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Date: 6/10/92
Subject: Forecast errors for aggregate demand

Consider a single product with stationary, stochastic demand, X_t. Then

$$X_t = \mu + e_t$$

where μ is the mean demand. Assume no bias so

$$E(e_t) = 0,$$

$$E(X_t) = \mu.$$

If we also assume constant variance, σ^2;

$$Var(X_t) = Var(e_t) = \sigma^2$$

Now the absolute error is $|e_t| = |D_t - F_t|$ with D_t the actual demand and F_t the forecasted demand. A measure of the "goodness" of our forecasting system over all observations is provided by the mean absolute deviation:

$$MAD = \sum_t |e_t| / N,$$

with N the number of periods of historical data.

· The standard deviation of the distribution of errors, σ, is related to MAD:

$$\sigma \cong k \cdot MAD$$

where k is a constant (1.25 for normally distributed errors).

While MAD is certainly a reasonable measure it does not convey a great deal of information. An MAD of say 20 for a product is one thing if the mean demand is 1000 and quite another if the mean demand is 50.

So a better measure of the "goodness" of a forecasting system would be a relative measure. Lets call it the mean absolute error relative to the mean demand, or for short, the mean proportional error

$$MPE = \frac{MAD}{\mu} = \frac{k^{-1}\sigma}{\mu}$$

Now suppose we have many products each with their unique set of μ_i and σ_i. We would like to form an aggregate product as the (unweighted) sum of all of the individual products. Then the mean demand of the aggregate products is

$$\mu_A = \sum_{i=1}^{p} \mu_i \quad , \qquad p = \# \text{ of products}.$$

If the demands of the p products are statistically independent (a rather stringent assumption, but at least we can proceed), the aggregate variance is then the sum of the individual variances:

$$\sigma_A^2 = \sum_{i=1}^{p} \sigma_i^2$$

The aggregate MPE is then

$$MPE_A = \frac{k^{-1}\sqrt{\sum_{i=1}^{p} \sigma_i^2}}{\sum_{i=1}^{p} \mu_i} = \frac{\cancel{k^{-1}}\,\cancel{k}\sqrt{\sum_{i=1}^{p} MAD_i^2}}{\sum_{i=1}^{p} \mu_i}$$

where we have assumed all individuals products had similar error distributions, permitting k to be factored.

The mean proportional error for the aggregated products will be smaller than that of any individual product. To see this, we will look at a simple example. Suppose we have p products all with ~~constant~~ the same mean μ and the same standard deviation, σ.

$$\mu_A = p\mu$$

$$\sigma_A^2 = p\sigma^2$$

$$MPE_A = \frac{k\sqrt{p}\,\sigma}{p\,\mu} = \frac{k\sigma}{\sqrt{p}\,\mu}$$

Form the ratio

$$\boxed{\frac{MPE_A}{MPE} = \frac{\frac{k\sigma}{\sqrt{p}\,\mu}}{\frac{k\sigma}{\mu}} = \frac{1}{\sqrt{p}}} \qquad MPE_A = \frac{MPE}{\sqrt{p}}$$

To be even more concrete, suppose we have 5 products, ~~and~~ that all ~~ee~~ errors are ~~normally~~ similarly distributed and

$$\mu_1 = \mu_2 = \cdots = \mu_5 = 100$$

$$\cancel{\sigma_1 = \sigma_2 = \cdots = \sigma_5 = 30}$$

$$MAD_1 = MAD_2 = \cdots = MAD_5 = 30$$

$$\cancel{\sigma_1 = \sigma_2 = \cdots = \sigma_5 = k(30)}$$

The relative errors for each product are equal

$MPE_i = .30$,

i.e. forecast error is on average 30% of the mean demand.
Now lets look at the aggregate relative error.

$$\mu_A = 5(100) = 500$$

$$\sigma_A^2 = 5\sigma^2 = 5(k)^2(30)^2$$

$$\sigma_A = 30k\sqrt{5}$$

$$MPE_A = \frac{30\cancel{k}\sqrt{5}}{5(100)\cancel{k}} = \frac{1}{\sqrt{5}}(.30)$$

$$MPE_A = .134 ,$$

i.e. aggregate forecast error for 5 products is on 13.4% of the average aggregate demand.