. The larger the magnitude number, the *dimmer* is the observed star. (The vertical scales of the two graphs differ by a standard magnitude, which moves the curve up and down without changing its shape.) The horizontal scale, also logarithmic, is calibrated in what is called the redshift factor z, defined implicitly in the following equation.

$$\lambda_{\text{observed}} \equiv (1 + z)\lambda_{\text{emitted}} \ . \tag{45}$$

Here $\lambda_{\text{observed}}$ is the wavelength of light observed from Earth, while λ_{emitted} is the wavelength of the light emitted from the source. The emitted wavelength is known if one knows the emitting atom, identified from the pattern of wavelengths in the spectrum. As we saw in Section 4.2, light is redshifted as it travels through the expanding universe — its wavelength increases in proportion to $R(t)$. Since $R = 1$ when the light is observed today, the time at which the light is emitted is related to redshift by

$$\frac{1}{1 + z} = R(t_{\text{emitted}}) \ . \tag{46}$$

Cosmologists routinely use redshift as a measure of cosmic time through this equation because redshift is directly measurable while the age of the universe at the time of emission is not.

If the data points in the figures lie along a straight line, then the expansion velocity is proportional to distance, as one would expect if nothing slowed down matter blasted out of the Big Bang. But the data points in the figure appear to lie slightly above the straight line for the most distant supernovae (upper right on the diagrams). This could mean that the most distant supernovae (highest on the vertical axis) are moving away slower than expected (farther to the left on the horizontal axis than expected). We are watching motions of these most distant supernovas as they were long ago, because it takes light such a long time to reach us. These data are taken to mean that the expansion rate long ago was slower than it is now. In other words, the expansion is accelerating.

8.3   Cosmic Microwave Background Radiation

The Universe is filled with a nearly uniform glow of microwaves having a **blackbody** spectrum called the cosmic microwave background radiation. The intensity as a function of frequency for such radiation is given by the Planck Law discovered in 1900 by Max Planck:

$$I(f) = \frac{2hf^3}{c^3} \frac{1}{e^{hf/k_{\mathrm{B}}T} - 1} \ . \tag{47}$$

Radiation with this spectrum (dependence on frequency) is produced by an opaque medium with temperature T. The microwave background radiation fits a Planck law stunningly well — the *CO*smic *B*ackground *E*xplorer (COBE) satellite measured the spectrum to match the Planck Law to about 1 part in 10^4 in the early 1990s. Figure 6 shows the measured spectrum; the estimate of the best-fit temperature has increased by 0.001 K to $T_0 = 2.726$ K since this figure was made in 1998.

At first glance, the microwave background radiation is absurd — the Universe is not opaque, and the matter we see is much hotter than 3 degrees above absolute zero. However, the microwave background radiation is a messenger from the early universe, and it has aged and become stretched out along the trip. The frequency of every light wave decreases in proportion to $1/R(t)$ as the universe expands. Remarkably, the form of the Planck law is preserved by the cosmic redshift. However, as a result of the cosmic redshift the radiation temperature also decreases in proportion to $1/R(t)$. In other words, at redshift z the radiation temperature was higher:

$$T(z) = (1 + z)T_0 \ . \tag{48}$$

This is an example of how cosmologists use redshift as a proxy for time since the Big Bang.

Most of the gas filling the universe is hydrogen. Neutral atomic hydrogen gas is transparent to microwaves, to infrared light, and to optical light – only when the photon energy becomes large enough to ionize hydrogen does the gas become opaque. For the conditions
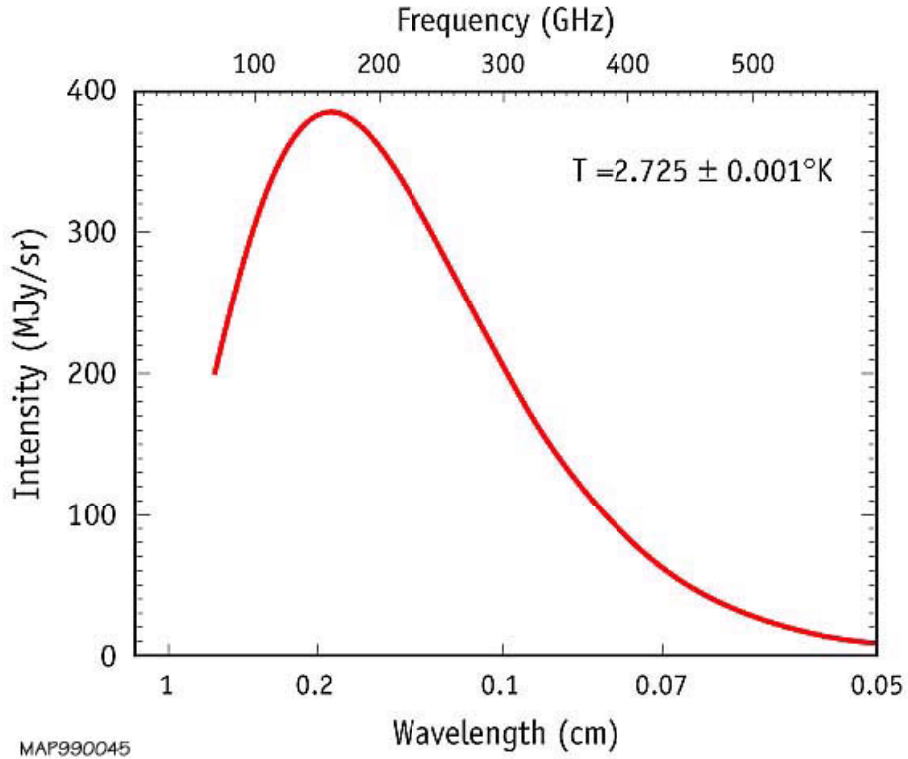
Figure 6: Spectrum of the cosmic microwave background radiation measured in the 1990s by the FIRAS instrument aboard the COBE satellite. (From the WMAP website.)

prevailing in the Universe, hydrogen gas ionizes at a temperature comparable to the surface layer of cool stars, $T \approx 3000$ K. Conclusion: the microwave background radiation was produced at a redshift $z \approx 3000/2.726 = 1100$. The age of the universe at the time when hydrogen became transparent, t_{CMB}, follows from $R(t_{CMB}) \approx 2.726/3000$, yielding $t_{CMB} \approx 380,000$ years. The CMB radiation gives us a picture of the Universe nearly 13.7 billion years ago. This is our earliest view of the universe.

What do we see when we look at the early universe? The spectrum tells only part of the story. To see the rest, we can look at images of the sky in microwaves. The first sensitive all-sky maps of the microwave background radiation were made in the early 1990s by the COBE satellite. In 2001 a new microwave telescope called the *W*ilkinson *M*icrowave *A*nisotropy *P*robe (WMAP) was launched into orbit. It has greatly refined our picture of
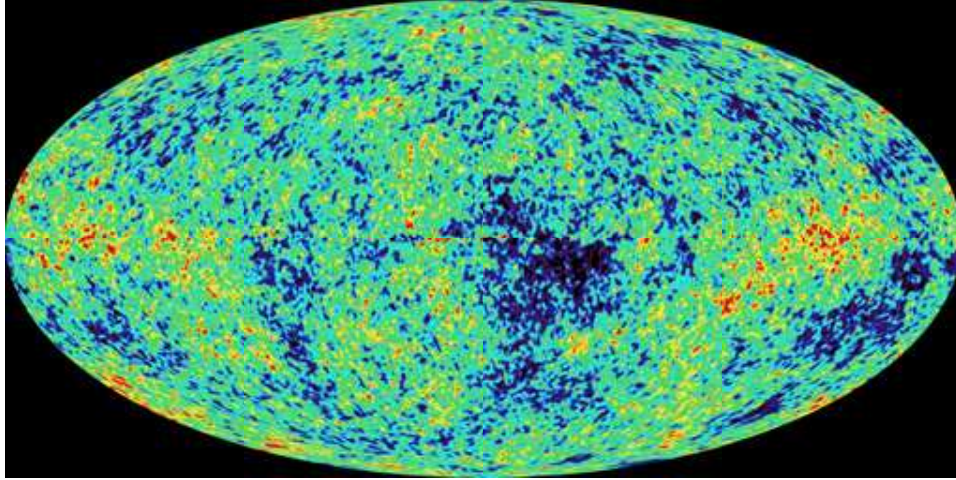
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Figure 7: An all-sky map of the cosmic microwave background at high contrast made by the WMAP satellite. The oval is a projection of the entire sky onto the page. The colors are "false color" and indicate the temperature of the radiation ranging from $T_0 - 2 \times 10^{-4}$ K (black) to $T_0 + 2 \times 10^{-4}$ K (red) where T_0 is the average temperature. The early universe had slight temperature variations. (Image courtesy of the WMAP Science Team, from the WMAP website.)

the early universe.

Figure 7 shows an image of the microwave brightness around the sky made by WMAP. The Planck law is an excellent fit to the spectrum in a fixed direction of the sky; however, the temperature is not the same in all directions. The temperature varies by a few parts in 10^5 from place to place in the early universe. These fluctuations are, we believe, the seeds from which galaxies, stars, and all cosmic structure formed during the past 13 billion years.

In this project we focus on the average properties of the universe rather than the fluctuations. However, the map of fluctuations is also a treasure trove of information about the cosmic parameters. This is because the pattern of fluctuations in the sky provides a kind of fingerprint of the early universe. For example, Figure 7 shows that the fluctuations have a characteristic angular size of about 1 degree.

The 1 degree scale has a direct physical significance and can be used to measure the curvature of the universe. The fluctuations in temperature are due to sound waves in the hot gas of the early Universe: the Universe is filled with a super low frequency static created in the aftermath of the Big Bang. Sound waves compress and rarefy gas, changing its temperature. Sound waves oscillate in time but they also oscillate in space at a fixed moment

of time. The temperature fluctuations we see in the microwave background give a snapshot of the spatial variation of these sound waves 380,000 years after the Big Bang!

The 1 degree scale corresponds to the distance that a sound wave could travel from the time the sound waves were created in the Big Bang until 380,000 years later when they were revealed to us as fluctuations in the cosmic microwave background radiation. If we know the size of this "acoustic yardstick" in meters and the distance the radiation has since travelled to reach our telescopes — *and* we know the spatial geometry (open, closed, or Euclidean) — then we can predict the angular size of the fluctuations. In practice, we measure the angular size and other quantities enabling us to accurately determine the acoustic yardstick size and the distance travelled. The details are beyond the level of this class, but the result is not. The angular size measurement implies that the cosmic spatial curvature K is very small and consistent with zero. The spatial geometry of the Universe appears to be the simplest possible: Euclidean space. On the other hand, dark matter and dark energy curve space*time* in such a way that the cosmic expansion accelerates. What a strange Universe we live in!

9   The Universe NOW: The Omega Diagram

Supernovae and the microwave background radiation constrain the values of $\Omega_{\mathrm{mat},0}$ and $\Omega_{\Lambda,0}$. We have already seen that radiation contributes very little to the critical density today. The major contributors are thus matter (including baryons) and dark energy, which we model as a cosmological constant.

During the last five years, our knowledge of the density parameter values has gone from shadowy outline to measurements of 10% accuracy. Figure 8 illustrates our current knowledge about the key parameters based on Type Ia supernovae, the cosmic microwave background radiation, and the abundance of clusters of galaxies. The microwave background data clearly show that the Universe is close to flat, if not exactly so. They also imply a nonzero dark energy contribution, especially when combined with the abundance of clusters of galaxies. The latter measurement is most sensitive to $\Omega_{\mathrm{mat},0}$ and indicates that there is too little matter to close the Universe. The combination of microwave background and cluster data independently support the radical claim made by the supernova observers in 1998 that the universe is accelerating. We found out earlier that the expansion accelerates if $\Omega_{\Lambda,0} > \frac{1}{2}\Omega_{\mathrm{mat},0}$.

Figure 8 does not include all of the constraints on the Omegas. When they are applied, the result is equations (17) and (18). Future satellite missions should shrink the uncertainties in the Omegas to less than 0.01. Once they do, we may still be left with two outstanding mysteries: What are the dark matter and dark energy?
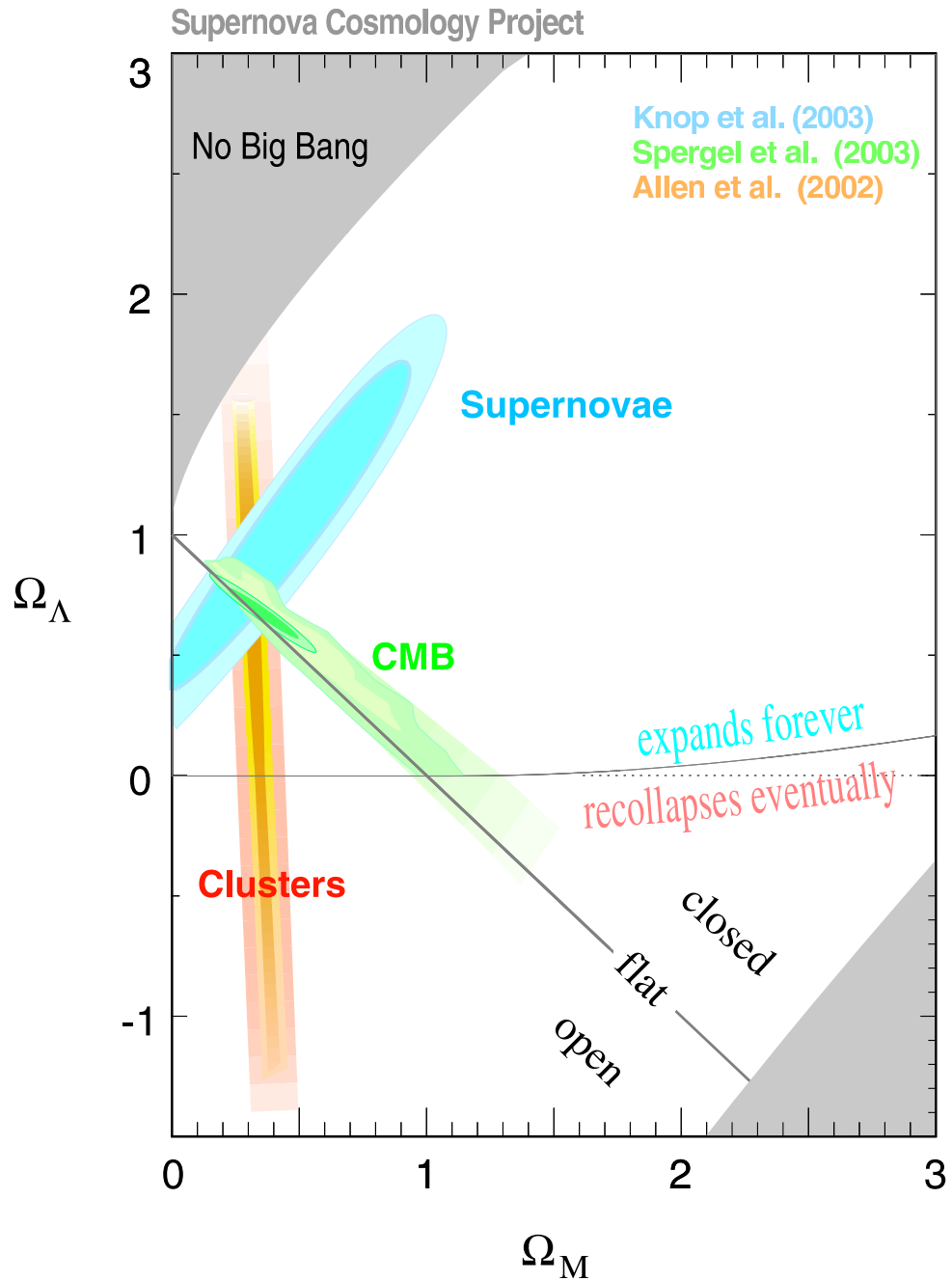
Figure 8: The Omega Diagram. The parameters Ω_M and Ω_Λ are called $\Omega_{mat,0}$ and $\Omega_{\Lambda,0}$ in this project. The colored regions are 68% and 95% confidence regions based on three different sets of measurements — Type Ia supernovae, the cosmic microwave background radiation, and the abundance of clusters of galaxies. Curves show the boundary between open and closed universes and the boundary between models that expand forever and those that eventually recollapse. Figure from the Supernova Cosmology Project, Perlmutter et al. (2005).