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Measuring the Metric, and Curvature versus Acceleration

1 Introduction

These notes show how observers can set up a coordinate system and measure the spacetime geometry using clocks and lasers. The approach is similar to that of special relativity, but the reader must not be misled. Spacetime diagrams with rectilinear axes do not imply flat spacetime any more than flat maps imply a flat earth.

Cartography provides an excellent starting point for understanding the metric. Terrestrial maps always provide a scale of the sort "One inch equals 1000 miles." If the map is of a sufficiently small region and is free from distortion, one scale will suffice. However, a projection of the entire sphere requires a scale that varies with location and even direction. The Mercator projection suggests that Greenland is larger than South America until one notices the scale difference. The simplest map projection, with latitude and longitude plotted as a Cartesian grid, has a scale that depends not only on position but also on direction. Close to the poles, one degree of latitude represents a far greater distance than one degree of longitude.

The map scale is the metric. The spacetime metric has the same meaning and use: it translates coordinate distances and times ("one inch on the map") to physical ("proper") distances and times.

The terrestrial example also helps us to understand how coordinate systems can be defined in practice on a curved manifold. Let us consider how coordinates are defined on the Earth. First pick one point and call it the north pole. The pole is chosen along the rotation axis. Now extend a family of geodesics from the north pole, called meridians of longitude. Label each meridian by its longitude ϕ. We choose the meridian going through Greenwich, England, and call it the"prime meridian," $\phi = 0$. Next, we define latitude λ as an affine parameter along each meridian of longitude, scaled to $\pi/2$ at the north pole and decreasing linearly to $-\pi/2$ at the point where the meridians intersect again (the south pole). With these definitions, the proper distance between the nearby points with coordinates (λ, ϕ) and $(\lambda + d\lambda, \phi + d\phi)$ is given by $ds^2 = R^2(d\lambda^2 + \cos^2\lambda\, d\phi^2)$. In this way, every point on the sphere gets coordinates along with a scale which converts coordinate intervals to proper distances.

This example seems almost trivial. However, it faithfully illustrates the concepts involved in setting up a coordinate system and measuring the metric. In particular, coordinates are numbers assigned by obsevers who exchange information with each other. There is no conceptual need to have the idealized dense system of clocks and rods filling spacetime. Observe any major civil engineering project. The metric is measured by two surveyors with transits and tape measures or laser ranging devices. Physicists can do the same, in principle and in practice. These notes illustrate this through a simple thought experiment. The result will be a clearer understanding of the relation between curvature, gravity, and acceleration.

2 The metric in 1+1 spacetime

We study coordinate systems and the metric in the simplest nontrivial case, spacetime with one space dimension. This analysis leaves out the issue of orientation of spatial axes. It also greatly reduces the number of degrees of freedom in the metric. As a symmetric 2 matrix, the metric has three independent coefficients. Fixing two coordinates imposes two constraints, leaving one degree of freedom in the metric. This contrasts with the six metric degrees of freedom in a 3+1 spacetime. However, if one understands well the 1+1 example, it is straightforward (albeit more complicated) to generalize to 2+1 and 3+1 spacetime.

We will construct a coordinate system starting from one observer called A. Observer A may have any motion whatsoever relative to other objects, including acceleration. But neither spatial position nor velocity is meaningful for A before we introduce other observers or coordinates ("velocity relative to what?") although A's acceleration (relative to a local inertial frame!) is meaningful: A stands on a scale, reads the weight, and divides by rest mass. Observer A could be you or me, standing on the surface of the earth. It could equally well be an astronaut landing on the moon. It may be helpful in this example to think of the observers as being stationary with respect to a massive gravitating body (e.g. a black hole or neutron star). However, we are considering a completely general case, in which the spacetime may not be at all static. (That is, there may not be any Killing vectors whatsoever.)

We take observer A's worldine to define the t-axis: A has spatial coordinate $x_A \equiv 0$. A second observer, some finite (possibly large) distance away, is denoted B. Both A and B carry atomic clocks, lasers, mirrors and detectors.

Observer A decides to set the spacetime coordinates over all spacetime using the following procedure, illustrated in Figure 1. First, the reading of A's atomic clock gives

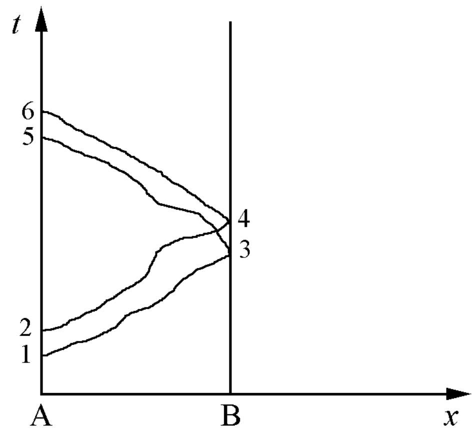

Figure 1: Setting up a coordinate system in curved spacetime. There are two timelike worldlines and two pairs of null geodesics. The appearance of flat coordinates is misleading; the metric varies from place to place.

the t-coordinate along the t-axis ($x = 0$). Then, A sends a pair of laser pulses to B, who reflects them back to A with a mirror. If the pulses do not return with the same time separation (measured by A) as they were sent, A deduces that B is moving and sends signals instructing B to adjust her velocity until $t_6 - t_5 = t_2 - t_1$. The two continually exchange signals to ensure that this condition is maintained. A then declares that B has a constant space coordinate (by definition), which is set to half the round-trip light-travel time, $x_B \equiv \frac{1}{2}(t_5 - t_1)$. A sends signals to inform B of her coordinate.

Having set the spatial coordinate, A now sends time signals to define the t-coordinate along B's worldline. A's laser encodes a signal from Event 1 in Figure 1, "This pulse was sent at $t = t_1$. Set your clock to $t_1 + x_B$." B receives the pulse at Event 3 and sets her clock. A sends a second pulse from Event 2 at $t_2 = t_1 + \Delta t$ which is received by B at Event 4. B compares the time difference quoted by A with the time elapsed on her atomic clock, the proper time $\Delta\tau_B$. To her surprise, $\Delta\tau_B \neq \Delta t$.

At first A and B are sure something went wrong; maybe B has begun to drift. But repeated exchange of laser pulses shows that this cannot be the explanation: the round-trip light-travel time is always the same. Next they speculate that the lasers may be traveling through a refractive medium whose index of refraction is changing with time. (A constant index of refraction wouldn't change the differential arrival time.) However, they reject this hypothesis when they find that B's atomic clock continually runs at a different rate than the timing signals sent by A, while the round-trip light-travel time

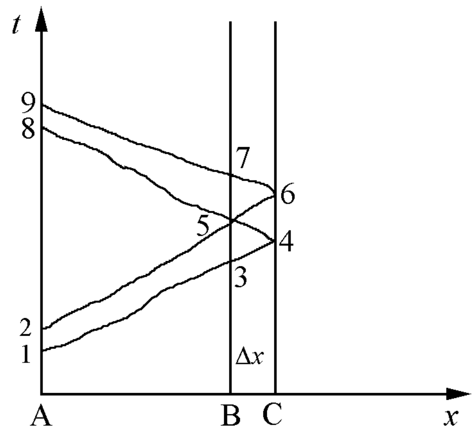

Figure 2: Testing for space curvature.

measured by A never changes. Moreover, laboratory analysis of the medium between them shows no evidence for any change.

Becoming suspicious, B decides to keep two clocks, an atomic clock measuring τ_B and another set to read the time sent by A, denoted t. The difference between the two grows increasingly large.

The observers next speculate that they may be in a non-inertial frame so that special relativity remains valid despite the apparent contradiction of clock differences ($g_{tt} \neq 1$) with no relative motion ($dx_B/dt = 0$). We will return to this speculation in Section 3. In any case, they decide to keep track of the conversion from coordinate time (sent by A) to proper time (measured by B) for nearby events on B's worldline by defining a metric coefficient:

$$g_{tt}(t, x_B) \equiv \lim_{\Delta t \to 0} -\left(\frac{\Delta\tau_B}{\Delta t}\right)^2 \ . \tag{1}$$

The observers now wonder whether measurements of spatial distances will yield a similar mystery. To test this, a third observer is brought to help in Figure 2. Observer C adjusts his velocity to be at rest relative to A. Just as for B, the definition of rest is that the round-trip light-travel time measured by A is constant, $t_8 - t_1 = t_9 - t_2 = 2x_C \equiv 2(x_B + \Delta x)$. Note that the coordinate distances are expressed entirely in terms of readings of A's clock. A sends timing signals to both B and C. Each of them sets one clock to read the time sent by A (corrected for the spatial coordinate distance x_B and x_C, respectively) and also keeps time by carrying an undisturbed atomic clock. The former is called coordinate time t while the latter is called proper time.

The coordinate time synchronization provided by A ensures that $t_2 - t_1 = t_5 - t_3 = t_6 - t_4 = t_7 - t_5 = t_9 - t_8 = 2\Delta x$. Note that the procedure used by A to set t and x relates the coordinates of events on the worldlines of B and C:

$$\begin{aligned}(t_4, x_4) &= (t_3, x_3) + (1,1)\Delta x \ , \quad (t_5, x_5) = (t_4, x_4) + (1,-1)\Delta x \ , \\ (t_6, x_6) &= (t_5, x_5) + (1,1)\Delta x \ , \quad (t_7, x_7) = (t_6, x_6) + (1,-1)\Delta x \ . \end{aligned} \tag{2}$$

Because they follow simply from the synchronization provided by A, these equations are exact; they do not require Δx to be small. However, by themselves they do not imply anything about the *physical* separations between the events. Testing this means measuring the metric.

To explore the metric, C checks his proper time and confirms B's observation that proper time differs from coordinate time. However, the metric coefficient he deduces, $g_{tt}(x_C, t)$, differs from B's. (The difference is first-order in Δx.)

The pair now wonder whether spatial coordinate intervals are similarly skewed relative to proper distance. They decide to measure the proper distance between them by using laser-ranging, the same way that A set their spatial coordinates in the first place. B sends a laser pulse at Event 3 which is reflected at Event 4 and received back at Event 5 in Figure 2. From this, she deduces the proper distance of C,

$$\Delta s = \frac{1}{2}(\tau_5 - \tau_3) \tag{3}$$

where τ_i is the reading of her atomic clock at event i. To her surprise, B finds that Δx does not measure proper distance, not even in the limit $\Delta x \to 0$. She defines another metric coefficient to convert coordinate distance to proper distance,

$$g_{xx} \equiv \lim_{\Delta x \to 0} \left(\frac{\Delta s}{\Delta x} \right)^2 \ . \tag{4}$$

The measurement of proper distance in equation (4) must be made at fixed t; otherwise the distance must be corrected for relative motion between B and C (should any exist). Fortunately, B can make this measurement at $t = t_4$ because that is when her laser pulse reaches C (see Fig. 2 and eqs. 2). Expanding $\tau_5 = \tau_B(t_4 + \Delta x)$ and $\tau_3 = \tau_B(t_4 - \Delta x)$ to first order in Δx using equations (1), (3), and (4), she finds

$$g_{xx}(x, t) = -g_{tt}(x, t) \ . \tag{5}$$

The observers repeat the experiment using Events 5, 6, and 7. They find that, while the metric may have changed, equation (5) still holds.

The observers are intrigued to find such a relation between the time and space parts of their metric, and they wonder whether this is a general phenomenon. Have they discovered a modification of special relativity, in which the Minkowski metric is simply multipled by a conformal factor, $g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$?

They decide to explore this question by measuring g_{tx}. A little thought shows that they cannot do this using pairs of events with either fixed x or fixed t. Fortunately, they have ideal pairs of events in the lightlike intervals between Events 3 and 4:

$$ds^2_{34} \equiv \lim_{\Delta t, \Delta x \to 0} g_{tt}(t_4 - t_3)^2 + 2g_{tx}(t_4 - t_3)(x_4 - x_3) + g_{xx}(x_4 - x_3)^2 \ . \tag{6}$$

Using equations (2) and (5) and the condition $ds = 0$ for a light ray, they conclude

$$g_{tx} = 0 \ . \tag{7}$$

Their space and time coordinates are orthogonal but on account of equations (5) and (7) all time and space intervals are stretched by $\sqrt{g_{xx}}$.

Our observers now begin to wonder if they have discovered a modification of special relativity, or perhaps they are seeing special relativity in a non-inertial frame. However, we know better. Unless the Riemann tensor vanishes identically, the metric they have determined cannot be transformed everywhere to the Minkowski form. Instead, what they have found is simply a consequence of how A fixed the coordinates. Fixing two coordinates means imposing two gauge conditions on the metric. A defined coordinates so as to make the problem look as much as possible like special relativity (eqs. 2). Equations (5) and (7) are the corresponding gauge conditions.

It is a special feature of 1+1 spacetime that the metric can always be reduced to a conformally flat one, i.e.

$$ds^2 = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

for some function $\Omega(x^\mu)$ called the conformal factor. In two dimensions the Riemann tensor has only one independent component and the Weyl tensor vanishes identically. Advanced GR and differential geometry texts show that spacetimes with vanishing Weyl tensor are conformally flat.

Thus, A has simply managed to assign conformally flat coordinates. This isn't a coincidence; by defining coordinate times and distances using null geodesics, he forced the metric to be identical to Minkowski up to an overall factor that has no effect on null lines. Equivalently, in two dimensions the metric has one physical degree of freedom, which has been reduced to the conformal factor $\Omega \equiv \sqrt{g_{xx}} = \sqrt{-g_{tt}}$.

This does not mean that A would have had such luck in more than two dimensions. In n dimensions the Riemann tensor has $n^2(n^2-1)/12$ independent components (Wald p. 54) and for $n \geq 3$ the Ricci tensor has $n(n+1)/2$ independent components. For $n = 2$ and $n = 3$ the Weyl tensor vanishes identically and spacetime is conformally flat. Not so for $n > 3$.

It would take a lot of effort to describe a complete synchronization in 3+1 spacetime using clocks and lasers. However, even without doing this we can be confident that the metric will not be conformally flat except for special spacetimes for which the Weyl tensor vanishes. Incidentally, in the weak-field limit conformally flat spacetimes have no deflection of light (can you explain why?). The solar deflection of light rules out conformally flat spacetime theories including ones proposed by Nordstrom and Weyl.

It is an interesting exercise to show how to transform an arbitrary metric of a 1+1 spacetime to the conformally flat form. The simplest way is to compute the Ricci scalar. For the metric of equation (8), one finds

$$R = \Omega^{-2}(\partial_t^2 - \partial_x^2)\ln\Omega^2 \ . \tag{9}$$

Starting from a 1+1 metric in a different form, one can compute R everywhere in spacetime. Equation (9) is then a nonlinear wave equation for $\Omega(t,x)$ with source $R(t,x)$. It can be solved subject to initial Cauchy data on a spacelike hypersurface on which $\Omega = 1$, $\partial_t\Omega = \partial_x\Omega = 0$ (corresponding to locally flat coordinates).

We have exhausted the analysis of 1+1 spacetime. Our observers have discerned one possible contradiction with special relativity: clocks run at different rates in different places (and perhaps at different times). If equation (9) gives Ricci scalar $R = 0$ everywhere with $\Omega = \sqrt{-g_{tt}}$, then the spacetime is really flat and we must be seeing the effects of accelerated motion in special relativity. If $R \neq 0$, then the variation of clocks is an entirely new phenomenon, which we call gravitational redshift.

3 The metric for an accelerated observer

It is informative to examine the problem from another perspective by working out the metric that an arbitrarily accelerating observer in a flat spacetime would deduce using the synchronization procedure of Section 2. We can then more clearly distinguish the effects of curvature (gravity) and acceleration.

Figure 3 shows the situation prevailing in special relativity when observer A has an arbitrary timelike trajectory $x_A^\mu(\tau_A)$ where τ_A is the proper time measured by his atomic clock. While A's worldline is erratic, those of light signals are not, because here $t = x^0$ and $x = x^1$ are flat coordinates in Minkowski spacetime. Given an arbitrary worldline $x_A^\mu(\tau_A)$, how can we possibly find the worldines of observers at fixed coordinate displacement as in the preceding section?

The answer is the same as the answer to practically all questions of measurement in GR: use the metric! The metric of flat spacetime is the Minkowski metric, so the paths of laser pulses are very simple. We simply solve an algebra problem enforcing that Events 1 and 2 are separated by a null geodesic (a straight line in Minkowski spacetime) and likewise for Events 2 and 3, as shown in Figure 3. Notice that the lengths (i.e. coordinate differences) of the two null rays need not be the same.

The coordinates of Events 1 and 3 are simply the coordinates along A's worldine, while those for Event 2 are to be determined in terms of A's coordinates. As in Section 2, A defines the spatial coordinate of B to be twice the round-trip light-travel time. Thus, if event 0 has $x^0 = t_A(\tau_0)$, then Event 3 has $x^0 = t_A(\tau_0 + 2L)$. For convenience we will

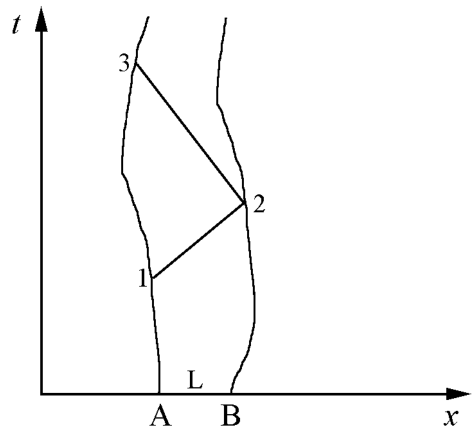

Figure 3: An accelerating observer sets up a coordinate system with an atomic clock, laser and detector.

set $\tau_0 \equiv \tau_A - L$. Then, according to the prescription of Section 2, A will assign to Event 2 the coordinates (τ_A, L). The coordinates in our flat Minkowksi spacetime are

$$\begin{aligned}
\text{Event 1:} \quad x^0 &= t_A(\tau_A - L) \ , \quad x^1 = x_A(\tau_A - L) \ , \\
\text{Event 2:} \quad x^0 &= t(\tau_A, L) \ , \quad x^1 = x(\tau_A, L) \ , \\
\text{Event 3:} \quad x^0 &= t_A(\tau_A + L) \ , \quad x^1 = x_A(\tau_A + L) \ . \qquad (10)
\end{aligned}$$

Note that the argument τ_A for Event 2 is not an affine parameter along B's wordline; it is the clock time sent to B by A. A second argument L is given so that we can look at a family of worldlines with different L. A is setting up coordinates by finding the spacetime paths corresponding to the coordinate lines $L =$ constant and $\tau_A =$ constant. We are performing a coordinate transformation from (t, x) to (τ_A, L).

Requiring that Events 1 and 2 be joined by a null geodesic in flat spacetime gives the condition $x_2^\mu - x_1^\mu = (C_1, C_1)$ for some constant C_1. The same condition for Events 2 and 3 gives $x_3^\mu - x_2^\mu = (C_2, -C_2)$ (with a minus sign because the light ray travels toward decreasing x). These conditions give four equations for the four unknowns C_1, C_2, $t(\tau_A, L)$, and $x(\tau_A, L)$. Solving them gives the coordinate transformation between (τ_A, L) and the Minkowski coordinates:

$$\begin{aligned}
t(\tau_A, L) &= \frac{1}{2}\left[t_A(\tau_A + L) + t_A(\tau_A - L) + x_A(\tau_A + L) - x_A(\tau_A - L)\right] \ , \\
x(\tau_A, L) &= \frac{1}{2}\left[x_A(\tau_A + L) + x_A(\tau_A - L) + t_A(\tau_A + L) - t_A(\tau_A - L)\right] \ . \qquad (11)
\end{aligned}$$

Note that these results are exact; they do not assume that L is small nor do they restrict A's worldline in any way except that it must be timelike. The student may easily evaluate C_1 and C_2 and show that they are not equal unless $x_A(\tau_A + L) = x_A(\tau_A - L)$.

Using equations (11), we may transform the Minkowski metric to get the metric in the coordinates A has set with his clock and laser, (τ_A, L):

$$ds^2 = -dt^2 + dx^2 = g_{tt} d\tau_A^2 + 2g_{tx} d\tau_A dL + g_{xx} dL^2 \ . \tag{12}$$

Substituting equations (11) gives the metric components in terms of A's four-velocity components,

$$-g_{tt} = g_{xx} = \left[V_A^t(\tau_A + L) + V_A^x(\tau_A + L)\right]\left[V_A^t(\tau_A - L) - V_A^x(\tau_A - L)\right] \ , \quad g_{tx} = 0 \ . \tag{13}$$

This is precisely in the form of equation (8), as it must be because of the way in which A coordinatized spacetime.

It is straightforward to work out the Riemann tensor from equation (13). Not surprisingly, it vanishes identically. Thus, an observer *can* tell, through measurements, whether he or she lives in a flat or nonflat spacetime. The metric is measurable.

Now that we have a general result, it is worth simplifying to the case of an observer with constant acceleration g_A in Minkowski spacetime. Problem 3 of Problem Set 1 showed that one can write the trajectory of such an observer (up to the addition of constants) as $x = g_A^{-1} \cosh g_A \tau_A$, $t = g_A^{-1} \sinh g_A \tau_A$. Equation (13) then gives

$$ds^2 = e^{2g_A L}\left(-d\tau_A^2 + dL^2\right) \ . \tag{14}$$

One word of caution is in order about the interpretation of equation (14). Our derivation assumed that the acceleration g_A is constant for observer A at $L = 0$. However, this does not mean that other observers (at fixed, nonzero L) have the same acceleration. To see this, we can differentiate equations (11) to derive the 4-velocity of observer B at (τ_A, L) and the relation between coordinate time τ_A and proper time along B's worldline, with the result

$$V_B^\mu(\tau_A, L) = (\cosh g_A \tau_A, \sinh g_A \tau_A) = (\cosh g_B \tau_B, \sinh g_B \tau_B) \ , \quad \frac{d\tau_B}{d\tau_A} = \frac{g_A}{g_B} = e^{gL} \ . \tag{15}$$

The four-acceleration of B follows from $a_B^\mu = dV_B^\mu / d\tau_B = e^{-gL} dV^\mu / d\tau_A$ and its magnitude is therefore $g_B = g_A e^{-gL}$. The proper acceleration varies with L precisely so that the proper distance between observers A and B, measured at constant τ_A, remains constant.

4 Gravity versus acceleration in 1+1 spacetime

Equation (14) gives one form of the metric for a flat spacetime as seen by an accelerating observer. There are many other forms, and it is worth noting some of them in order to gain some intuition about the effects of acceleration. For simplicity, we will restrict our discussion here to static spacetimes, i.e. metrics with $g_{0i} = 0$ and $\partial_t g_{\mu\nu} = 0$. In 1+1 spacetime this means the line element may be written

$$ds^2 = -e^{2\phi(x)}dt^2 + e^{-2\psi(x)}dx^2 \ . \tag{16}$$

(The metric may be further transformed to the conformally flat form, eq. 8, but we leave it in this form because of its similarity to the form often used in $3 + 1$ spacetime.)

Given the metric (16), we would like to know when the spacetime is flat. If it is flat, we would like the explicit coordinate transformation to Minkowski. Both of these are straightforward in 1+1 spacetime. (One might hope for them also to be straightforward in more dimensions, at least in principle, but the algebra rapidly increases.)

The definitive test for flatness is given by the Riemann tensor. Because the Weyl tensor vanishes in 1+1 spacetime, it is enough to examine the Ricci tensor. With equation (16), the Ricci tensor has nonvanishing components

$$R_{tt} = e^{\phi+\psi}\,\frac{d\tilde{g}}{dx} \ , \quad R_{xx} = -e^{-(\phi+\psi)}\frac{d\tilde{g}}{dx} \quad \text{where} \quad \tilde{g}(x) = e^{\phi}\,g(x) = e^{\phi+\psi}\,\frac{d\phi}{dx} \ . \tag{17}$$

The function $g(x)$ is the proper acceleration along the x-coordinate line, along which the tangent vector (4-velocity) is $V^\mu_x = e^{-\phi}(1,0)$. This follows from computing the 4-acceleration with equation (16) using the covariant prescription $a^\mu(x) = \nabla_V V^\mu = V^\nu_x \nabla_\nu V^\mu_x$. The magnitude of the acceleration is then $g(x) \equiv (g_{\mu\nu}a^\mu a^\nu)^{1/2}$, yielding $g(x) = e^\psi d\phi/dx$. The factor e^ψ converts $d\phi/dx$ to $g(x) = d\phi/dl$ where $dl = \sqrt{g_{xx}}\,dx$ measures proper distance.

A stationary observer, i.e. one who remains at fixed spatial coordinate x, feels a time-independent effective gravity $g(x)$. Nongravitational forces (e.g. a rocket, or the contact force from a surface holding the observer up) are required to maintain the observer at fixed x. The gravity field $g(x)$ can be measured very simply by releasing a test particle from rest and measuring its acceleration relative to the stationary observer. For example, we measure g on the Earth by dropping masses and measuring their acceleration in the lab frame.

We will see following equation (18) below why the function $\tilde{g}(x) = (d\tau/dt)g(x)$ rather than $g(x)$ determines curvature. For now, we simply note that equation (17) implies that spacetime curvature is given (for a static 1+1 metric) by the gradient of the gravitational redshift factor $\sqrt{-g_{tt}} = e^\phi$ rather than by the "gravity" (i.e. acceleration) gradient dg/dx.

In linearized gravitation, $g = \tilde{g}$ and so we deduced (in the notes *Gravitation in the Weak-Field Limit*) that a spatially uniform gravitational (gravitoelectric) field can be transformed away by making a global coordinate transformation to an accelerating frame. For strong fields, $g \neq \tilde{g}$ and it is no longer true that a uniform gravitoelectric field can be transformed away. Only if the gravitational redshift factor $e^{\phi(x)}$ varies linearly with proper distance, i.e. $\tilde{g} \equiv d(e^{\phi})/dl$ is a constant, is spacetime is flat, enabling one to transform coordinates so as to remove all evidence for acceleration. If, on the other hand, $d\tilde{g}/dx \neq 0$ — even if $dg/dx = 0$ — then the spacetime is not flat and no coordinate transformation can transform the metric to the Minkowski form.

Suppose we have a line element for which $\tilde{g}(x) =$ constant. We know that such a spacetime is flat, because the Ricci tensor (hence Riemann tensor, in 1+1 spacetime) vanishes everywhere. What is the coordinate transformation to Minkowski?

The transformation may be found by writing the metric as $g = \Lambda^T \eta \Lambda$ where $\Lambda^{\bar{\mu}}{}_{\nu} = \partial \bar{x}^{\bar{\mu}}/\partial x^{\nu}$ is the Jacobian matrix for the transformation $\bar{x}(x)$. (Note that here g is the matrix with entries $g_{\mu\nu}$ and not the gravitational acceleration!) By writing $\bar{t} = \bar{t}(t,x)$ and $\bar{x} = \bar{x}(t,x)$, substituting into $g = \Lambda^T \eta \Lambda$, using equation (16) and imposing the integrability conditions $\partial^2 \bar{t}/\partial t \partial x = \partial^2 \bar{t}/\partial x \partial t$ and $\partial^2 \bar{x}/\partial t \partial x = \partial^2 \bar{x}/\partial x \partial t$, one finds

$$\bar{t}(t,x) = \frac{1}{g} \sinh \tilde{g}\, t \ , \quad \bar{x}(t,x) = \frac{1}{g} \cosh \tilde{g}\, t \quad \text{if} \quad \frac{d\tilde{g}}{dx} = 0 \ , \tag{18}$$

up to the addition of irrelevant constants. We recognize this result as the trajectory in flat spacetime of a constantly accelerating observer.

Equation (18) is easy to understand in light of the discussion following equation (14). The proper time τ for the stationary observer at x is related to coordinate time t by $d\tau = \sqrt{-g_{tt}(x)}\, dt = e^{\phi} dt$. Thus, $g(x)\tau = e^{\phi}\, g\, t = \tilde{g}\, t$ or, in the notation of equation (15), $g_B \tau_B = g_A \tau_A$ (since τ_A was used there as the global t-coordinate). The condition $e^{\phi}\, g = \tilde{g}(x) =$ constant amounts to requiring that all observers be able to scale their gravitational accelerations to a common value for the observer at $\phi(x) = 0$, $\tilde{g}$. If they cannot (i.e. if $d\tilde{g}/dx \neq 0$), then the metric is not equivalent to Minkowski spacetime seen in the eyes of an accelerating observer.

With equations (16)–(18) in hand, we can write the metric of a flat spacetime in several new ways, with various spatial dependence for the acceleration of our coordinate observers:

$$\begin{aligned} ds^2 &= e^{2\tilde{g}\,x}(-dt^2 + dx^2) \ , \quad g(x) = \tilde{g}\, e^{-\tilde{g}\,x} && (19) \\ &= -\tilde{g}^2 (x - x_0)^2 dt^2 + dx^2 \ , \quad g(x) = \frac{1}{x - x_0} && (20) \\ &= -[2\tilde{g}(x - x_0)] dt^2 + [2\tilde{g}(x - x_0)]^{-1} dx^2 \ , \quad g(x) = \sqrt{\frac{\tilde{g}}{2(x - x_0)}} \ . && (21) \end{aligned}$$

The first form was already given above in equation (14). The second and third forms are peculiar in that there is a coordinate singularity at $x = x_0$; these coordinates only work for $x > x_0$. This singularity is very similar to the one occuring in the Schwarzschild line element. Using the experience we have obtained here, we will remove the Schwarzschild singularity at $r = 2GM$ by performing a coordinate transformation similar to those used here. The student may find it instructive to write down the coordinate transformations for these cases using equation (18) and drawing the (t, x) coordinate lines on top of the Minkowski coordinates $(\bar{t}, \bar{x})$. While the singularity at $x = x_0$ can be transformed away, it does signal the existence of an event horizon. Equation (20) is called Rindler spacetime. Its event horizon is discussed briefly in Schutz (p. 150) and in more detail by Wald (pp. 149–152).

Actually, equation (21) is closer to the Schwarzschild line element. Indeed, it becomes the r-t part of the Schwarzschild line element with the substitutions $x \to r$, $-2\tilde{g}x_0 \to 1$ and $\tilde{g} \to -GM/r^2$. These identifications show that the Schwarzschild spacetime differs from Minkowski in that the acceleration needed to remain stationary is radially directed and falls off as $e^{-\phi}\, r^{-2}$. We can understand many of its features through this identification of gravity and acceleration.

For completeness, I list three more useful forms for a flat spacetime line element:

$$ds^2 = -dt^2 + \tilde{g}^2(t-t_0)^2 dx^2 \ , \quad g(x) = 0 \tag{22}$$
$$= -dU dV \tag{23}$$
$$= -e^{v-u} du dv \ . \tag{24}$$

The first is similar to Rindler spacetime but with t and x exchanged. The result is suprising at first: the acceleration of a stationary observer vanishes. Equation (22) has the form of Gaussian normal or synchronous coordinates (Wald, p. 42). It represents the coordinate frame of a freely-falling observer. It is interesting to ask why, if the observer is freely-falling, the line element does not reduce to Minkowski despite the fact that this spacetime is flat. The answer is that different observers (i.e., worldlines of different x) are in uniform motion relative to one another. In other words, equation (22) is Minkowski spacetime in expanding coordinates. It is very similar to the Robertson-Walker spacetime, which reduces to it (short of two spatial dimensions) when the mass density is much less than the critical density.

Equations (23) and (24) are Minkowski spacetime in null (or light-cone) coordinates. For example, $U = \bar{t} - \bar{x}$, $V = \bar{t} + \bar{x}$. These coordinates are useful for studying horizons.

Having derived many results in $1 + 1$ spacetime, I close with the cautionary remark that in $2 + 1$ and $3 + 1$ spacetime, there are additional degrees of freedom in the metric that are quite unlike Newtonian gravity and cannot be removed (even locally) by transformation to a linearly accelerating frame. Nonetheless, it should be possible to extend the treatment of these notes to account for these effects — gravitomagnetism and gravitational radiation. As shown in the notes *Gravitation in the Weak-Field Limit*, a uniform gravitomagnetic field is equivalent to uniformly rotating coordinates. Gravitational radiation is different; there is no such thing as a spatially uniform gravitational wave. However, one can always choose coordinates so that gravitational radiation strain s_{ij} and its first derivatives vanish at a point.