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# Gravitation in the Weak-Field Limit



## 1 Introduction

In special relativity, electromagnetism is described by a one-form field $A_\mu(x)$ in flat spacetime. Similarly, in the weak-field limit gravitation is described by a symmetric tensor field $h_{\mu\nu}(x)$ in flat spacetime. Pursuing the analogy can lead us to many insights about GR. These notes detail linearized GR, discussing particle motion via Hamiltonian dynamics, the gravitational field equations, the transverse gauge (giving the closest thing to an inertial frame in GR), gauge transformations, motion in accelerated and rotating frames, Mach's principle, and more.

Linear theory is also useful for most practical computations in general relativity. Linear theory suffices for nearly all experimental applications of general relativity performed to date, including the solar system tests (light deflection, perihelion precession, and Shapiro time delay measurements), gravitational lensing, and gravitational wave detection. The Hulse-Taylor binary pulsar offers some tests of gravity beyond lineaer theory (Taylor et al 1992), as do (in principle) cosmological tests of space curvature.

Some of this material is found in Thorne et al (1986) and some in Bertschinger (1996) but much of it is new. The notation differs slightly from chapter 4 of my Les Houches lectures (Bertschinger 1996); in particular, $\phi$ and $\psi$ are swapped there, and $h_{ij}$ in those notes is denoted $s_{ij}$ here (eq. 11 below).

Throughout this set of notes, the Minkowski metric $\eta_{\mu\nu}$ is used to raise and lower indices. In this set of notes we refer to gravity as a field in flat spacetime as opposed to the manifestation of curvature in spacetime. With one important exception, this pretense can be made to work in the weak-field limit (although it breaks down for strong gravitational fields). As we will see, gravitational radiation is the exception — it can only be understood properly as a traveling wave of space curvature.

## 2 Particle Motion and Gauge Dependence

We begin by studying an analogue of general relativity, the motion of a charged particle. The covariant action for a particle of mass $m$ and charge $q$ has two terms: one for the free particle and another for its coupling to electromagnetism:

$$S[x^\mu(\tau)] = \int -m \left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} d\tau + \int qA_\mu \frac{dx^\mu}{d\tau}\, d\tau \ . \tag{1}$$

Varying the trajectory and requiring it to be stationary, with $\tau$ being an affine parameter such that $V^\mu = dx^\mu/d\tau$ is normalized $\eta_{\mu\nu}V^\mu V^\nu = -1$, yields the equation of motion

$$\frac{dV^\mu}{d\tau} = \frac{q}{m}F^\mu{}_\nu V^\nu \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2}$$

Regarding gravity as a weak (linearized) field on flat spacetime, the action for a particle of mass $m$ also has two terms, one for the free particle and another for its coupling to gravity:

$$S[x^\mu(\tau)] = \int -m \left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} d\tau + \int \frac{m}{2}\, h_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{-1/2} d\tau \ . \tag{3}$$

This result comes from using the free-field action with metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and linearizing in the small quantities $h_{\mu\nu}$. Note that for this to be valid, two requirements must be satisfied: First, the curvature scales given by the eigenvalues of the Ricci tensor (which have units of inverse length squared) must be large compared with the length scales under consideration (e.g. one must be far from the Schwarzschild radius of any black holes). Second, the coordinates must be nearly orthonormal. One cannot, for example, use spherical coordinates; Cartesian coordinates are required. (While this second condition can be relaxed, it makes the analysis much simpler. If the first condition holds, then coordinates can always be found such that the second condition holds also.)

Requiring the gravitational action to be stationary yields the equation of motion

$$\frac{dV^\mu}{d\tau} = -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right)V^\alpha V^\beta = -\Gamma^\mu{}_{\alpha\beta}V^\alpha V^\beta \ . \tag{4}$$

The object multiplying the 4-velocities on the right-hand side is just the linearized Christoffel connection (with $\eta^{\mu\nu}$ rather than $g^{\mu\nu}$ used to raise indices).

Equations (2) and (4) are very similar, as are the actions from which they were derived. Both $F_{\mu\nu}$ and $\Gamma^\mu{}_{\alpha\beta}$ are tensors under Lorentz transformations. This fact ensures that equations (2) and (4) hold in any Lorentz frame. Thus, in the weak field limit it is straightforward to analyze arbitrary relativistic motions of the sources and test particles, as long as all the components of the Lorentz-transformed field, $h_{\bar\mu\bar\nu} = \Lambda^\mu{}_{\bar\mu}\Lambda^\nu{}_{\bar\nu}h_{\mu\nu}$ are small compared with unity (otherwise the linear theory assumption breaks down). A simple example of the Lorentz transformation of weak gravitational fields was given in problem 2 of Problem Set 6.

From these considerations one might conclude that regarding linearized gravity as a field in flat spacetime with gravitational field strength tensor $\Gamma^{\mu}{}_{\alpha\beta}$ presents no difficulties. However, there is a very important stumbling block: the electromagnetic force law is gauge-invariant while the gravitational one is not.

The electromagnetic field strength tensor, hence equation (2), is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Phi(x) \ . \tag{5}$$

The Christoffel connection is, however, *not* invariant under the gravitational gauge transformation

$$h_{\mu\nu}(x) \to h_{\mu\nu} + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \ . \tag{6}$$

(Note that in both special relativity and linearized GR, $\nabla_\mu = \partial_\mu$.) While $F_{\mu\nu}$ is a tensor under general coordinate transformations, $\Gamma^{\mu}{}_{\alpha\beta}$ is not. Because the gravitational gauge transformation is simply an infinitesimal coordinate transformation, our putative gravitational field strength tensor is not gauge-invariant. While the form of equation (4) is unchanged by Lorentz transformations, it is not preserved by arbitrary coordinate transformations.

Try to imagine the Lorentz force law if the electromagnetic fields were not gauge-invariant. We would be unable to get a well-defined prediction for the motion of a particle.

The situation in gravity is less bleak because we recognize that the gauge transformation is equivalent to shifting the coordinates, $x^\mu \to x^\mu - \xi^\mu(x)$. If the coordinates are deformed, fictitious forces (like the Coriolis force) are introduced by the change in the Christoffel symbols. But while this perspective is natural in general relativity, it doesn't help one trying to obtain trajectories in the weak-field limit.

Can one ignore the gauge-dependence of $\Gamma^{\mu}{}_{\alpha\beta}$ by simply regarding $h_{\mu\nu}(x)$ as a given field? Yes, up to a point. However, as we will see later, the gauge-dependence rears its ugly head when one tries to solve the linearized field equations for $h_{\mu\nu}$. The Einstein equations contain extra degrees of freedom arising from the fact that a gauge-transformation of any solution is also a solution. Gravitational fields can mimic fictitious forces. In the full theory of GR this is no problem in principle, because gravity itself is a fictitious force — gravitational deflection arises from the use of curvilinear coordinates. (Of course, in a curved manifold we have no choice — we must use curvilinear coordinates!)

Regardless of how we interpret gravity, in practice we must eliminate the gauge freedom somehow. There are two ways to do this: one may form gauge-invariant quantities akin to the electromagnetic field strength tensor or impose gauge conditions that fix the potentials $h_{\mu\nu}$.

It happens that while the Christoffel connection is not gauge-invariant, in linearized gravity (but not in general) the Riemann tensor is gauge-invariant. Thus one way to form gauge-invariant quantities is to replace equation (4) by the geodesic deviation equation,

$$\frac{d^2(\Delta x)^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\nu} V^\alpha V^\beta (\Delta x)^\nu \tag{7}$$

where $(\Delta x)^\nu$ is the infinitesimal separation vector between a pair of geodesics. While this tells us all about the local environment of a freely-falling observer, it fails to tell us where the observer goes. In most applications we need to know the trajectories. Thus we will have to find other strategies for coping with the gauge problem.

# 3 Hamiltonian Formulation and Gravitomagnetism

Some aid in solving the gauge problem comes if we abandon manifest covariance and use $t = x^0$ to parameterize trajectories instead of the proper time $d\tau$. This yields the added benefit of highlighting the similarities between linearized gravity and electromagnetism. In particular, it illustrates the phenomenon of gravitomagnetism.

Changing the parameterization in equation (1) from $\tau$ to $t$ and performing a Legendre transformation gives the Hamiltonian

$$H(x^i, \pi_i, t) = \left(p^2 + m^2\right)^{1/2} + q\phi \ , \quad p^i \equiv \pi_i - qA_i \ , \quad \phi \equiv -A_0 \ . \tag{8}$$

Here we denote the conjugate momentum by $\pi^i$ to distinguish it from the mechanical momentum $p^i$. (Note that $p^i$ and $\pi_i$ are the components of 3-vectors in Euclidean space, so that their indices may be raised or lowered without change.) It is very important to treat the Hamiltonian as a function of the conjugate momentum and not the mechanical momentum, because only in this way do Hamilton's equations give the correct equations of motion:

$$\frac{dx^i}{dt} = \frac{p^i}{E} \equiv v^i \ , \quad \frac{d\pi_i}{dt} = q\left(-\partial_i\phi + v^j\partial_i A_j\right) \ , \quad E \equiv \sqrt{p^2 + m^2} = \frac{m}{\sqrt{1 - v^2}} \ . \tag{9}$$

Combining these gives the familiar form of the Lorentz force law,

$$\frac{dp^i}{dt} = q\,(\underline{E} + \underline{v} \times \underline{B})_i \ , \quad \underline{E} \equiv -\underline{\nabla}\phi - \partial_t \underline{A} \ , \quad \underline{B} = \underline{\nabla} \times \underline{A} \tag{10}$$

where underscores denote standard 3-vectors in Euclidean space. The dependence of the fields on the potentials ensures that the equation of motion is still invariant under the gauge transformation $\phi \to \phi - \partial_t\Phi$, $\underline{A} \to \underline{A} + \underline{\nabla}\Phi$.

Now we repeat these steps for gravity, starting from equation (3). For convenience, we first decompose $h_{\mu\nu}$ as

$$h_{00} = -2\phi \ , \quad h_{0i} = w_i \ , \quad h_{ij} = -2\psi\delta_{ij} + 2s_{ij} \ , \quad \text{where} \ \ s^j{}_j = \delta^{ij}s_{ij} = 0. \tag{11}$$

The ten degrees of freedom in $h_{\mu\nu}$ are incorporated into two scalars under spatial rotations ($\phi$ and $\psi$), one 3-vector, and one symmetric 2-index tensor, the traceless strain $s_{ij}$. Notice that $w_i$ and $s_{ij}$ generalize the weak-field metric used previously in 8.962.

To first order in $h_{\mu\nu}$, the Hamiltonian may now be written

$$\begin{aligned} H(x^i,\pi_i,t) &= (1+\phi)E \ , \quad E \equiv (\delta^{ij}p_ip_j + m^2)^{1/2} \ , \\ p_i &\equiv (1+\psi)\pi_i - (\delta^{ij}\pi_i\pi_j + m^2)^{1/2}w_i - s^j{}_i\pi_j \ . \end{aligned} \tag{12}$$

Here, $\pi_i$ is the conjugate momentum while $p_i$ and $E$ are the proper 3-momentum and energy measured by an observer at fixed $x^i$, just as they are in equation (8). To prove this, we construct an orthonormal basis for such an observer:

$$\vec{e}_{\hat{0}} = \frac{1}{\sqrt{-g_{00}}}\,\vec{e}_0 = (1-\phi)\vec{e}_0 \ , \quad \vec{e}_{\hat{i}} = \vec{e}_i + g_{0i}\vec{e}_0 - \frac{1}{2}h^j{}_i\vec{e}_j = (1+\psi)\vec{e}_i + w_i\vec{e}_0 - s^j{}_i\vec{e}_j \ . \tag{13}$$

This basis is constructed by first setting $\vec{e}_{\hat{0}} \parallel \vec{e}_0$ and normalizing it with $\vec{e}_{\hat{0}}\cdot\vec{e}_{\hat{0}} = -1$. Next, $\vec{e}_{\hat{i}}$ is required to be orthogonal to $\vec{e}_{\hat{0}}$, giving the $g_{0i}\vec{e}_0$ term (to first order in the metric perturbations). Requiring $\vec{e}_{\hat{i}}\cdot\vec{e}_{\hat{j}} = \delta_{ij}$ gives the remaining term. Now, using the results from the notes *Hamiltonian Dynamics of Particle Motion*, the spacetime momentum one-form is $\widetilde{P} = -H\tilde{e}^0 + \pi_i\tilde{e}^i$. Setting $E = -\widetilde{P}(\vec{e}_{\hat{0}})$ and $p_i = \widetilde{P}(\vec{e}_{\hat{i}})$ gives the desired results (to first order in the metric perturbations).

Equation (12) has the simple Newtonian interpretation that the Hamiltonian is the sum of $E$, the kinetic plus rest mass energy, and $E\phi$, the gravitational potential energy. This result is remarkably similar to equation (8), with just two differences. In place of charge $q$, the gravitational coupling is through the energy $E$. Gravitation also has a rank $(0,2)$ spatial tensor $h_{ij}$ in addition to spatial scalar and vector potentials.

Although the gravitational potentials represent physical metric perturbations, having obtained the Hamiltonian we can forget about this for the moment in order to gain intuition about weak-field gravity by applying our understanding of analogous electromagnetic phenomena.

Hamilton's equations applied to equation (12) give

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H}{\partial \pi_i} = (1+\phi+\psi)v^j\left(\delta_{ij} - v_iw_j - s_{ij}\right) \ , \quad v^i \equiv \frac{p^i}{E} \ , \\ \frac{d\pi_i}{dt} &= -\frac{\partial H}{\partial x^i} = E\left[-\partial_i\phi + v^j\partial_iw_j - (\partial_i\psi)v^2 + (\partial_is_{jk})v^jv^k\right] \end{aligned} \tag{14}$$

Let us compare our result with equation (9). The equation for $dx^i/dt$ is more complicated than the corresponding equation for electromagnetism because of the more complicated momentum-dependence of the gravitational "charge" $E$. Alternatively, one may adopt the curved spacetime perspective and note that $dx^i$ and $dt$ are coordinate differentials and not proper distances or times, so that the coordinate velocity $dx^i/dt$ must be corrected to give the proper 3-velocity $v^i$ measured by an observer at fixed $x^i$ in an orthonormal frame, with the same result. The Newtonian and curved spacetime interpretations are consistent.

Equations (14) may be combined to give the weak-field gravitational force law

$$\begin{aligned}\frac{dp^i}{dt} &= E\left(\underline{g}+\underline{v}\times\underline{H}\right)_i + \frac{1}{2}E\left[-v^j\partial_t h_{ij} + v^j v^k(\partial_i h_{jk}-\partial_k h_{ij})\right] \ , \\ \underline{g} &\equiv -\underline{\nabla}\phi - \partial_t \underline{w} \ , \quad \underline{H} = \underline{\nabla}\times\underline{w} \ . \end{aligned} \tag{15}$$

As before, underscores denote standard 3-vectors in Euclidean space. (The terms involving $h_{ij}$ may be expanded by substituting $h_{ij} = -2\psi\delta_{ij} + 2s_{ij}$. No simplification results, so they are left in a more compact form above.) This equation, the gravitational counterpart of the Lorentz force law, is exact for linearized GR (though it is not valid for strong gravitational fields). Combined with the first of equations (14), it is equivalent to the geodesic equation for timelike or null geodesics in a weakly perturbed Minkowski spacetime.

Equation (15) is remarkably similar to the Lorentz force law. It reveals electric-type forces (present for particles at rest) and magnetic-type forces (force perpendicular to velocity). In addition there are velocity-dependent forces arising from the tensor potentials, i.e. from the spatial curvature terms in the metric. The Newtonian limit is obvious when $v \ll 1$. But equation (15) is correct also for relativistic particles and for relativistically moving gravitational sources, as long as the fields are weak, i.e. $|h_{\mu\nu}| \ll 1$.

It is straightforward to check that equation (15) is invariant under a gauge transformation generated by shifting the time coordinate, equation (6) with $\xi^0 = \Phi$ and $\xi^i = 0$. However, the force law is not invariant under gauge (coordinate) transformations generated by $\xi^i$. Thus, the Hamiltonian formulation has not solved the gauge problem, although it has isolated it. As a result, it has provided important insight into the nature of relativistic gravitation.

The fields $g_i = -\partial_i\phi - \partial_t w_i$ and $H^i = \epsilon^{ijk}\partial_j w_k$ are called the gravitoelectric and gravitomagnetic fields, respectively. (Here, $\epsilon^{ijk}$ is the fully antisymmetric three-dimensional Levi-Civita symbol, with $\epsilon^{123} = +1$.) They are invariant under the gauge transformation generated by $\xi^0 = \Phi$ and therefore are not sensitive to how one chooses hypersurfaces of constant time, although they do depend on the parameterization of spatial coordinates within these hypersurfaces. Once those coordinates are fixed, the gravitoelectric and gravitomagnetic fields have a clear meaning given by equation (15). Noting that $\underline{p} = E\underline{v}$, these fields contribute to the acceleration $d\underline{v}/dt = \underline{g} + \underline{v}\times\underline{H}$.

There are four distinct gravitational phenomena present in equation (15). They are

- The quasi-Newtonian gravitational field $\underline{g}$.
- The gravitomagnetic field $\underline{H}$, which is responsible for Lense-Thirring precession and the dragging of inertial frames.

- The scalar part of $h_{ij}$, i.e. $h_{ij} = -2\psi\delta_{ij}$, which (for $\psi = \phi$) doubles the deflection of light by the sun compared with the simple Newtonian calculation.

- The transverse-traceless part of $h_{ij}$, or gravitational radiation, described by the transverse-traceless strain matrix $s_{ij}$.

The rest of these notes will explore these phenomena in greater detail.

# 4 Field Equations

Greater understanding of the physics of weak-field gravitation comes from examining the Einstein equations and comparing them with the Maxwell equations. This will allow us to solve the gauge problem and thereby to explore the phenomena mentioned above with confidence that we are not being misled by coordinate artifacts.

Starting from equation (11), we obtain the linearized Christoffel symbols

$$\begin{aligned}
&\Gamma^0{}_{00} = \partial_t\phi \ , \quad \Gamma^0{}_{i0} = \partial_i\phi \ , \quad \Gamma^0{}_{ij} = -\partial_{(i}w_{j)} + \partial_t(s_{ij} - \delta_{ij}\psi) \ , \\
&\Gamma^i{}_{00} = \partial_i\phi + \partial_t w_i \ , \quad \Gamma^j{}_{i0} = \partial_{[i}w_{j]} + \partial_t(s_{ij} - \delta_{ij}\psi) \ , \\
&\Gamma^k{}_{ij} = \delta_{ij}\partial_k\psi - 2\delta_{k(i}\partial_{j)}\psi - \partial_k s_{ij} + 2\partial_{(i}s_{j)k} \ . \qquad (16)
\end{aligned}$$

(Notice that the Kronecker delta is used to raise and lower spatial components.) The Ricci tensor has components

$$\begin{aligned}
R_{00} &= \partial^2\phi + \partial_t(\partial_i w^i) + 3\partial_t^2\psi \ , \\
R_{0i} &= -\frac{1}{2}\partial^2 w_i + \frac{1}{2}\partial_i(\partial_j w^j) + 2\partial_t\partial_i\psi + \partial_t\partial_j s^j{}_i \ , \\
R_{ij} &= -\partial_i\partial_j(\phi - \psi) - \partial_t\partial_{(i}w_{j)} + (\partial_t^2 - \partial^2)(s_{ij} - \psi\delta_{ij}) + 2\partial_k\partial_{(i}s_{j)}{}^k \qquad (17)
\end{aligned}$$

where $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. The Einstein tensor components are

$$\begin{aligned}
G_{00} &= 2\partial^2\psi + \partial_i\partial_j s^{ij} \ , \\
G_{0i} &= -\frac{1}{2}\partial^2 w_i + \frac{1}{2}\partial_i(\partial_j w^j) + 2\partial_t\partial_i\psi + \partial_t\partial_j s^j{}_i \ , \\
G_{ij} &= (\delta_{ij}\partial^2 - \partial_i\partial_j)(\phi - \psi) + \partial_t\left[\delta_{ij}(\partial_k w^k) - \partial_{(i}w_{j)}\right] + 2\delta_{ij}(\partial_t^2\psi) \\
&\quad + (\partial_t^2 - \partial^2)s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}(\partial_k\partial_l s^{kl}) \ . \qquad (18)
\end{aligned}$$

It is fascinating that the time-time part of the Einstein tensor contains only the spatial parts of the metric, and $h_{00} = -2\phi$ appears only in $G_{ij}$. Although the equation of motion for nonrelativistic particles in the Newtonian limit is dependent only on $h_{00}$ (through $\Gamma^i{}_{00}$), the Newtonian gravitational field equation (the Poisson equation) is sensitive only to $h_{ij}$! I do not know if this is a merely a coincidence; it is not true for the fully nonlinear Einstein equations.

It is also fascinating that $G_{00}$ contains no time derivatives and $G_{0i}$ contains only first time derivatives. If the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ are to provide evolution equations for the metric, we would have expected a total of ten independent second-order in time equations, one for each component of $g_{\mu\nu}$. (After all, typical mechanical systems have, from the Euler-Lagrange equations, second-order time evolution equations for each generalized coordinate.) What is going on?

A clue comes from similar behavior of the Maxwell equations:

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu \ , \quad \partial_{[\kappa} F_{\mu\nu]} = 0 \ . \tag{19}$$

The substitution $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ automatically satisfies the source-free Maxwell equations and gives

$$\partial^2 A^0 + \partial_t(\partial_j A^j) = -4\pi J^0 \ , \quad \partial_t(\partial_i A^0 + \partial_t A^i) + \partial_i(\partial_j A^j) - \partial^2 A^i = 4\pi J^i \ , \tag{20}$$

where once again $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. Only the spatial parts of the Maxwell equations provide second-order time evolution equations. Does this mean that $A^i$ evolves dynamically but $A^0$ does not?

The answer to that question is clearly no, because $A_\mu$ is gauge-dependent and one can easily choose a gauge in which $\partial_\nu F^{0\nu}$ contains second time derivatives of $A^0$. (The Lorentz gauge, with $\partial_\mu A^\mu = 0$, is a well-known example.)

However, there is a sense in which the time part of the Maxwell equations (the first of eqs. 20) is redundant and therefore need not provide an equation of motion for the field. As the reader may easily verify, the time derivative of this equation, when subtracted from the spatial divergence of the spatial equations (the second of eqs. 20), enforces charge conservation, $\partial_\mu J^\mu = 0$. (We are working in flat spacetime so there is no need for the covariant derivative symbol.) This is another way of expressing the statement that gauge-invariance implies charge conservation. We are perfectly at liberty to choose a gauge such that $\partial_j A^j = 0$ (the Coulomb or transverse gauge), in which case only $A^i$ need be solved for by integrating a time evolution equation. Coulomb's law, $\partial^2 A^0 = -4\pi J^0$, may be regarded as a constraint equation to ensure conservation of charge.

Similarly, general relativity has a conservation law following from gauge-invariance: $\partial_\mu T^{\mu\nu} = 0$. Now there are four conserved quantities, the energy and momentum. (In the weak-field limit, but not in general, $T^{\mu\nu}$ can be integrated over volume to obtain a globally conserved energy and momentum.) The reader can easily verify the redundancy in equations (18): $\partial_t G^{00} + \partial_i G^{0i} = 0$, $\partial_t G^{0i} + \partial_j G^{ij} = 0$. Thus, if the matter evolves so as to conserve stress-energy $T^{\mu\nu}$, then the $G_{00}$ and $G_{0i}$ Einstein equations are redundant. They are present in order to enforce stress-energy conservation. In the literature they are known as the (linearized) Arnowitt-Deser-Misner (ADM) energy and momentum constraints (Arnowitt et al. 1962).

The Ricci and Einstein tensors are invariant (in linearized theory) under gauge transformations (eq. 6). This follows from the fact that a gauge transformation is a diffeomorphism and changes each tensor by the addition of a Lie derivative term: $R_{\mu\nu} \to R_{\mu\nu} + \mathcal{L}_{\vec{\xi}} R_{\mu\nu}$ and similarly for $G_{\mu\nu}$. The Lie derivative is first-order in both the shift vector $\vec{\xi}$ and the Ricci tensor, and therefore vanishes in linear theory. Put another way, because the Ricci tensor vanishes for the flat background spacetime, its Lie derivative vanishes.

Although $R_{\mu\nu}$ and $G_{\mu\nu}$ are gauge-invariant, their particular forms in equations (17) and (18) are not, because of the appearance of the metric perturbations $(\phi, w_i, \psi, s_{ij})$. Part of this dependence, in $G_{0i}$ and $G_{ij}$, can be eliminated by using the gravitoelectromagnetic fields, giving

$$\begin{aligned} G_{0i} &= \frac{1}{2}(\underline{\nabla} \times \underline{H})_i + 2\partial_t\partial_i\psi + \partial_t\partial_j s^j{}_i \ , \\ G_{ij} &= \partial_{(i}g_{j)} - \delta_{ij}(\partial_k g^k) + (\partial_i\partial_j - \delta_{ij}\partial^2)\psi + 2\delta_{ij}(\partial_t^2\psi) \\ &\quad + (\partial_t^2 - \partial^2)s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}(\partial_k\partial_l s^{kl}) \ . \end{aligned} \tag{21}$$

Note that the potentials $\phi$ and $w_i$ (from $h_{00}$ and $h_{0i}$) enter into both the equations of motion and the Einstein equations only through the fields $g_i$ and $H_i$, giving strong support to the interpretation of $\underline{g}$ and $\underline{H}$ as physical fields for linearized GR. But what of $\psi$ and $s_{ij}$? We explore this question in the next section.

## 5 Gauge-fixing: Transverse Gauge

Up to this point, we have imposed no gauge conditions at all on the metric tensor potentials. However, we have four coordinate variations at our disposal. Under the gauge transformation (6), the potentials change by

$$\delta\phi = \partial_t\xi^0 \ , \quad \delta w_i = -\partial_i\xi^0 + \partial_t\xi^i \ , \quad \delta\psi = -\frac{1}{3}\partial_i\xi^i \ , \quad \delta s_{ij} = \partial_{(i}\xi_{j)} - \frac{1}{3}\delta_{ij}(\partial_k\xi^k) \ . \tag{22}$$

Examing equations (21), it is clear that substantial simplification would result if could choose a gauge such that

$$\partial_j s^j{}_i = 0 \ . \tag{23}$$

Indeed, this is possible, by gauge-transforming any $s_{ij}$ which does not obey this condition using the spatial shift vector $\xi^i$ obtained by solving $\partial_j(s^j{}_i + \delta s^j{}_i) = 0$, or

$$\partial^2\xi^i + \frac{1}{3}\partial_i(\partial_j\xi^j) = -2\partial_j s^j{}_i \ . \tag{24}$$

This is an elliptic equation which may be solved by decomposing $\xi^i$ into longitudinal (curl-free) and transverse (divergence-free) parts. Solutions to this equation always exist;

indeed, suitable boundary conditions must be specified in order to yield a unique solution. In Section 8 we will discuss the physical meaning of the extra solutions.

Equation (23) is called the transverse-traceless gauge condition. It is widely used when studying gravitational radiation, but we will see that it is also useful for other applications.

Similarly, although we have hidden the vector potential $w_i$ in the gravitoelectromagnetic fields, the gauge may be fixed by requiring it to be transverse:

$$\partial_i w^i = 0 \ . \tag{25}$$

(The equations of motion depend only on $\underline{\nabla} \times \underline{w}$, so we expect to lose no physics by setting the longitudinal part to zero.) To convert a coordinate system that does not satisfy equation (25) to one that does, one solves the following elliptic equation for $\xi^0$:

$$\partial^2 \xi^0 - \partial_t(\partial_i \xi^i) = \partial_i w^i \ . \tag{26}$$

Once again, this equation (in combination with eq. 24 for $\xi^i$) may have multiple solutions depending on boundary conditions. (For given $\xi^i$, this is simply a Poisson equation for $\xi^0$.)

The combination of gauge conditions given by equations (23) and (25) imposes four conditions on the coordinates. They generalize the Coulomb gauge conditions of electromagnetism, $\partial_i A^i = 0$. As a result, both $w_i$ and the traceless part of $h_{ij}$ (i.e., $s_{ij}$) are transverse. The gauge condition on $s_{ij}$ is well-known and is almost always used in studies of gravitational radiation; it reduces the number of degrees of freedom of $s_{ij}$ from five to two, corresponding to the two orthogonal polarizations of gravitational radiation. However, the metric is not fully constrained until a gauge condition is imposed on $w_i$ as well. Equation (25) reduces the number of degrees of freedom of $w_i$ from three to two. The total number of physical degrees of freedom is six: one each for the spatial scalar fields $\phi$ and $\psi$, two for the transverse vector field $w_i$, and two for the transverse-traceless tensor field $s_{ij}$.

Based on its similarity with the Coulomb gauge of electromagnetism, Bertschinger (1996) dubbed these gauge conditions the Poisson gauge. Here we will call them **transverse gauge**. In transverse gauge, the Einstein equations become

$$\begin{aligned}
G_{00} &= 2\partial^2\psi = 8\pi G T_{00} \ , \\
G_{0i} &= \frac{1}{2}(\underline{\nabla} \times \underline{H})_i + 2\partial_t\partial_i\psi = 8\pi G T_{0i} \ , \\
G_{ij} &= (\delta_{ij}\partial^2 - \partial_i\partial_j)(\phi - \psi) - \partial_t\partial_{(i}w_{j)} + 2\delta_{ij}(\partial_t^2\psi) + (\partial_t^2 - \partial^2)s_{ij} \\
&= \partial_{(i}g_{j)} - \delta_{ij}(\partial_k g^k) + (\partial_i\partial_j - \delta_{ij}\partial^2)\psi + 2\delta_{ij}(\partial_t^2\psi) + (\partial_t^2 - \partial^2)s_{ij} = 8\pi G T_{ij} \ .
\end{aligned} \tag{27}$$

The $G_{00}$ equation is precisely the Newtonian Poisson equation, justifying the alternative name Poisson gauge.

# 6 Scalar, Vector and Tensor Components

Having reduced the number of degrees of freedom in the metric to six, let us now reexamine the statement made at the end of Section 3 that there are four distinct gravitational phenomena. They may be classified by the form of the metric variables as scalar ($\phi$ and $\psi$), vector ($w_i$) and tensor ($s_{ij}$). The scalar-vector-tensor decomposition was first performed by Lifshitz (1946) in the context of perturbations of a Robertson-Walker (cosmological) spacetime, but it works (at least) for perturbations of any spacetime (such as Minkowski) with sufficient symmetry (i.e. with sufficient number of Killing vector fields). See Section 4.2 of Bertschinger (1996) for the cosmological application.

The scalar-vector-tensor decomposition is based on decomposing both the metric and stress-energy tensor components into longitudinal and transverse parts. Three-vectors like $\underline{w}$ (regarded as a three-vector in Euclidean space) and $T_{0i}$ are decomposed as follows:

$$w^i = w^i_{\parallel} + w^i_{\perp} \ , \quad \underline{\nabla} \times \underline{w}_{\parallel} = \vec{e}_i \epsilon^{ijk} \partial_j w_{k,\parallel} = 0 \ , \quad \underline{\nabla} \cdot \underline{w}_{\perp} = \delta^{ij} \partial_i w_{j,\perp} = 0 \ . \tag{28}$$

In the transverse gauge, $\underline{w}_{\parallel} = 0$ but we are retaining it here for purposes of illustration.

The terms "longitudinal" and "transverse" come from the Fourier transform representation. Because $\underline{w}_{\parallel} = \underline{\nabla}\Phi_w$ for some scalar field $\Phi_w$, the Fourier transform of $\underline{w}_{\parallel}$ is parallel to the wavevector $\underline{k}$. Similarly, $\underline{w}_{\perp} = \underline{\nabla} \times \underline{A}_w$ for some vector field $\underline{A}_w$, hence its Fourier transform is perpendicular (i.e. transverse) to $\underline{k}$. A spatial constant vector may be regarded as being either longitudinal or transverse.

Jackson (1975, Section 6.5) gives explicit expressions for the longitudinal and transverse parts of a three-vector field in flat space:

$$\underline{w}_{\parallel} = -\frac{1}{4\pi} \underline{\nabla} \int \frac{\underline{\nabla}' \cdot \underline{w}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d^3x' \ , \quad \underline{w}_{\perp} = \frac{1}{4\pi} \underline{\nabla} \times \underline{\nabla} \times \int \frac{\underline{w}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d^3x' \ . \tag{29}$$

Note that this decomposition is nonlocal, i.e. the longitudinal and transverse parts carry information about the vector field everywhere. Thus, if $\underline{w}$ is nonzero only in a small region of space, its longitudinal and transverse parts will generally be nonzero everywhere. One cannot deduce causality by looking at $\underline{w}_{\parallel}$ or $\underline{w}_{\perp}$ alone.

Similarly, a symmetric two-index tensor may be decomposed into three parts depending as to whether its divergence is longitudinal, transverse, or zero:

$$h_{ij} = h_{ij,\parallel} + h_{ij,\perp} + h_{ij,\mathrm{T}} \ . \tag{30}$$

We will refer to these parts as longitudinal (or scalar), rotational (or solenoidal or vector) and transverse (or tensor) parts of $h_{ij}$. In the transverse gauge $h_{ij} = h_{ij,\mathrm{T}}$, but we retain the other parts here for purpose of illustration.

The longitudinal and rotational parts are defined in terms of a scalar field $h_{\parallel}(x)$ and a transverse vector field $\underline{h}_{\perp}(x)$ such that

$$h_{ij,\parallel} = \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\partial^2\right) h_{\parallel} \ , \quad h_{ij,\perp} = \partial_{(i} h_{j),\perp} \ . \tag{31}$$

As stated above, the divergences of $h_{ij,\|}$ and $h_{ij,\perp}$ are longitudinal and transverse vectors, respectively, and the divergence of $h_{ij,\mathrm{T}}$ vanishes identically:

$$\delta^{jk}\partial_k h_{ij,\|} = \frac{2}{3}\partial_i(\partial^2 h_\|) \ , \quad \delta^{jk}\partial_k h_{ij,\perp} = \frac{1}{2}\partial^2 h_{i,\perp} \ , \quad \delta^{jk}\partial_k h_{ij,\mathrm{T}} = 0 \ . \tag{32}$$

Thus, the longitudinal part is obtainable from a scalar field, the rotational part is obtainable only from a (transverse) vector field, and the transverse part is obtainable only from a (transverse traceless) tensor field. The reader may find it a useful exercise to construct integral expressions for these parts, similar to equations (29).

The stress-energy tensor may be decomposed in a similar way. Doing this, the linearized Einstein equations (27) in transverse gauge give field equations for the physical fields $(\psi, g_i, H_i, s_{ij})$:

$$\begin{aligned}
&\partial^2\psi = 4\pi G T_{00} \ , \\
&\underline{\nabla}\cdot\underline{g} - 3\partial_t^2\psi = -4\pi G(T_{00} + T^i{}_i) \ , \\
&\underline{\nabla}\times\underline{H} = -16\pi G \underline{f}_\perp \ , \quad \underline{f} \equiv T^{0i}\vec{e}_i \ , \\
&(\partial_t^2 - \partial^2)s_{ij} = 8\pi G T_{ij,\mathrm{T}}
\end{aligned} \tag{33}$$

plus constraint equations to ensure $\partial_\mu T^{\mu\nu} = 0$:

$$\begin{aligned}
&\partial_t\underline{\nabla}\psi = -4\pi G \underline{f}_\| \ , \\
&\partial_{(i}g_{j)} - \frac{1}{3}\delta_{ij}(\partial_k g^k) + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2)\psi = 8\pi G(\Pi_{ij} - \Pi_{ij,\mathrm{T}}) \ , \\
&\text{where} \ \ \Pi_{ij} \equiv T_{ij} - \frac{1}{3}\delta_{ij}T^k{}_k \ .
\end{aligned} \tag{34}$$

A third constraint equation may be further decomposed into longitudinal and rotational parts as follows:

$$(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2)(\psi - \phi) = 8\pi G\,\Pi_{ij,\|} \ , \quad -\partial_t\partial_{(i}w_{j)} = 8\pi G\,\Pi_{ij,\perp} \ . \tag{35}$$

Equations (33)–(35) may be regarded as the fundamental Einstein equations in linear theory. No approximations have been made in deriving them, aside from $|h_{\mu\nu}| \ll 1$.

# 7 Physical Content of the Einstein equations

Equations (33)–(35) are remarkable in bearing similarities to both Newtonian gravity and electrodynamics. They exhibit precisely the four physical features mentioned at the end of Section 3: the quasi-Newtonian gravitational field $\underline{g}$, the gravitomagnetic field $\underline{H}$, the spatial potential $\psi$, and the transverse-traceless strain $s_{ij}$.

To see the effects of these fields, let us rewrite the gravitational force law, equation (15), using equation (11) with the transverse gauge conditions (23) and (25):

$$\frac{d\underline{p}}{dt} = E\left[\underline{g} + \underline{v}\times\underline{H} + \underline{v}(\partial_t\psi) - v^2\underline{\nabla}_\perp\psi\right] + E\left[-v^j\partial_t s^i{}_j + \epsilon^{ijk}v_j v^l \Omega^k{}_l\right]\vec{e}_i \tag{36}$$

where $\underline{\nabla}_\perp \equiv (\delta^{ij} - v^i v^j/v^2)\vec{e}_i\partial_j$ is the gradient perpendicular to $\underline{v}$ and

$$\Omega^{kl} \equiv \partial_m s_n{}^{(k}\epsilon^{l)mn} \tag{37}$$

is the "curl" of the strain tensor $s^{kl}$. (We define the curl of a symmetric two-index tensor by this equation).

We can build intuition about each component of the gravitational field $(\underline{g}, \underline{H}, \psi, s_{ij})$ by comparing equations (33)–(36) with the corresponding equations of Newtonian gravitation and electrodynamics.

First, the gravitoelectric field $\underline{g}$ is similar to the static Newtonian gravitational field in its effects, but its field equation (33b) differs from the static Poisson equation. While the potential $\psi$ obeys the Newtonian Poisson equation (33a), its time derivative enters the equations of motion for both $\underline{g}$ and for particle momenta. Why? Note first that we've regarded $\phi$ as the more natural generalization of the Newtonian potential because it gives the deflection for slowly-moving particles; the terms with $\psi$ in equation (36) all vanish when $v^i = 0$. Under what conditions then do we have $\phi \neq \psi$ and why does equation (33b) differ from the Newtonian Poisson equation?

The answers lie in source motion and causality. If the sources are static (or their motion is negligible), $\partial_t\psi = 0$ from equation (33a). The first of equations (35) shows that if the shear stress is small (compared with $T_{00}$), then $\phi \approx \psi$ (up to solutions of $\partial_i\partial_j(\phi - \psi) = 0$). Small stresses imply slow motions, so we deduce that the gravitational effects are describable by static gravitational fields in the Newtonian limit. Thus, one cannot argue that the Einstein equations violate causality because $\psi$ is the solution of a static elliptic equation. The gravitational effects on slowly moving particles come not from $\psi$ but from $\underline{g}$, whose source depends on the $\partial_t^2\psi$ as well as on the pressure.

It is instructive to compare the field equations for $\underline{g}$ and $\underline{H}$ with the Maxwell equations for $\underline{E}$ and $\underline{B}$:

$$\begin{aligned} \underline{\nabla}\cdot\underline{E} = 4\pi\rho_c\ ,\quad \underline{\nabla}\times\underline{E} + \partial_t\underline{B} = 0\ ,\\ \underline{\nabla}\cdot\underline{B} = 0\ ,\quad \underline{\nabla}\times\underline{B} - \partial_t\underline{E} = 4\pi\underline{J}_c \end{aligned} \tag{38}$$

were $J^\mu = (\rho_c, \underline{J}_c)$ is the four-current density. By comparison, $\underline{g}$ and $\underline{H}$ obey

$$\begin{aligned} \underline{\nabla}\cdot\underline{g} - 3\partial_t^2\psi = -4\pi G(T_{00} + T^i{}_i)\ ,\quad \underline{\nabla}\times\underline{g} + \partial_t\underline{H} = 0\ ,\\ \underline{\nabla}\cdot\underline{H} = 0\ ,\quad \underline{\nabla}\times\underline{H} = -16\pi G\underline{f}_\perp\ . \end{aligned} \tag{39}$$

How do we interpret these?

Gauss' law for the gravitoelectric field differs from its electrostatic counterpart because of the time-dependence of $\psi$ and the inclusion of spatial stress as a source. (The electromagnetic source, being a vector rather than a rank-2 tensor, has no such possibility.) We already noted that the $\partial_t^2\psi$ term is needed to ensure causality. (The proof of this is somewhat detailed, requiring a transformation to the Lorentz gauge where all metric components obey wave equations.)

The source-free equations, of both electrodynamics and gravitodynamics ensure that magnetic field lines have no ends. Faraday's law of induction $\partial_t\underline{B} + \underline{\nabla} \times \underline{E} = 0$ (and its gravitational counterpart) ensures $\underline{\nabla} \cdot \underline{B} = 0$ persists when the current sources evolve in Ampere's law.

So far gravitation and electrodynamics appear similar. However, Ampere's law reveals a fundamental difference between the two theories. *There is no gravitational displacement current.* The gravitomagnetic field does not obey a causal evolution equation — it is determined by the instantaneous energy current. Moreover, it is not the whole current $\underline{f} = T^{0i}\vec{e}_i$ that appears as its source but rather only the transverse current. (The longitudinal current would be incompatible with the transverse field $\underline{\nabla} \times \underline{H}$.)

Recall that the Maxwell equations enforce charge conservation through the time derivative of Gauss' law combined with the divergence of Ampere's law. Gravitation is completely different: $\partial_\mu T^{\mu 0} = 0$ is enforced by equations (33a) and (34a), which are not even present in our gravitational "Maxwell" equations. So gravitation doesn't need a displacement current to enforce energy conservation. However, the displacement current plays another fundamental role in electromagnetism, which was recognized by Maxwell before there was any experimental evidence for this term: it leads to wave equations for the electromagnetic fields.

The conclusion is inescapable — $\underline{g}$ and $\underline{H}$ do not obey causal wave equations. This does not mean GR violates causality, because one must include the effects of $\psi$ and $s_{ij}$ on any particle motion (eq. 36). This is left as an extended exercise for the reader. However, it is worth noting that one cannot simply deduce causality from the fact that $s_{ij}$ evolves according to a causal wave equation (eq. 33d). The source for $s_{ij}$ is the transverse-traceless stress, which extends over all space even if $T^{\mu\nu} = 0$ outside a finite region. (This gives rise to "near-field" contributions from gravitational radiation sources similar to the near-field electromagnetic fields of radiating charges.)

So far we have discussed the physics of $\underline{g}$ and $\underline{H}$ in detail but there are some aspects of the spatial metric perturbation fields $\psi$ and $s_{ij}$ remaining to be discussed. Starting with equation (36), we see that $\psi$ plays two roles. The first was discussed in the notes *Hamiltonian Dynamics of Particle Motion*: $\psi$ doubles the deflection of light (or any particle with $v = 1$). Its effect on the proper 3-momentum is to produce a transverse force $-Ev^2\underline{\nabla}_\perp\psi$. However, a time-varying potential also changes the proper energy of a particle through the longitudinal force $E\underline{v}(\partial_t\psi) = \underline{p}\partial_t\psi$. This effect is not the same as a time-varying gravitational (or electric) field; the Lorentz force law contains no such term as $\underline{p}\partial_t\phi$. It is purely a relativistic effect arising from the tensorial nature of gravity.

Finally, the best-known relativistic phenomenon of gravity is gravitational radiation, described (in transverse gauge) by the transverse-traceless potential $s_{ij}$. One could deduce the whole set of linearized Einstein equations by starting from the premise that gravitational radiation should be represented by a traceless two-index tensor (physically representing a spin-two field) and, because static gravitational fields are long-ranged, the graviton must be massless hence gravitational radiation must be transverse. (These statements will not be proven; doing so requires some background in field theory.) All the other gravitational fields may be regarded as auxiliary potentials needed to enforce gauge-invariance (local stress-energy conservation). In a similar way, Maxwell's equations may be built up starting from the premise that the transverse vector potential $\underline{A}_\perp$ obeys a wave equation with source given by the transverse current.

Gravitational radiation affects particle motion in three ways. The first two are apparent in equation (36). Noting that $v^l \Omega^k{}_l$ appears in the equation of motion the same way as the gravitomagetic field $\underline{H}$, we conclude that gravitational radiation contributes a force perpendicular to the velocity. However, that force is quadratic rather than linear in the velocity (for a given energy). Second, gravitational radiation contributes a term to the force that is linear in the velocity but dependent on the time derivative: $-v^j \partial_t s_{ij}$.

Both of these effects appear only in motion relative to the coordinate system. Because gravitational radiation produces no "force" on particles at rest in the coordinates, particles at rest remain at rest. The Christoffel symbol $\Gamma^i{}_{00}$ receives no contribution from $h_{ij}$.

Does this mean that gravitational radiation has no effect on static particles? No — it means instead that gravitational radiation cannot be understood as a force in flat spacetime; it is fundamentally a wave of space curvature. One cannot deduce its effects from the coordinates alone; one must also use the metric. The proper spatial separation between two events (e.g. points on two particle worldlines) with small coordinate separation $\Delta x^i = (\Delta x) n^i$ is $(g_{ij}\Delta x^i \Delta x^j)^{1/2} = (\Delta x)(1 + s_{ij} n^i n^j)$. (Note that Schutz and most other references used $h_{ij} = \frac{1}{2} s_{ij}$.) We see that $s_{ij}$ is the true strain — the change in distance divided by distance due to a passing gravitational wave. This strain effect, and not the velocity-dependent forces appearing in equation (36), is what is being sought by LIGO and other gravitational radiation detectors. The velocity-dependent forces do make a potentially detectable signature in the cosmic microwave background anisotropy, however, which provides a way to search for very long wavelength gravitational radiation.

# 8 Residual Gauge Freedom: Accelerating, Rotating, and Inertial Frames

Before concluding our discussion of linear theory, it is worthwhile examining equations (24) and (26) to deduce the gauge freedom remaining after we impose the transverse gauge conditions (23) and (25). Doing so will help to clarify the differences between gravity, acceleration, and rotation.

The gauge conditions are unaffected by linear transformations of the spatial coordinates, which are homogeneous solutions of equations (24) and (26):

$$\xi^0 = a^0(t) + b_i(t)x^i \ , \quad \xi^i = a^i(t) + \Big[c_{(ij)} + c_{[ij]}(t)\Big]\, x^j \ , \quad c_{(ij)} = \text{constant} \ . \qquad (40)$$

Equations (24) and (26) also have quadratic solutions in the spatial coordinates, but these are excluded because gauge transformations require that the coordinate transformation $x^\mu \to y^\mu = x^\mu - \xi^\mu$ be one-to-one and invertible. (The symmetric tensor $c_{ij}$ must be constant because otherwise $\xi^0$ would have a contribution $\frac{1}{2}\dot{c}_{ij}x^i x^j$.)

The various terms in equation (40) have straightforward physical interpretations: $a^0(t)$ represents a global redefinition of the time coordinate $t \to t - a^0(t)$, $b_i(t)$ is a velocity which tilts the $t$-axis as in an infinitesimal Lorentz transformation ($t' = t - vx$), $da^i/dt$ is the other half of the Lorentz transformation (e.g. $x' = x - vt$), $c_{(ij)}$ represents a static stretching of the spatial coordinates, and $c_{[ij]}$ is a spatial rotation of the coordinates about the axis $\epsilon^{ijk}c_{jk}$.

Notice that the class of coordinate transformations allowed under a gauge transformation is broader than the Lorentz transformations of special relativity. Transformations to accelerating ($d^2a^i/dt^2$) and rotating ($dc_{[ij]}/dt$) frames occur naturally because the formulation of general relativity is covariant. That is, the equations of motion have the same form in any coordinate system. (However, the assumption $|h_{\mu\nu}| \ll 1$ greatly limits the coordinates allowed in linear theory.)

Using equations (22) and (40), the changes in the fields are

$$\delta\underline{g} = -\underline{\ddot{a}} + \underline{\dot{\omega}} \times \underline{r} \ , \quad \delta\underline{H} = -2\,\underline{\omega} \ , \quad \delta\psi = -\frac{1}{3}c^k{}_k \ , \quad \delta s_{ij} = c_{(ij)} - \frac{1}{3}\delta_{ij}c^k{}_k \qquad (41)$$

where $\underline{r} \equiv x^i\vec{e}_i$ is the "radius vector" (which has the same meaning here as in special relativity) and the angular velocity $\omega^i$ is defined through

$$\frac{dc_{[ij]}}{dt} \equiv \epsilon_{ijk}\omega^k \ . \qquad (42)$$

The spatial curvature force terms in equation (36) are invariant because the residual gauge freedom of transverse gauge in equation (41) allows only for constant spatial deformations (i.e., time-independent $\delta h_{ij}$). Gravitational radiation is necessarily time-dependent, so it is completely fixed by the transverse-traceless gauge condition equation (23). The spatial curvature potential $\psi$ is arbitrary up to the addition of a constant. Thus, only the gravitoelectric and gravitomagnetic fields have physically relevant gauge freedom after the imposition of the transverse gauge conditions.

Note that equations (41) leave the Einstein equations (33)–(34) and (39) invariant. The Riemann, Ricci and Einstein tensors are gauge-invariant for a weakly perturbed Minkowski spacetime.

However, the gravitational force equation (36) is not gauge-invariant. Under the gauge transformation of equations (40) and (41) it acquires additional terms:

$$\delta\left(\frac{dp}{dt}\right) = E\left(\delta \underline{g} + \underline{v} \times \delta \underline{H}\right) = E\left(-\underline{\ddot{a}} + \underline{\dot{\omega}} \times \underline{r} + 2\underline{\omega} \times \underline{v}\right) \ . \tag{43}$$

The reader will recognize these terms as exactly the fictitious forces arising from acceleration and rotation relative to an inertial frame. The famous Coriolis acceleration is $2\underline{\omega} \times \underline{v}$. (The centrifugal force term is absent because it is quadratic in the angular velocity and it vanishes in linear theory.)

The Weak Equivalence Principle is explicit in equation (43): acceleration is equivalent to a uniform gravitational (gravitoelectric) field $\underline{g}$. Moreover, we have also discovered that rotation is equivalent to a uniform gravitomagnetic field $\underline{H}$. Uniform fields are special because they can be transformed away while remaining in transverse gauge.

The observant reader may have noticed the word "inertial" used above and wondered about its meaning and relevance here. Doesn't GR single out no preferred frames? That is absolutely correct; GR distinguishes no preferred frames. However, *we* singled out a class of frames (i.e. coordinates) by imposing the transverse gauge conditions (23) and (25). **Transverse gauge provides a relativistic counterpart to inertial frames.** This is not just one frame but a class of frames because equation (36) is invariant under (small constant velocity) Lorentz transformations: $b_i$ and $da^i/dt$ are absent from equation (43). Thus, the Galilean-invariance of Newton's laws is extended to the Lorentz-invariance of the relativistic force law in transverse gauge. However, the gravitational force now includes magnetic and other terms not present in Newton's laws.

Although the gravitational force equation is not *invariant*, it is *covariant*. Fictitious forces are automatically incorporated into existing terms ($\underline{g}$ and $\underline{H}$); the *form* of equation (36) is invariant under the residual gauge freedom of equations (40) even though the *values* of each term are not. This points out a profound fact of gravity in general relativity: **nothing in the equations of motion distinguishes gravity from a fictitious force**.

Indeed, the curved spacetime perspective regards gravitation entirely as a fictitious force. Nonetheless, we can, by imposing the transverse gauge (or other gauge) conditions, make our own separation between physical and fictitious forces. (Here I must note the caveat that transverse gauge has not been extended to strong gravitational fields so I don't know whether all the conclusions obtained here are restricted to nearly flat spacetimes.) Uniform gravitoelectric or gravitomagnetic fields can always be transformed away, hence they may be regarded as being due to acceleration or rotation rather than gravity. Spatially varying gravitoelectric and gravitomagnetic fields cannot be transformed away. They can only be caused by the stress-energy tensor and they are not coordinate artifacts.

This separation between gravity and fictitious forces is somewhat unnatural in GR (and it requires a tremendous amount of preparation!), but it is helpful for building intuition by relating GR to Newtonian physics.

This discussion also sheds light on GR's connection to Mach's principle, which states that inertial frames are determined by the rest frame of distant matter in the universe (the "fixed stars"). This is not strictly true in GR. Non-rotating inertial frames would be ones in which $\underline{H} = 0$ everywhere. Locally, any nonzero $\underline{H}$ can be transformed away by a suitable rotation, but the rotation rates may be different at different places in which case there can exist no coordinate system in which $\underline{H} = 0$ everywhere. (In this case the coordinate lines would quickly tangle, cross and become unusable.) Transverse energy currents, due for example to rotating masses, produce gravitomagnetic fields that cannot be transformed away. (See the gravitational Ampere's law in eqs. 39.) However, the gravitomagnetic fields may be very small, in which case there do exist special frames in which $\underline{H} \approx 0$ and there are no Coriolis terms in the force law.

We happen to live in a universe with small transverse energy currents: the distant matter is not rotating. (Sensitive limits are placed by the isotropy of the cosmic microwave background radiation.) Thus, due to good fortune, Mach was partly correct. However, were he to stand close to a rapidly rotating black hole, and remain fixed relative to the distant stars, he would get dizzy from the gravitomagnetic field. (Mach would literally feel like his head was spinning.) Thus, Mach's principle is not built into GR but rather is a consequence of the fact that we live in a non-rotating (or very slowly rotating) universe.

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