Massachusetts Institute of Technology
Department of Physics

Physics 8.224 Spring 2005

Week 6 Problem Set Solutions

Problem 1: GRorbits Software: Right Angle Turn for Light

I get $1/b = 0.161$.

Problem 2: GRorbits Software: Einstein Ring

I get $1/b = 0.0475M$.

Problem 3: Energy Production by a Quasar?

a) From Exercise 2D of Chapter 4 of EBH, we know that the radius of innermost stable orbit, $r_{ISCO} = 6M$. Exercise 3A gives the result that, for a stone orbiting at this radius, the shell speed is given by equation (48):

$$v_{shell} = \sqrt{\frac{M}{r\left(1 - \frac{2M}{r}\right)}} = \frac{1}{2}$$

The energy of the orbiting stone, as measured by the shell observer is:

$$\frac{E_{shell}}{m} = \gamma_{shell} = \frac{1}{\sqrt{1 - v_{shell}^2}} = \frac{2}{\sqrt{3}} \approx 1.155$$

The shell kinetic energy is then:

$$\frac{K_{orbit}}{m} = \frac{E_{shell}}{m} - 1 = \left(\frac{2}{\sqrt{3}} - 1\right) \approx 0.155$$

b) Chapter 5, equation (48) gives the shell energy of the stone plunging from rest at infinity:

$$\frac{E_{shell}}{m} = \frac{E/m}{\sqrt{1 - \frac{2M}{r}}} = \sqrt{\frac{3}{2}} \approx 1.225$$

The shell kinetic energy will then be:

$$\frac{K_{plunging}}{m} = \left(\sqrt{\frac{3}{2}} - 1\right) \approx 0.225$$

c) The total kinetic energy of the two stones is just sum of the individual kintetic energies, which is then the energy of the collision flash. Equation (48) works for *anything*, so let's run it backwards:

$$\frac{E_{remote}}{m} = \frac{E_{flash}}{m}\sqrt{1 - \frac{2M}{r}} = \frac{2\sqrt{2}}{3}\left(4 - \sqrt{3}\right) \approx 0.310$$

d) The fraction of rest energy $2m$ converted to light energy at infinity is $0.310/2 = 0.155 = 15.5\%$.

e) We now drop the remaining mass (they are at rest with respect to the shell, so they have shell energy $2m$) into the black hole, so M increases (as measured from infinity) by:

$$\Delta M = 2m\sqrt{1 - \frac{2M}{r}} = 2\sqrt{\frac{2}{3}}m \approx 1.633m$$

Problem 4: The Horizon as a One-Way Barrier

a) Equation (48) plus conservation of energy tells us that:

$$E_\infty = E(r_1)_{shell}\sqrt{1 - \frac{2M}{r_1}} = E(r_2)_{shell}\sqrt{1 - \frac{2M}{r_2}}$$

Combining, we obtain:

$$E(r_2)_{shell} = \sqrt{\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}}}E(r_1)_{shell}$$

b) As $r \rightarrow 2M$, the above expression gives $E(r_2)_{shell} \rightarrow 0$ for any fixed value of $r_2 > 2M$. It is infinitely redshifted. This shows that a light flash launched from the horizon cannot be detected at any radius outside it.

c) For $r_1 > 2M$ and $r_2 \rightarrow 2M$, we get $E(r_2)_{shell} \rightarrow \infty$. It is infinitely blueshifted. This makes sense because, by reversing the argument, it would mean that we need an infinite amount of energy in order that a light pulse launched at the horizon could be detected outside of it.

Problem 5: Plunger Wink-Out Time

a) i. From photon energy formula $E = hf$ (or Equation [C] of the Selected Formulas):

$$f_{far} = \frac{E_{far}}{h} = \frac{E_{shell}}{h}\sqrt{1-\frac{2M}{r}} = f_{shell}\sqrt{1-\frac{2M}{r}} = f_{shell}\sqrt{\frac{5}{7}} \approx 0.845 f_{shell}$$

ii. Equation (24) of Ch. 3 gives the shell speed of the plunging beacon $v_{shell} = \sqrt{2M/r} \approx 0.535$, and the relativistic doppler shift gives:

$$f_{shell} = f_0\sqrt{\frac{1-v_{shell}}{1+v_{shell}}} = f_0\sqrt{\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}}} = f_0\sqrt{\frac{1-\sqrt{2/7}}{1+\sqrt{2/7}}} \approx 0.551 f_0$$

b) Just note that:

$$1-\frac{2M}{r} = \left(1+\sqrt{\frac{2M}{r}}\right)\left(1-\sqrt{\frac{2M}{r}}\right)$$

Combining the results of part a) and using this fact, we see that:

$$f_{far} = f_0\sqrt{1-\frac{2M}{r}}\sqrt{\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}}} = f_0\left(1-\sqrt{\frac{2M}{r}}\right) = f_0\left(1-\sqrt{\frac{2}{7}}\right) \approx 0.465 f_0$$

c) Solve the result of part b) for general r:

$$r = \frac{2M}{\left(1-\frac{f_{far}}{f_0}\right)^2} \qquad \text{and} \qquad r_1 = r_{0.9} = 200M$$

d) Reuse the result from c to get $r_2 = r_{0.1} \approx 2.469M$.

e) Integrate Equation (53):

$$\Delta t = -\int_{r_2}^{r_1}\frac{dr}{1-\frac{2M}{r}} = \int_{u_1=\sqrt{r_1/2M}}^{u_2=\sqrt{r_2/2M}}\frac{4Mu^4du}{1-u^2} = \left[-\frac{u^3}{3} - u + \frac{1}{2}ln\left|\frac{u+1}{u-1}\right|\right]_{u_1}^{u_2}$$

Plugging in the numerical values from parts c) and d), we get the far-away time taken by the beacon in its descent $\Delta t_{beacon} \approx 1373M$.

f) The time measured by the far-away observer for the light to rise from the inner to the outer radius is:

$$\Delta t_{light} = \int_{r_2}^{r_1}\frac{dr}{1-\frac{2M}{r}} = (r_2 - r_1) + 2M ln\left[\frac{r_1-2M}{r_2-2M}\right] \approx 210M$$

g) The total wink-out time is $\Delta t_{far} = \Delta t_{beacon} + \Delta t_{light} \approx 1580M$.

h) For $M = 50M_\odot$, we get $\Delta t_{far} \approx 1.17 \cdot 10^8\text{m} \approx 0.391\text{s}$.