Massachusetts Institute of Technology
Department of Physics

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Week 5 Problem Set Solutions

Problem 1: Circular Orbit

a) From equation (50):

$$v_{bkkpr} = \left(\frac{M}{r}\right)^{1/2} = \frac{1}{\sqrt{8}} \approx 0.354$$

Thus, one orbit takes:

$$T_{bkkpr} = \frac{2\pi r}{v_{bkkpr}} = 32\sqrt{2}\pi M \approx 142M$$

b) From equation (48):

$$v_{shell} = \sqrt{\frac{M}{r-2M}} = \frac{1}{\sqrt{6}} \approx 0.408$$

Similarly:

$$T_{shell} = \frac{2\pi r}{v_{shell}} = 16\sqrt{6}\pi M \approx 123M$$

c) The orbiter sees the shell going past her at the same rate the shell observer sees the orbiter going past him. Using special relativity, the orbiting clock runs slow (compared to the shell clock) so the two times are related by:

$$T_{orbiter} = \tau = \left(1 - v_{shell}^2\right)^{1/2} T_{shell} = 16\sqrt{5}\pi M \approx 112M$$

d) From the earlier parts of this problem:

$$\frac{d\tau}{dt_{bkkpr}} = \frac{\tau}{T_{bkkpr}} = \sqrt{\frac{5}{8}} \approx 0.791$$

For flat spacetime:

$$\frac{d\tau}{dt} = \sqrt{1 - v^2}$$

The corresponding speed would be given by:

$$v_{flat} = \sqrt{\frac{3}{8}} = 0.612$$

Problem 2: Newtonian Orbits as a Limiting Case

a) Equation (28) gives the effective potential for Newtonian gravity:

$$\frac{V(r)}{m} = -\frac{M}{r} + \frac{(L/m)^2}{2r^2}$$

Circular orbits will be at the minimum of this effective potential. Differentiating $V(r)/m$ with respect to r and setting the result equal to zero, we get:

$$r_{Newt} = \frac{(L/m)^2}{M}$$

b) Equation (43) (the positive sign represents the minimum) can be recast as:

$$r_{GR} = r_{Newt}\left[\frac{1}{2} + \frac{1}{2}\left(1 - \frac{12M^2}{(L/m)^2}\right)^{1/2}\right] = r_{Newt}(1+q)$$

Taylor expanding this result to first order in M/r_{Newt}:

$$q = -\frac{3M}{r_{Newt}}$$

c) For $q = -0.01$:

$$r_{min} = 300M$$

d) Since $r_{\odot} \approx 1.477 \times 10^3$ m:

$$r_{min} \approx 4.431 \times 10^5 \text{ m}$$

which is much less than:

$$R_{\odot} \approx 6.960 \times 10^8 \text{ m}$$

e) For the planet Mercury:

$$q = -\frac{3M}{r_{Mercury}} \approx 7.627 \times 10^{-8}$$

Problem 3: Time to the Center from Any Initial Radius

a) From equations (12) and (18) (from EBH pages 3-9, and 3-12, respectively) for a particle at r_0:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = \left(1 - \frac{2M}{r_0}\right)^{1/2}$$

is the conserved energy as seen by a far away observer. Manipulating:

$$dt^2 = \frac{\left(1 - \frac{2M}{r_0}\right)}{\left(1 - \frac{2M}{r}\right)^2} d\tau^2$$

Substitute this into the Schwarzschild metric for radial motion to obtain:

$$d\tau^2 = \frac{\left(1 - \frac{2M}{r_0}\right)}{\left(1 - \frac{2M}{r}\right)} d\tau^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)}$$

This yields, after some manipulation:

$$dr^2 = \left(\frac{2M}{r} - \frac{2M}{r_0}\right) d\tau^2$$

This leads to an expression to be integrated from r_0 to 0:

$$d\tau = \left(\frac{r_0}{2M}\right)^{1/2} \frac{r^{1/2} dr}{(r_0 - r)^{1/2}}$$

Make the substitution $r = z^2$ so that $dr = 2z dz$. The integrand becomes:

$$d\tau = \left(\frac{r_0}{2M}\right)^{1/2} \frac{2z^2 dz}{(r_0 - z^2)^{1/2}}$$

Integrating with a standard table of integrals:

$$\tau = \left(\frac{r_0}{2M}\right)^{1/2} \left[-z\left(r_0 - z^2\right)^{1/2} - r_0 \sin^{-1}\left(\frac{z}{r_0^{1/2}}\right)\right]_{r_0^{1/2}}^{0} = \left(\frac{r_0}{2M}\right)^{1/2} \frac{\pi r_0}{2}$$

This is consistent with equation (3), page 3-21 of EBH when $r_0 = 2M$.

b) For $r_0 = 8M$, the answer to part a) becomes:

$$\tau = 8\pi M$$

For our sun, $M = 1477$ m (from the back cover of EBH), so for the black hole at the center of our galaxy (3.7 million solar masses), the falling time is about 460s, or about seven and a half minutes.

Problem 4: Falling in the Rain

a) The metric in the rain frame (with $G = 1$ and $\theta = \pi/2$) is:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - 2\sqrt{\frac{2M}{r}} dt dr - dr^2 - r^2 d\phi^2$$

The integrand of the proper time is:

$$f(r, \frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\phi}{d\lambda}) = \left[\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - 2\sqrt{\frac{2M}{r}} \frac{dt}{d\lambda}\frac{dr}{d\lambda} - \left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2\right]^{1/2}$$

Notice immediately that the integrand does not depend explicitly on two of the variables, i.e. $\partial f/\partial t = \partial f/\partial \phi = 0$. So there will be two constants of motion. They follow at once from the Euler-Lagrange equations,

$$\frac{\partial f}{\partial(dt/d\lambda)} = \frac{1}{f}\left(\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} - \sqrt{\frac{2M}{r}}\frac{dr}{d\lambda}\right) = C_1$$

$$\frac{\partial f}{\partial(d\phi/d\lambda)} = -\frac{r^2}{f}\frac{d\phi}{d\lambda} = -C_2$$

Now using $\tau = \int f d\lambda$ we can substitute $f d\lambda = \tau$, and rename the constants of motion:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} - \sqrt{\frac{2M}{r}}\frac{dr}{d\tau} \qquad\qquad \frac{L}{m} = r^2\frac{d\phi}{d\tau}$$

b) For a radially moving mass, we have $d\theta = d\phi = 0$ so the metric reduces to:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - 2\sqrt{\frac{2M}{r}} dt dr - dr^2$$

Dividing through by $d\tau^2$ and using the results of part a), we can express $dr/d\tau$ as:

$$\frac{dr}{d\tau} = -\sqrt{\frac{r}{2M}}\left(\frac{E}{m} - \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\right)$$

Plugging this into the equation for the metric above and doing a lot of algebra, we obtain:

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - 2\left(\frac{E}{m}\right)\left(\frac{dt}{d\tau}\right) + \left(\frac{2M}{r} + \left(\frac{E}{m}\right)^2\right) = 0$$

The solution is

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \left[\frac{E}{m} - \sqrt{\frac{2M}{r}\left(\left(\frac{E}{m}\right)^2 + \frac{2M}{r} - 1\right)}\right]$$

where we've chosen the minus sign in the quadratic formula in order to get radial *in*fall.

c) Set the left-hand side of the equation for $dr/d\tau$ to zero at r_0, thereby obtaining:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)\frac{dt}{d\tau}$$

Combine this with the metric (with $dr/d\tau = 0$) to immediately derive:

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \qquad \text{and} \qquad \frac{E}{m} = \sqrt{1 - \frac{2M}{r_0}}$$

Now suppose the falling body started at a great distance from the black hole. Then, as we saw already, for the Schwarzschild-frame bookkeeper, dr/dt approaches zero at the horizon, so the body never seems to reach it. Our intuition tells us that for the rain observer, this is not so. The rain observer himself started out from rest at some far-away shell, so a friend on a nearby inner shell starting slightly ahead of him should cross the horizon slightly before he does. Let's see what the formulas say. We should take the limit of $dt/d\tau$ carefully as $r \to 2M$. Define $2M/r = 1 - \epsilon$ where $\epsilon > 0$ is a small number. Substituting these and our energy expression, the equation from part b) now reads:

$$\begin{aligned} \frac{dt}{d\tau} &= \epsilon^{-1}\left[\sqrt{1 - \frac{2M}{r_0}} - \sqrt{(1-\epsilon)\left(1 - \epsilon - \frac{2M}{r_0}\right)}\right] \\ &\approx \left(1 - \frac{M}{r_0}\right)\left(1 - \frac{2M}{r_0}\right)^{-1/2} + O(\epsilon) \end{aligned}$$

We see that $dt/d\tau$ remains finite as the body approaches the horizon.

Problem 5: Falling from Rest at $r_0 = 8M$

From the first equation in the solution to Problem 3, calculate:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} = \frac{\sqrt{3}}{2} \approx 0.866025$$

Set this value by hand after click on E/m in the GRorbits program. Then set $r_0 = 8M$. With GRorbits, I get $t_{rain} = 26.85M$ and $\tau = 25.13M$. The latter is the same as the prediction of Problem 4.

Problem 6: Circular Orbit at $r_0 = 8M$

For Schwarzschild, I get $T_{bkkpr} = 142M$ and $\tau = 112.27M$, which are the same as computed in Problem 1.

Problem 7: Advance of the Periastron

I get the desired precession for $L/m \approx 7.9M$. This works for a wide range of radii. It does not work for Newton, who predicts zero advance.