

Week 4 Problem Set Solutions

Problem 1: Plunging from Rest at Infinity

- a) Black hole Alpha has a mass $M = 8\text{km}$. Using equation [24] with $r = 50\text{km}$, we see that as the stone falls past the stationary shell observer, the observer would measure the stone's velocity as:

$$v_{shell} = \frac{dr_{shell}}{dt_{shell}} = \left(\frac{2M}{r}\right)^{1/2} = \left(\frac{16}{50}\right)^{1/2} = 0.566$$

- b) The bookkeeper speed is given by equation [21]:

$$v = \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r}\right)^{1/2} = \left(1 - \frac{16}{50}\right) \left(\frac{16}{50}\right)^{1/2} = 0.385$$

- c) At $r = 32\text{km}$ we get $v_{shell} = 0.707$.
- d) At $r = 32\text{km}$ we get $v = 0.354$.
- e) Equation [24] shows that the shell speed increases as r decreases for $r > 2M$. The bookkeeper speed, on the other hand (see eq.[21]), increases with decreasing r for $r > 6M$ and decreases for $r < 6M$, falling to zero at the horizon. The radii in the question above are all less than $6M$. See also Figure 5 on page 3-18.

In words: As the stone approaches the horizon, the Schwarzschild metric shows that the stone as experiencing some serious warpage of time and space, which in the limiting case of equation [21] results in the bookkeeper velocity falling to zero. However, the strange behavior at $r = 2M$ is not due to the warping of spacetime but rather to the extreme warping of the bookkeeper coordinates.

Shell observers at rest very close to the horizon (but at the same instantaneous position as the stone!) record the velocity of the stone in a region of their coordinate system free of such a pathology. Their neighborhood is flat! They will experience something more in line with our familiar physics that a falling stone accelerates with time (to the maximum allowed speed, c).

Problem 2: Energy Conversion Using a Black Hole

- a) The shell observer with this machine stationed at $r_0 = 30\text{km}$ measures the energy of the incoming bag of garbage (mass m) to be (Equation [27]):

$$\frac{E_{shell}}{m} = \frac{1}{\left(1 - \frac{2M}{r_0}\right)^{1/2}}$$

After the bag of garbage is brought to rest, the shell observer measures its energy to be m , the rest energy. The difference between these two is the kinetic energy of the bag that is converted to the energy of the flash of light:

$$\frac{E_{flash}}{m} = \frac{E_{shell}}{m} - 1 = \frac{1}{\left(1 - \frac{2M}{r_0}\right)^{1/2}} - 1 = \left(\sqrt{3} - 1\right) \approx 0.732$$

- b) As mentioned in the text, equation [27] is very general and turns out to work even for light, relating the energy of light measured at the shell to that measured far away.

$$\frac{E_{recovered}}{m} = \frac{E_{flash}}{m} \left(1 - \frac{2M}{r_0}\right)^{1/2} = 1 - \left(1 - \frac{2M}{r_0}\right)^{1/2} = \left(1 - \frac{1}{\sqrt{3}}\right) \approx 0.423$$

- c) The total energy of the mass m of garbage at rest on the shell r_0 , as measured by a far-away observer, is NOT m , but is given by equation [18]:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} = \frac{1}{\sqrt{3}} \approx 0.577$$

When the garbage is released, this energy is swallowed by the black hole and added to its total mass M , as observed from far away.

Checking Conservation:

The far-away observer drops an object of energy m from rest into the black hole and receives back a light flash of energy $0.423m$. She also observes that the black hole increases in mass by $0.577m$. Sum these last two energies to obtain a final total energy m , equal to the original garbage rest mass. Conservation of total energy measured by the far-away observer survives in general relativity!

- d) According to the far-away observer, the energy input is m and the energy output in the form of light is $E_{recovered}$, so the efficiency is

$$\frac{|E_{recovered} - m|}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} = \frac{1}{\sqrt{3}} \approx 0.577$$

Note that as the stopping shell approaches the horizon $r_0 \rightarrow 2M$, the energy received in the flash approaches the initial rest energy $E_{recovered} \rightarrow m$, so the process comes closer to perfect efficiency.

Problem 3: Hitting a Neutron Star

- a) The mass of the neutron star is $1.35 \times 1.477\text{km} = 1.994\text{km}$. The r -value of the horizon of a black hole with this mass is $2M = 2 \times 1.994\text{km} = 3.988\text{km}$. So the surface of the neutron star with a radius of 10km is not very close to the potential horizon of the black hole of equal mass.
- b) Use equation [24] to find the shell speed at the surface of the star:

$$v_{shell} = \left(\frac{2M}{r}\right)^{1/2} \approx 0.631$$

- c) Use equation [21].

$$v_{book} = \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r}\right)^{1/2} \approx 0.380$$

which is smaller than the speed measured by the shell observer.

- d) All particles have the same speed striking the surface. For a given speed, kinetic energy is proportional to rest energy. The rest energy (mass) of an electron is smaller than that of the nucleon by a factor of 2000. Therefore the kinetic energy of each electron just after the impact is negligible compared with that of each nucleon.
- e) The 56 nucleons (neutrons plus protons) share their kinetic energy with the 26 electrons for a total of 82 particles per iron atom that collides with the surface of the neutron star. This average kinetic energy of the final plasma is thus $56/82$ of the initial kinetic energy of individual nucleons. What is that initial kinetic energy? The solution to part B gives the speed of the nucleon as it hits the surface. From this the shell observer calculates its kinetic energy using the expressions of special relativity ($m \approx 1 \text{ GeV}$):

$$KE_{nucleon} = \frac{m}{\sqrt{1 - v_{shell}^2}} - m = \left[\left(1 - \frac{2M}{r}\right)^{-1/2} - 1 \right] m \approx 0.29 \text{ GeV}$$

When this average kinetic energy of the nucleons is shared with the electrons, the result is an average kinetic energy of the plasma particle:

$$KE_{plasma} = \frac{56}{82} \times 0.29 \text{ GeV} \approx 0.20 \text{ GeV}$$

The temperature in energy units is $2/3$ this value, or 130 MeV .

- f) The average temperature of a plasma particle is converted to the average energy of photons. As these photons move outward, climbing out of the gravitational well, the light flash gets gravitationally red-shifted as it finally arrives at a great distance. Equation [27] expresses this in terms of energy. Photons (light pulses) can be thought of as having ONLY kinetic energy; stop a photon and it disappears. Rearranging that equation we obtain:

$$E = \left(1 - \frac{2M}{r}\right)^{1/2} E_{shell} = 0.775 \times 130 \text{ MeV} \approx 100 \text{ MeV}$$

Problem 4: The Plunger

- a) We can use equation [12] of chapter 2:

$$dr_{shell} = dr \left(1 - \frac{2M}{r}\right)^{-1/2}$$

This is the proper distance between the shells according to the shell observer.

- b) Since we are dealing with small lengths and time periods, spacetime looks locally flat, and we can use special relativity. As usual, the plunger sees the shell observer's ruler contracted in the radial direction. So to the plunger the distance between the shells looks smaller than dr_{shell} . To find the contraction factor, we need to find the relative speed v with which the plunger is moving past the shell. This is the same as the speed of the plunger, measured by the shell observer, and is given by equation [24]:

$$v = \frac{dr_{shell}}{dt_{shell}} = \left(\frac{2M}{r}\right)^{1/2}$$

Therefore, the plunger measures the distance between shells

$$dr_{plunger} = dr_{shell} \sqrt{1 - v^2} = dr_{shell} \sqrt{1 - \frac{2M}{r}} = dr$$

- c) The shell observer measures the time of plunger's travel between two shells of distance dr_{shell} apart as:

$$dt_{shell} = \frac{dr_{shell}}{v} = dr \left(1 - \frac{2M}{r}\right)^{-1/2} \left(\frac{2M}{r}\right)^{-1/2}$$

- d) Since, according to the plunger, the shells move past her with speed v , the time it takes to pass them is

$$dt_{plunger} = \frac{dr_{plunger}}{v} = dr \left(\frac{2M}{r} \right)^{-1/2}$$

- e) As required by special relativity, the relative speeds measured by the two observers are consistent over very small lengths and time intervals, where spacetime is locally flat:

$$v = \frac{dr_{shell}}{dt_{shell}} = \frac{dr_{plunger}}{dt_{plunger}}$$

Problem 5: Can the Bookkeeper Be a Real Observer?

- a) Starting from equation [10] of Chapter 2, which gives the Schwarzschild metric in the equatorial plane,

$$d\tau^2 = \left(1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\phi^2$$

we set $d\tau = dt$ to obtain the constraint:

$$0 = -\frac{2M}{r} dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\phi^2$$

that must be satisfied in order for the observer to read the bookkeeper time. Dividing both sides through by dt^2 and introducing radial and tangential velocities $v_r = dr/dt$ and $v_\phi = r d\phi/dt$ we can re-write this constraint as

$$\frac{v_r^2}{1 - \frac{2M}{r}} + v_\phi^2 = -\frac{2M}{r}$$

This equation tells how an observer must be moving in bookkeeper coordinates in order to read the far-away time as his proper time. Note that for finite r , the only way to get a negative quantity on the right-hand side is to be inside the horizon $r < 2M$! Setting in addition $dr = d\phi = 0$ we get:

$$dt^2 = \left(1 - \frac{2M}{r} \right) dt^2$$

which implies that $2M/r = 0$, or $r = \infty$. This means that the only possibility for a stationary observer to read the far-away time dt is to be far away indeed.

b) Starting from equation [15] on page B-13, and replacing dt_{rain} by dt' ,

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt'^2 - 2\left(\frac{2M}{r}\right)^{1/2} dt' dr - dr^2 - r^2 d\phi^2$$

and setting $d\tau = dt'$ we obtain

$$0 = -\left(\frac{2M}{r}\right) dt'^2 - 2\left(\frac{2M}{r}\right)^{1/2} dt' dr - dr^2 - r^2 d\phi^2$$

Dividing as before by dt'^2 ,

$$2v'_r \left(\frac{2M}{r}\right)^{1/2} + v_r'^2 + v_\phi'^2 = -\frac{2M}{r}$$

and collecting terms with v'_r

$$\left(v'_r + \sqrt{\frac{2M}{r}}\right)^2 + v_\phi'^2 = 0$$

The solution is $v'_r = -(2M/r)^{1/2}$ and $v'_\phi = 0$, since all the terms are positive. So we see that the only possibility is to move radially. Note that now the rain observer is free to explore all of spacetime while reading rain time, but he must move in radial direction inwards. Also, v'_r is given by the same expression as the shell velocity v_{shell} of equation [24]. This is as it should be, because while passing each shell on the way to the black hole, the rain observer measures the speed of the shell moving past him to be the same as the speed of the rain observer, as measured by the shell observer. This is so because at each frame the spacetime is locally flat and special relativity results hold. Finally, we restrict the observer to be stationary $dr = d\phi = 0$ to obtain:

$$dt'^2 = \left(1 - \frac{2M}{r}\right) dt'^2$$

which implies that the only stationary observers that measure rain time are those infinitely far away.

Problem 6: One Way Motion Inside the Horizon

a)-c) These are simple derivations with results given.

d) The taillight flash corresponds to the plus sign in equation [17], leading to:

$$\frac{dr}{dt_{rain}} = - \left(\frac{2M}{r} \right)^{1/2} + 1$$

Inside the horizon, where $r < 2M$, the magnitude of the first term on the right is greater than 1. Therefore, the resulting sign of the right-hand side is negative. Conclusion: Even the “radially outward” taillight flash move inward!

e) You ride on the raindrop moving radially inward inside the horizon. Think of throwing a stone toward the center. This stone cannot move inward faster than a light flash launched in the same inward direction. Now think of throwing a stone away from the center. This stone cannot move “outward” faster than a light flash launched in the same direction. Therefore, if both “inward” and “outward” light flashes move inward, so must any stone that you launch in either direction. Otherwise, you will see the stone fly backwards faster than light in your frame, which contradicts Einstein’s postulates.

Problem 7: Lagrangian Mechanics

a) Let’s write down the Lagrangian explicitly

$$\mathcal{L} = T - U = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m\Phi_N(x, y, z)$$

Since the Lagrangian treats all three coordinates equally, we can derive the Euler-Lagrange equation of motion for x and infer the results for y and z .

$$\frac{\partial \mathcal{L}}{\partial x} = -m \frac{\partial \Phi_N(x, y, z)}{\partial x} \equiv F_x$$

Where we used equation (1) of the notes *Gravity, Metrics and Coordinates*.

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{d}{dt} (m\dot{x}) = m\ddot{x}$$

Equating the two pieces and putting all three components together we obtain the familiar vector equation:

$$\vec{F} = m\vec{\ddot{x}}$$

- b) The flat 3D spacetime metric in spherical coordinates is given by equation (7) of the notes *Gravity, Metrics and Coordinates*:

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2$$

In the equatorial plane $\theta = \pi/2$ it is simplified to

$$d\tau^2 = dt^2 - dx^2 - dy^2 = dt^2 - dr^2 - r^2 d\phi^2$$

Dividing everything by dt^2 , and identifying $v^2 = \dot{x}^2 + \dot{y}^2$ we find:

$$v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$$

- c) In spherical coordinates, the kinetic energy T of a particle moving in the equatorial plane looks like

$$T = \frac{m}{2} v^2 = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2)$$

Assuming now that Φ_N depends only on r , the Lagrangian is

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) - m\Phi_N(r)$$

Let's derive the Euler-Lagrange equations of motion for each coordinate. The equation of motion for r

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right)$$

becomes

$$mr\dot{\phi}^2 - m \frac{\partial \Phi_N(r)}{\partial r} = m\ddot{r}$$

For the angular coordinate ϕ the equation of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right)$$

becomes

$$0 = \frac{d}{dt} (mr^2 \dot{\phi}) = 2mr\dot{r}\dot{\phi} + mr^2 \ddot{\phi}$$

which simplifies for nonzero r to

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0$$

Notice the time independence of the quantity $mr^2 \dot{\phi}$. This is a constant of the motion because we assumed the potential Φ_N to be independent of the coordinate ϕ . It is the familiar conserved angular momentum.