Massachusetts Institute of Technology
Department of Physics

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Week 7 Problem Set Solutions

Problem 1: Limiting Values of R

In this problem we will use the extreme Kerr metric (3) and note that the dr^2 term in the metric should blow up at the horizon. Therefore, $r_H = M$ and using equation (4), we get:

$$R_H^2 = M^2 + M^2 + \frac{2M^3}{M} = 4M^2 \qquad \text{so} \qquad R_H = 2M$$

Now find the values of r for which R differs from r by one part in a million:

$$\frac{R-r}{r} = 10^{-6} = \frac{\sqrt{r^2 + M^2 + \frac{2M^3}{r}} - r}{r} = \sqrt{1 + \frac{M^2}{r^2} + \frac{2M^3}{r^3}} - 1 \approx 1 + \frac{M^2}{2r^2} - 1 = \frac{M^2}{2r^2}$$

where we expanded the square root using the fact that in order to satisfy this condition we must have $r >> r_H = M$. The condition is satisfied for $r > 707M$.

Problem 2: Deriving Energy-at-Infinity and Angular Momentum

The Kerr metric in the equatorial plane is:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4Ma}{r} dt d\phi - \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1} dr^2 - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right) r^2 d\phi^2$$

The integrand of the proper time is:

$$f = \left[\left(\frac{d\tau}{d\lambda}\right)^2\right]^{1/2}$$

We note immediately that the integrand does not depend explicitly on two of the variables, i.e.:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \phi} = 0$$

Therefore, we can determine two constants of motion. To find them, let's get the other side of the Euler-Lagrange equation:

$$\frac{f}{\partial\left(dt/d\lambda\right)} = \frac{1}{f}\left(\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}+\frac{2Ma}{r}\frac{d\phi}{d\lambda}\right) = C_1$$

$$\frac{f}{\partial\left(d\phi/d\lambda\right)} = \frac{1}{f}\left(1+\frac{a^2}{r^2}+\frac{2Ma^2}{r^3}\right)r^2\frac{d\phi}{d\lambda}+\frac{1}{f}\frac{2Ma}{r}\frac{dt}{d\lambda} = -C_2$$

Now we can substitute $\lambda \rightarrow \tau$ (which sets $f \rightarrow 1$) and rename the constants of motion.

$$\frac{E_{Kerr}}{m} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}+\frac{2Ma}{r}\frac{d\phi}{d\tau}$$

$$\frac{L_{Kerr}}{m} = \left(1+\frac{a^2}{r^2}+\frac{2Ma^2}{r^3}\right)r^2\frac{d\phi}{d\tau}-\frac{2Ma}{r}\frac{dt}{d\tau}$$

These quantities are conserved along any path determined by the principle of extremal aging (any geodesic path). If we set $a = 0$, we see that the constants of motion reduce to the familiar expressions of the non-rotating case.

Problem 3: Bookkeeper Speed in the "Final Circle"

Multiply equation (22) by R on both sides and substitute the values at the horizon $r_H = M$ and $R_H = 2M$. The result is:

$$R_H\frac{d\phi}{dt} = \frac{2M^2}{M\cdot 2M} = 1$$

Hence, the stone moves in the direction of rotation with the $Rd\phi/dt = 1$. One might say that the bookkeeper speed is the speed of light. However, one should bear in mind that $Rd\phi/dt$, like any bookkeeper speed, is not a physical speed at all. It's a coordinate speed.

Problem 4: Ring Speed According to the Bookkeeper

The units of ω in equation (22) are in meters^{-1}. This makes sense since the increment of the angle $d\phi$ has no units and the increments of time dt has units of meter.

r	$R\omega$
$100M$	0.00020
$10M$	0.0199
$2M$	0.408
M	1

Problem 5: Available Energy from the Monster Black Hole in Our Galaxy

For an extreme Kerr black hole $a = M$, so equation (40) becomes $M_{irr}^2 = M^2/2$ or $M_{irr} = M/\sqrt{2}$. The fraction of the mass that can be, in principle, extracted as energy is thus:

$$\frac{\Delta M}{M} = 1 - \frac{1}{\sqrt{2}} \approx 0.293$$

So the available energy is $E \approx 0.293 \times 2.6 \times 10^6 M_\odot \approx 7.6 \times 10^5 M_\odot$. That is a great deal of energy.

Problem 6: Energies of a Particle

a) Use equation (44) with values $r = 1.5M$, $R = 2.141M$:

$$\frac{E}{m} = \frac{r - M}{R} \approx \frac{0.5M}{2.141M} = 0.234$$

So the energy-at-infinity required is $(k + 0.234)m$.

b) At the static limit, $r = 2M$ and $R = \sqrt{6}M \approx 2.449M$.

$$\frac{E}{m} = \frac{r - M}{R} \approx \frac{M}{2.449M} \approx 0.408$$

So the energy-at-infinity that must be added to the original particle is $(k + 0.408)m$.

c) For a particle at rest at $r = 4M$, use equation (19) with $d\phi/dt = 0$.

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{1/2} = \frac{1}{\sqrt{2}} \approx 0.707$$

So we would have to add an energy-at-infinity of $(k + 0.707)m$.

d) At rest at infinity, the particle will have energy $E = m$, so we have to add an energy-at-infinity of $(k + 1)m$ to remove it to infinity.

Problem 7: Energy Extracted in Phase 1

For $r = 1.5M$ and $R = 2.141M$, equation (44) gives:

$$\frac{E}{2M} = \frac{r - M}{R} \approx \frac{0.5M}{2.141M} \approx 0.234$$

so the energy $E_{phase1} = 2m - 2m \times 0.234 = 1.532m$ has been milked off.

Problem 8: Energy Extracted in Phase 2

a)-b) The ring observer combines matter with anti-matter of total rest mass $2m$ and produces two opposite light flashes each of energy m.

c) Using equation (45) for $v_{ring} = 1$ gives the energy measured far away of the forward moving light flash with $E_{ring} = m$:

$$\frac{E_{phase2}}{m} = \frac{r - M}{R} + \frac{2M^2}{rR} \approx \frac{0.5M}{2.141M} + \frac{2M^2}{1.5M \times 2.141M} \approx 0.856$$

The backward moving light flash has energy-at-infinity:

$$\frac{E}{m} = \frac{r - M}{R} + \frac{2M^2}{rR} \approx \frac{0.5M}{2.141M} - \frac{2M^2}{1.5M \times 2.141M} \approx -0.388$$

Problem 9: Total Energy Extracted

a) The total useful energy made available to the distant engineers as a result of this entire procedure is given by:

$$E_{phase1} + E_{phase2} = 1.532m + 0.856m = 2.388m$$

b) Energy taken away from the black hole during Phase 2 is the negative of the backward moving light flash that gets absorbed, $0.388m$. That is where the far-away engineers get the extra energy after starting with only $2m$.