Gravity, Metrics and Coordinates

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La perfection est atteinte non quand il ne reste rien ajouter, mais quand il ne reste rien enlever.[1]

— Antoine de Saint-Exupery.

1. Introduction

These notes supplement Chapter 2 of EBH (*Exploring Black Holes* by Taylor and Wheeler). They provide several examples of metrics for realistic spacetimes and apply the concepts of coordinate transformations and local inertial frames discussed in the first set of notes, *Coordinates and Proper Time*. You should focus on the mathematical issues here first, and not expect the equations to provide you immediately with wonderful intuitive understanding. It takes time and practice to learn how to read a foreign language. Similarly it takes time and practice to learn how to read a metric. These notes will provide you with some of the mathematical background along with a few glittering examples of physical insight. First, however, we begin with a summary of Einstein's great insights that led to his revolutionary theory of gravity.

2. Gravity and Fields from Galileo to Einstein

Our introduction to general relativity begins with a review of the key discoveries in physics which inspired Einstein. It is a huge conceptual leap to go from Newtonian gravity to general relativity. Let us take a slow walk over the long bridge between these two theories.

Newtonian gravitation is described by an inverse square law of force. This force can be obtained from the gravitational potential $\Phi_{\rm N}(\vec{x},t)$ by means of simple calculus.[2] The gravitational potential is a scalar field, i.e. a single function defined at each spacetime event. The spatial gradient of the potential gives the gravitational force on a test particle:

$$\vec{F}_g = -m_g \vec{\nabla}\Phi_{\rm N} = -m_g \left(\frac{\partial\phi}{\partial x}\vec{e}_x + \frac{\partial\phi}{\partial y}\vec{e}_y + \frac{\partial\phi}{\partial z}\vec{e}_z\right) \ . \tag{1}$$

[1] Perfection is reached, not when there is no longer anything to add, but when there is no longer anything to take away.

[2] The subscript N reminds us that this is the Newtonian potential.

You should verify that the gravitational force due to a mass M arises from the potential $\Phi_{\rm N} = -GM/r$ with $r = \sqrt{x^2+y^2+z^2}$. We use the notation $\vec{e}_x$ for the unit vector in the x-direction (and similarly for y and z). The symbol m_g is used for the gravitational mass, i.e. the mass that determines the gravitational force on a body. This is by contrast with the inertial mass m_i appearing in Newton's second law of motion:

$$\vec{a} = m_i^{-1}\vec{F} \ , \tag{2}$$

where $\vec{a}$ is the acceleration of a body subjected to force $\vec{F}$. Combining equations (1) and (2) we see that the acceleration of a body in a gravitational field depends on the ratio m_g/m_i:

$$\vec{a} = -\frac{m_g}{m_i}\vec{\nabla}\Phi_{\rm N} \ . \tag{3}$$

Galileo showed experimentally 400 years ago that all bodies accelerate exactly the same way in a gravitational field.[3] Newton interpreted this result to imply the equivalence of gravitational and inertial mass, $m_i = m_g$.[4]

We are so used to the equivalence of inertial and gravitational masses that we drop the subscripts and refer only to the mass m. Einstein's first great insight was to realize that this result should not be taken for granted, as it provides an important clue to the nature of gravity.

To see why Galileo's result is special, consider electric fields, which obey an inverse square law very similar to gravity. The acceleration caused by an electric field is proportional to q/m_i where q is the test charge. However, the ratio q/m_i is *not* the same for all bodies, unlike m_g/m_i.[5] Einstein realized the deep significance of $m_i = m_g = m$ and adopted it as the centerpiece of his Equivalence Principle. By following the Equivalence Principle to its logical conclusion, Einstein was able to develop the theory of General Relativity from pure thought.

In this class we will not follow Einstein's path in all its mathematical sophistication. However, we will use the most important consequence of the Equivalence Principle, which provided the framework for Einstein's whole approach to combining gravity and relativity: *Gravitational forces result from the properties of spacetime itself.*

This assumption is radical from the perspective of Newtonian mechanics where gravity is given instantaneously by action at a distance and spacetime is nothing but the unchanging stage upon which all action takes place. According to Newton's laws, if you cross the street, the gravitational field a million light years away changes instantaneously. Yet according to the special theory of relativity, signals cannot travel infinitely fast. Newtonian gravity contradicts special relativity.

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[3]Galileo studied non-relativistically moving bodies. Had he been able to measure the accelerations of relativistically moving objects, he would have found that the acceleration depends on v/c.

[4]This conclusion assumes that the gravitational potential is independent of the mass of the test particle.

[5]Consider, for example, a proton, neutron, and electron.

A similar puzzle arises with electromagnetism and it had been resolved by the work of Faraday and Maxwell. The electric force between two charges obeys an inverse square law so that moving one charge in Boston would, according to Coulomb's law, instantaneously change the electric force everywhere in the universe.

Faraday introduced the concept of a field of force which transmits the action of electricity or magnetism (and gravity) through space. Maxwell showed that changes in the field are not propagated instantaneously but instead travel at the speed of light. The static Coulomb law, and by extension the static inverse square law of gravitation, need to be modified when the sources are in motion.

Einstein made a radical proposal: not only are gravitational effects carried by a field, but that field is intimately related to the spacetime geometry itself. *In general relativity, the spacetime metric itself plays the role of the field conveying all gravitational effects.*[6] The various terms in the metric are now to be regarded as functions of the spacetime coordinates following from the distribution of gravitational sources. We will devote the rest of this semester to exploring the metric and its effect on motion. John Wheeler summarizes general relativity in one elegant sentence: "Spacetime tells matter how to move; matter tells spacetime how to curve."

Spacetime is no longer the eternal, unchanging stage upon which all the world's dramas are played out. Spacetime itself is an actor. The choreography of stage and players makes general relativity one of the most challenging — and rewarding — physical theories to master.

Having argued that all gravitational effects must be encoded in the metric itself, we are led to ask how they are encoded in the metric. To answer this question we will examine a simple class of spacetime metrics around the structure of most interest in this course, the spherical black hole.

3. Static, Spherically Symmetric Spacetimes

The spacetimes around non-spinning stars and black holes are static and spherically symmetric to good accuracy. As a first step to studying the black hole metric, we provide a heuristic derivation of the general static, spherically symmetric metric.

We already know one static, spherically symmetric spacetime — the flat spacetime of special relativity. In Cartesian coordinates its metric is

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \ . \tag{4}$$

This metric is static because the coefficients of the differentials are independent of t. To show that

[6]The equivalence of inertial and gravitational masses crucially underlies this postulate. All bodies do *not* accelerate the same way in an electric field;, implying that electric effects cannot arise from the metric alone.

it is spherically symmetric, we transform from Cartesian to spherical polar coordinates:

$$x = r \sin\theta \cos\phi \ , \quad y = r \sin\theta \sin\phi \ , \quad z = r \cos\theta \ . \tag{5}$$

Taking the differentials (dx, dy, dz) and using the chain rule, one finds

$$\begin{aligned} dx &= dr\,\sin\theta\cos\phi + d\theta\, r\cos\theta\cos\phi - d\phi\, r\sin\theta\sin\phi \ , \\ dy &= dr\,\sin\theta\sin\phi + d\theta\, r\cos\theta\sin\phi + d\phi\, r\sin\theta\cos\phi \ , \\ dz &= dr\,\cos\theta - d\theta\, r\sin\theta \ . \end{aligned} \tag{6}$$

Substituting this into equation (4) gives

$$d\tau^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \ . \tag{7}$$

We now see the key feature of spherical symmetry: the polar angles (θ, ϕ) appear only in the combination of the metric for a sphere, $d\theta^2 + \sin^2\theta\, d\phi^2$.

The most general static metric with spherical symmetry is obtained by adding more terms to the metric (but none with θ and ϕ) and allowing the coefficients to depend on r:

$$d\tau^2 = e^{2\Phi(r)} dt^2 - 2A(r) dt dr - e^{2\Lambda(r)} dr^2 - r^2 e^{2B(r)} \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \ , \tag{8}$$

where Φ, Λ, A, and B are four arbitrary real functions of r. Note that Φ is not the same, in general, as the Newtonian gravitational potential $\Phi_{\rm N}$.

Equation (8) is actually *too* general. One of the most confusing aspects of general relativity for the novice is the fact that coordinates may be freely transformed without changing the physics. A terrestrial analogy might be when new telephone area codes are introduced and one's telephone number changes. Our labels may change but we must still be able to call our friends! In general relativity, however, the situation is more complicated: coordinates may be changed at any time, and sometimes several times during a calculation!

We can take advantage of this flexibility to simplify equation (8). In the homework, you will find equations that must be obeyed by coordinate transformations $t'(t, r)$ and $r'(r)$ such that, in the primed coordinates, equation (8) holds with $A = B = 0$. (The Φ and Λ fields may change but they will still depend only on r.) Dropping the primes, we conclude that the most general static, spherically symmetric spacetime may be written in the form

$$d\tau^2 = e^{2\Phi(r)} dt^2 - e^{2\Lambda(r)} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \ . \tag{9}$$

The radial coordinate r is sometimes called <u>angular radius</u> because it is used to related angular differentials to distance in the same way as in Euclidean space. For example, if we hold fixed t, r, and ϕ, the proper distance is $ds = r d\theta$. (Recall that $ds^2 \equiv -d\tau^2$.) It follows that the length of a great circle is $2\pi r$, motivating EBH to designate the r-coordinate <u>reduced circumference</u>. Note that *all* of the coordinates in the metric are bookkeeper coordinates — they have no meaning independent of the metric. The metric tells us how to measure distances and times – the coordinates do not! If you never confuse r or t with physical distance and time, you will save yourself a lot of grief! The authors of EBH are careful to put subscripts like *shell* on physical distances and times measured by shell observers. Bookkeeper coordinates have no subscripts.

4. Newtonian and Einstein Field Equations

In Newtonian gravity, there is only one potential $\Phi_{\rm N}(\vec{x},t)$. In general relativity, for static, spherically symmetric spacetimes, there are two: $\Phi(r)$ and $\Lambda(r)$. Before explaining why, we review the field equations of Newtonian gravity.

In Newtonian gravity, the gravitational potential obeys a second-order partial differential equation called the Poisson equation:

$$\nabla^2\Phi_{\rm N} = 4\pi G\rho \ . \qquad \text{[Newton]} \qquad (10)$$

Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator and $\rho(\vec{x},t)$ is the mass density. Equation (10) is similar to Gauss's law of electromagnetism

$$\vec{\nabla}\cdot E \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho_q \ , \qquad \text{[Coulomb]} \qquad (11)$$

where $\rho_q(\vec{x}\,)$ is the charge density. Static electric fields, like Newtonian gravitational fields, follow from an electrostatic potential Φ_E akin to equation (1): $\vec{E} = -\vec{\nabla}\Phi_E$. Combining this with Gauss's law gives $\nabla^2\Phi_E = -4\pi\rho_q$. Not surprisingly, gravity obeys the same equation except that the electric potential and charge are replaced by the gravitational potential and mass, with a sign change because like positive charges repel while positive masses attract. (The factor G in the gravitational equation may be eliminated by a choice of units.)

Equation (10) has a simple solution which we present without proof:

$$\Phi_{\rm N}(\vec{x},t) = -G\int\frac{\rho(\vec{x}\,',t)\,d^3x'}{|\vec{x}-\vec{x}\,'|} \ . \qquad \text{[Newton]} \qquad (12)$$

If you haven't seen equations (10) or (12) before, don't worry; we won't be using them. However, we will use the form of equation (10) that applies in spherical coordinates. If the mass density ρ is independent of angles, then the potential is spherically symmetric and the Laplace operator simplifies:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi_{\rm N}}{\partial r}\right) = 4\pi G\rho(r,t) \ . \qquad \text{[Newton]} \qquad (13)$$

You should be able to integrate this immediately to obtain

$$\Phi_{\rm N}(r,t) = \int_r^\infty \frac{GM(r',t)}{r'^2}\,dr' \ , \quad M(r,t) \equiv \int_0^r \rho(r,t)\,4\pi r'^2 dr' \ . \qquad \text{[Newton]} \qquad (14)$$

Note that $M(r,t)$ is the mass enclosed by a sphere of radius r. If the mass is concentrated at $r=0$, this immediately gives $\Phi_{\rm N} = -GM/r$. Note also that a prime is put on the dummy variable of integration because both $M(r,t)$ and $\Phi_{\rm N}(r,t)$ are given by definite integrals. (The upper limit of integration on $\Phi_{\rm N}$ is chosen so that $\Phi_{\rm N} \to 0$ as $r \to \infty$.)

In General Relativity, the metric functions Φ and Λ in equation (9) obey a set of nonlinear partial differential equations first derived by Einstein and called the Einstein field equations in his honor. We will neither present a derivation of the Einstein field equations nor even state their general form.[7] For the static, spherically symmetric metric of equation (9), these differential equations give the following:

$$\frac{1}{r^2}\frac{d}{dr}\left[r\left(1-e^{-2\Lambda}\right)\right] = 8\pi G\rho(r)\ , \qquad \text{[Einstein]} \tag{15a}$$

$$\frac{2}{r}\frac{d\Phi}{dr}e^{-2\Lambda} - \frac{1}{r^2}\left(1-e^{-2\Lambda}\right) = 8\pi G p(r)\ . \qquad \text{[Einstein]} \tag{15b}$$

These equations contain the mass-energy density $\rho(r)$ and pressure $p(r)$ of the medium responsible for producing the gravity. They illustrate a key difference between General Relativity and Newtonian gravity: *In General Relativity, pressure is a source of gravity.* The units of pressure are force per unit area, which is equivalent to energy per unit volume. Thus, p has the same units as ρc^2. (The factor c^2 is absent from eq. 15b because in these notes we always choose units so that $c=1$.)

Because the metric function $\Phi(r)$ depends on pressure, it follows that pressure causes gravitational effects. You might wonder whether these are related to the ordinary pressure force that a gas exerts on the walls of its container. The answer is no — the two effects are completely unrelated. The ordinary pressure force depends only on pressure *differences* arising from the pressure gradient $\vec{\nabla}p$. Equation (15b) says nothing at all about forces. It simply says that pressure causes spacetime to curve. We have said nothing yet about how gravitational force arises from spacetime curvature. That will be the subject of next week's material.

If pressure and density both contribute to gravity, why do Newton's laws include only density? The answer lies in the factor c^2 relating the units of the two quantities. Using Einstein's famous formula $E = mc^2$ and restoring the factors of c, the energy density of a nonrelativistic gas is ρc^2. The pressure has the same units, implying that pressure divided by density must be the square of some speed. For a nonrelativistic gas of molecules of mass m and number density n (molecules m^{-3}), elementary thermodynamics gives us

$$p = nkT = \rho\frac{kT}{m} \approx \rho c_{\rm s}^2\ . \tag{16}$$

Here, k is the Boltzmann constant and $c_{\rm s}$ is the sound speed of the gas.[8] For a nonrelativistic gas,

[7]After taking 8.224 and 8.07, you may wish to take 8.962 where the Einstein field equations and their solutions are investigated in detail.

[8]For an ideal gas, $c_{\rm s}^2 = \gamma kT/m$ where γ is a constant close to one. For a diatomic gas, $\gamma = 1.4$.

$c_s \ll c$ so that the right-hand side of equation (15b) is orders of magnitude less than the right-hand side of equation (15a).

5. Schwarzschild Metric and a Variation

Starting from equations (15), as a homework exercise you will derive the metric for a spacetime with $p = 0$ everywhere and a point mass M located at $r = 0$. The result is the famous Schwarzschild metric:

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \ . \tag{17}$$

The factors of G may be removed simply by appropriate choice of units, as discussed in Section 2-6 of EBH. EBH further restrict the discussion to the equatorial plane $\theta = \frac{\pi}{2}$ in which case $d\theta = 0$.

We will not say more about the Schwarzschild metric now, because that will be the main business for the next few weeks. However, we do wish to emphasize again that the coordinates (t, r, θ, ϕ) are simply bookkeeper coordinates with no physical significance by themselves. In General Relativity, the metric gives the coordinates meaning.

To show the non-uniqueness of the Schwarzschild coordinates, we will transform to a different set of coordinates which EBH call rain coordinates in Project B. (See pp. B-12 through B-14 in EBH.) The transformation proceeds as follows. First, we put primes on (t, r) in equation (17) — since they are just bookkeeper coordinates, we are free to call them anything we want to. Then we transform as follows:

$$t' = t + f(r) \ , \quad r' = r \ , \tag{18}$$

where $f(r)$ is a function to be determined. Next, take the differentials: $dt' = dt + (df/dr)dr$ and $dr' = dr$. Substituting into the Schwarzschild metric and grouping terms gives

$$d\tau^2 = Adt^2 + 2\frac{df}{dr}Adtdr - \left[A^{-1} - A\left(\frac{df}{dr}\right)^2\right] dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \ , \quad A \equiv 1 - \frac{2GM}{r} \ . \tag{19}$$

Now we impose a special condition on $f(r)$: we require that, for $dt = 0$, the metric reduce to the spatial metric of three-dimensional Euclidean space in spherical coordinates, as in equation (7). This condition cannot be met for an arbitrary spacetime but, remarkably, for a non-rotating black hole it is possible. It is accomplished by requiring the term in square brackets in equation (19) to equal 1. This condition is satisfied if

$$\pm f(r) = \int \frac{\sqrt{2GMr}\, dr}{r - 2GM} = 2\sqrt{2GMr} + 2GM \ln\left(\frac{\sqrt{r} - \sqrt{2GM}}{\sqrt{r} + \sqrt{2GM}}\right) \ . \tag{20}$$

We are free to choose either sign for f. With the choice of minus sign, the rain-frame metric becomes

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - 2\sqrt{\frac{2GM}{r}}\, dtdr - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \ . \tag{21}$$

Equation (21) looks quite different from equation (17). *Yet both metrics describe the same spacetime!* You must not assume that, because both metrics use the same symbols (t, r, θ, ϕ), these coordinates have the same meaning. The bookkeeper time for the two metrics is completely different, as shown by equations (18) and (20). EBH choose to call the bookkeeper time in the rain coordinates $t_{\rm rain}$ to avoid confusion with the bookkeeper time in Schwarzschild coordinates. Even without such subscripts, the bookkeeper coordinates become unambiguous once we specify the metric. To badly paraphrase Hamlet,

More relative than this: the metric's the thing
Wherein we'll catch the coordinates' meaning.

In the next few weeks we will focus on the interpretation of the metric and coordinates. We will find that the metric not only tells us almost everything that one can know about the structure of spacetime, but it also tells us everything we need to know about gravity.