Forrester Simulation

A. Demand

DEM(K-1, K) = units of demand during time period [K-1, K]. RATE - u/time

Now demand drives both warehouse shipping and production at the factory.

B. Factory Warehouse Distribution

1. INVENTORY

IAD(K) = Inventory Actual at Warehouse. LEVEL - u

SSD(K-1, K) = Shipment of units to customers during time period [K-1, K]. RATE - u/time

SRF(K-1, K) = Shipments received from factory

Then

$$IAD(K) = IAD(K-1) + DT\,[SRF(K-1,K) - SSD(K-1,K)] \quad (1)$$

where DT is the increment of time.

2. UNFILLED ORDERS

$UOD(K)$ = Unfilled orders at warehouse, LEVEL - u

$RRD(K-1,K)$ = Requisition received at warehouse distribution center during $[K-1, K]$

RATE - u/time

Then

$$UOD(K) = UOD(K-1) + DT\,[RRD(K-1,K) - SSD(K-1,K)] \quad (2)$$

We will look at RRD later.

3. SHIPPING RATE

Now distribution would like to fill orders at a rate proportional to the unfilled orders. However, inventory may limit the order filling rate.

Define

$STD(K)$ = Target shipping rate at Distributor

$$STD(K) = \frac{UOD(K)}{DFD(K)} \quad (3)$$

where

$DFD(K)$ = Delay (in time units) in filling orders at the distribution center.

The delay must depend on the inventory status.
Forrester takes

$$\boxed{DFD(K) = DHD + DUD\,\frac{(IDD(K))}{(IAD(K))}} \qquad (4)$$

where

DHD = Handling Delay

DUD = Delay in filling orders caused by out-of-stock when inventory is <u>normal</u>.

IDD(K) = Inventory level desired at distributor.

$$\boxed{IDD(K) = (AID)\,RSD(K)} \qquad (5)$$

with

AID = number of weeks of smoothed demand desired for inventory.

RSD(K) = Requistions Smoothed at Distributor.

$$\boxed{RSD(K) = RSD(K-1) + DT\left(\frac{1}{DRD}\right)\left[RRD(K-1,K) - RSD(K-1)\right]} \qquad (6)$$

where RRD(K-1, K) = Requistions received at distributor.

DRD = Delay time constant for smoothing reqs.

$$\boxed{RRD(K,K+1) = DELAY3\left(DEM(K-1,K),\ DOP\right)} \qquad (7)$$

where DEM is the given demand rate and DELAY3 is a 3rd order exponential delay operation which we will look at in detail later.

DOP = Delay in Order Processing at Customer Service

Next define the shipping rate which will exhaust the available inventory over the next time interval, DT.

$$\boxed{NID(K) = \frac{IAD(K)}{DT}} \qquad (8)$$

Now, finally, we can determine the shipping rate at the distributor

$$\boxed{SSD(K,K+1) = \begin{cases} STD(K), & \text{if } NID(K) \geq STD(K) \\ NID(K), & \text{if } NID(K) < STD(K) \end{cases}} \qquad (9)$$

4. CUSTOMER RECEIPT RATE

Define

SRR(K, K+1) = Shipments Received at Retail in period [K, K+1]. RATE – u/time

$$\boxed{SRR(K,K+1) = DELAY3(SSD(K-1,K), DTR)} \quad (10)$$

with

DTR = Delay in Transportation to Retail.

At this stage, we have governing relations for all variables except the Shipments Received from Factory, SRF, i.e the manufacturing output.

C. Factory Production

Define the Smoothed Demand as

$$\boxed{SDM(K) = SDM(K-1) + DT\left(\frac{1}{DDM}\right)\left(DEM(K-1,K) - SDM(K-1)\right)} \quad (11)$$

DDM = Delay constant for Demand Smoothing.

Let the Production rate Decision at Factory be

$$\boxed{PDF(K,K+1) = SDM(K) + \frac{1}{DID}\left[(IDD(K) - IAD(K)) + (UOD(K) - UND(K))\right]} \quad (12)$$

DID = Delay in Inventory Adjustment at Distributor.

Finally, we can close the loop by relating the Shipments Received from the Factory to the Production Decision at the Factory through a 3rd order delay:

$$\boxed{SRF(K,K+1) = DELAY3(PDF(K-1,K), DPF)} \quad (13)$$

with

DPF = Delay due to Production lead time at Factory.

This completes the simulation model. Next we consider initial conditions.

D. Initial Conditions

$DEM(-1,0) = 1000$; $DEM(0,1) = 1100$

$SDM(0) = 1000$

$RSD(0) = RRD(-1,0) = DEM(-1,0) = 1000$

$IAD(0) = IDD(0) = AID\,(RSD(0))$

$UOD(0) = RSD(0)\,(DHD + DUD) = UDD(0)$

$DFD(0) = DHD + DUD$

E. Constants (weeks)

DT = .05

DHD = 1.0

DUD = 0.6

AID = 6

DRD = 8

DOP = 1

DTR = 1

DDM = 8

DID = 4

F. Delays

1. First Order Exponential Delay

Let τ = delay in time units – known constant

$$\tau \frac{dx}{dt} + x = f(t) \qquad (*)$$

If $f(t)$ is the input, the output response is $x(t)$. Now with this relationship between $x(t)$ and $f(t)$, the response cannot immediately follow changes in $f(t)$.

Suppose

$$f(t) = 0 \quad , \quad t < 0$$

$$f(t) = F_o \quad , \quad t \geq 0.$$

The solution to (*) for general $f(t)$ is:

$$x = C_1 e^{-t/\tau} + \frac{1}{\tau} e^{-t/\tau} \int f(t) e^{t/\tau} dt$$

For the special case above and with $f(t) = F_o$, $t \geq 0$

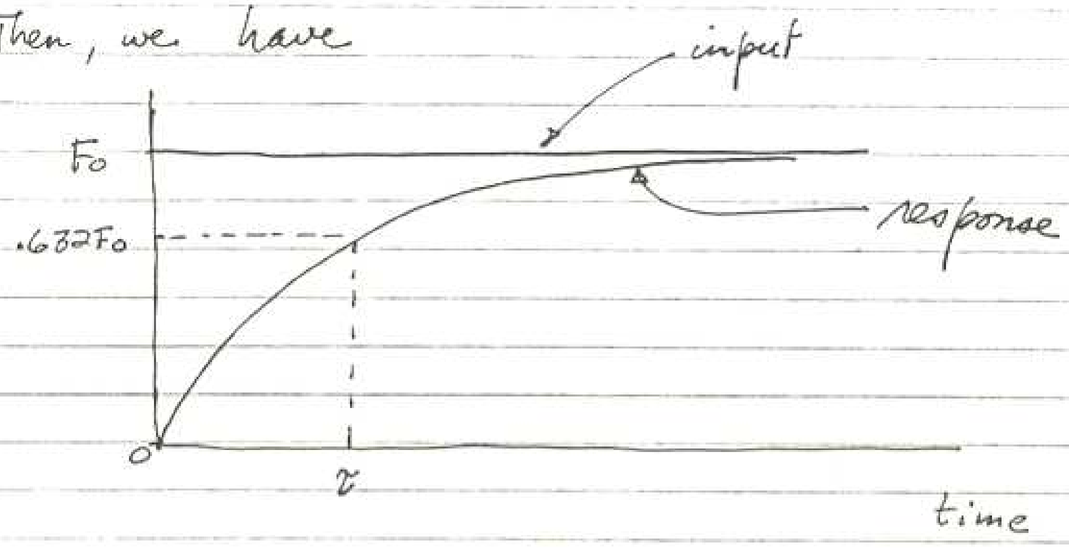
$$x(t) = C_1 e^{-t/\tau} + F_o$$

Applying the initial condition: $x(0) = 0$, gives

$$C_1 = -F_o$$

$$x(t) = F_o \left[1 - e^{-t/\tau} \right]$$

Then, we have



when $t = \tau$, (one "time constant")

$$x(\tau) = F_0 \left(1 - e^{-1}\right) = .632\ F_0$$

Note that the slope of $x(t)$ at $t = 0$ is positive indicating immediate response, but lagged.

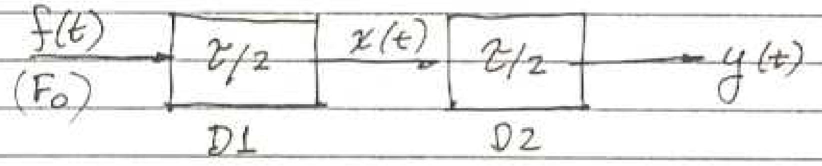
$$\frac{dx}{dt}(t) = \frac{F_0}{\tau} e^{-t/\tau}$$

so that at $t = 0$

$$\frac{dx}{dt}(0) = \frac{F_0}{\tau}$$

2. Second Order Delay

Here the output of the first delay is the input to the second delay. The delay time is split between the two.



$$\left(\frac{\tau}{2}\right)\frac{dy}{dt} + y = x(t) = F_0\left(1 - e^{-2t/\tau}\right)$$

Let $\hat{\tau} = \tau/2$. Then

$$y(t) = C_2 e^{-t/\hat{\tau}} + \frac{1}{\hat{\tau}} e^{-t/\hat{\tau}} \int F_0\left(1 - e^{-t/\hat{\tau}}\right) e^{t/\hat{\tau}}\, dt$$

$$\int F_0\left(1 - e^{-t/\hat{\tau}}\right) e^{t/\hat{\tau}}\, dt = F_0 \int \left(e^{t/\hat{\tau}} - 1\right) dt$$

$$= F_0\left[\hat{\tau} e^{t/\hat{\tau}} - t\right]$$

Then

$$y(t) = C_2 e^{-t/\hat{\tau}} + \frac{1}{\hat{\tau}} e^{-t/\hat{\tau}} F_0\left(\hat{\tau} e^{t/\hat{\tau}} - t\right)$$

$$y(t) = C_2 e^{-t/\hat{\tau}} + F_0 - \frac{F_0}{\hat{\tau}}\, t\, e^{-t/\hat{\tau}}$$

Require

$$y(0) = 0$$

$$C_2 = -F_0$$

$$y(t) = F_o\left(1 - e^{-t/\hat{\tau}} - \frac{t}{\hat{\tau}} e^{-t/\hat{\tau}}\right)$$

$$y(t) = F_o\left[1 - e^{-t/\hat{\tau}}(1 + t/\hat{\tau})\right]$$

Let's check $t = \hat{\tau}$. Note this is not the same as for D1.

$$y(\hat{\tau}) = F_o\,[1 - .736] = .264\,F_o$$

This rises less rapidly. Check slope:

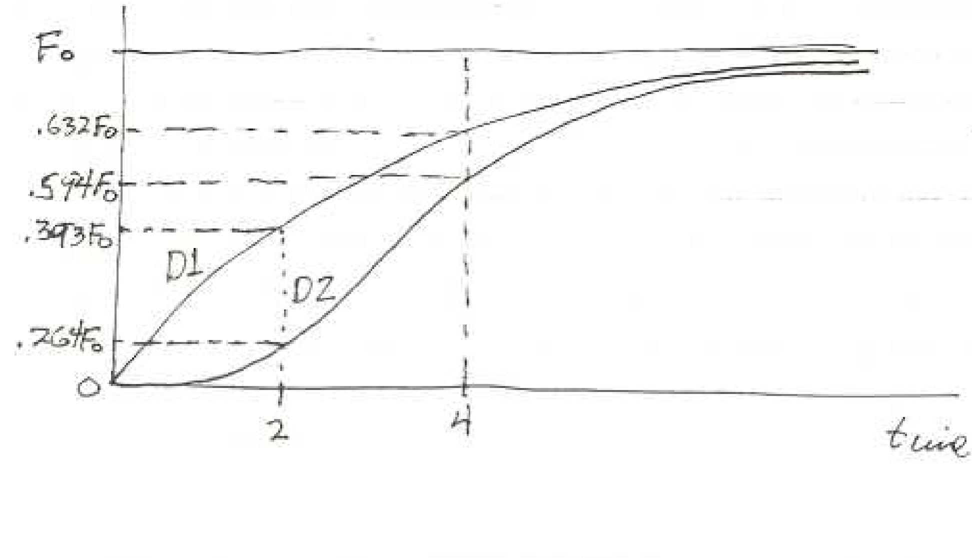
$$\hat{\tau}\frac{dy}{dt} = F_o(\cancel{1} - e^{-t/\hat{\tau}}) - F_o[\cancel{1} - e^{-t/\hat{\tau}}(1 + t/\hat{\tau})]$$

$$\hat{\tau}\frac{dy}{dt} = -F_o e^{-t/\hat{\tau}} + F_o e^{-t/\hat{\tau}}(\cancel{1} + t/\hat{\tau})$$

$$\frac{dy}{dt} = F_o \frac{t}{\hat{\tau}} e^{-t/\hat{\tau}}$$

So that

$$\frac{dy}{dt}(0) = 0$$



Lets compare DELAY1 and DELAY2 at the same time.

Suppose: $\tau = 4$

Then for DELAY1 at $t = \tau = 4$

$$x(t=4) = \underline{.632\ F_0}$$

For DELAY2, $\hat{\tau} = \tau/2 = 2$, so

$$y(t=4) = F_0\left[1 - e^{-4/2}(1 + 4/2)\right]$$

$$y(t=4) = F_0\,(1 - .406)$$

$$y(t=4) = \underline{.594\,F_0}$$

Lets check $t = 2$. For D1

$$x(t=2) = F_0\left[1 - e^{-.5}\right]$$

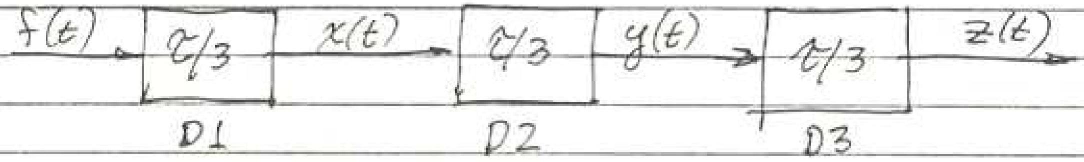
$$x(t=2) = \underline{.393\ F_0}$$

For D2:

$$y(t=2) = F_0\left[1 - e^{-1}(1+1)\right]$$

$$y(t=2) = \underline{.264\,F_0}$$

3. Third Order Delay



$$f(t) = F_0$$

Let $\hat{\tau} = \tau/3$

$$\hat{\tau}\frac{dz}{dt} + z = y(t) = F_0\left[1 - e^{-t/\hat{\tau}}(1 + t/\hat{\tau})\right]$$

Then

$$z(t) = C_3 e^{-t/\hat{\tau}} + \frac{1}{\hat{\tau}} e^{-t/\hat{\tau}} \int F_0\left[1 - e^{-t/\hat{\tau}}(1+t/\hat{\tau})\right] e^{t/\hat{\tau}} dt$$

$$\int F_0[\ ]e^{t/\hat{\tau}} dt = F_0 \int \left[e^{t/\hat{\tau}} - 1 - t/\hat{\tau}\right] dt$$

$$= F_0\left[\hat{\tau} e^{t/\hat{\tau}} - t - \frac{t^2}{2\hat{\tau}}\right]$$

$$z(t) = C_3 e^{-t/\hat{\tau}} + \frac{1}{\hat{\tau}} e^{-t/\hat{\tau}} F_0\left[\hat{\tau} e^{t/\hat{\tau}} - t - \frac{t^2}{2\hat{\tau}}\right]$$

$$z(t) = C_3 e^{-t/\hat{\tau}} + F_0\left[1 - e^{-t/\hat{\tau}}\left(\frac{t}{\hat{\tau}} + \frac{t^2}{2\hat{\tau}^2}\right)\right]$$

$$z(0) = 0 \Rightarrow C_3 = -F_0$$

$$z(t) = -F_0 e^{-t/\hat{\tau}} + F_0 - F_0 e^{-t/\hat{\tau}} \frac{t}{\hat{\tau}}\left(1 + \frac{t}{2\hat{\tau}}\right)$$

$$\boxed{z(t) = F_0\left\{1 - e^{-t/\hat{\tau}}\left[1 + \frac{t}{\hat{\tau}}\left(1 + \frac{t}{2\hat{\tau}}\right)\right]\right\}}$$

Now

$$\hat{\tau}\frac{dz}{dt} = y(t) - z(t) = F_0\left[\cancel{1} - e^{-t/\hat{\tau}}\cancel{\left(1 + t/\hat{\tau}\right)}\right]$$

$$- F_0\left\{\cancel{1} - e^{-t/\hat{\tau}}\left[\cancel{1} + \frac{t}{\hat{\tau}}\left(\cancel{1} + \frac{t}{2\hat{\tau}}\right)\right]\right\}$$

$$\tau\frac{dz}{dt} = F_0\, e^{-t/\hat{\tau}}\, \frac{t^2}{2\hat{\tau}^2}$$

So that both $\dot{z}(0) = \ddot{z}(0) = 0$.

We now have an analytical means of computing delays for the case of a <u>step-input</u>.

4. Finite - Difference Delays.

Let

DEL = Total time Delay. (Same as τ)

FIN(K-1,K) = Function input to the Delay

X1(K,K+1) = Output of 1st Delay.

Then for D1

$$X1(K,K+1) = X1(K-1,K) + \frac{DT}{DEL/3}\left(FIN(K-1,K) - X1(K-1,K)\right)$$

For D2

$$X2(K,K+1) = X2(K-1,K) + \frac{DT}{DEL/3}\left(X1(K-1,K) - X2(K-1,K)\right)$$

For D3

$$\boxed{OUT(K,K+1) = OUT(K-1,K) + \frac{DT}{DEL/3}\left(X2(K-1,K) - OUT(K-1,K)\right)}$$

Suppose

$DEL = \tau = 9$ time units

$DT = .05$ time units

$DEL/3 = \hat{\tau} = 3$ time units

At $t=0$ we have a step change, F_o. Then lets examine the response using both analytical expression and the finite difference method at some time t, say

$t = .3$ time units

Analytical

$$t/\hat{\tau} = .3/3 = .1$$

$$OUT(t=.3) = F_o\left\{1 - e^{-.1}\left[1 + .1\left(1 + .05\right)\right]\right\}$$

$$OUT(t=.3) = \underline{F_o\,(.0001546)}$$

Now, I am uncertain as to the exact timing of FIN. Lets take

$$FIN(-1,0) = 0 \qquad X1(-1,0) = 0 \qquad OUT(-1,0) = 0$$

$$FIN(0,1) = F_o \qquad X2(-1,0) = 0$$

Then $K=0$:
$(t=0)$

$$X1(0,1) = \overset{0}{X1(-1,0)} + \frac{.05}{3}\left(\overset{0}{FIN(-1,0)} - \overset{0}{X1(-1,0)}\right)$$

$$X1(0,1) = 0$$

$$X2(0,1) = \overset{0}{X2(-1,0)} + \frac{.05}{3}\left(\overset{0}{X1(-1,0)} - \overset{0}{X2(-1,0)}\right)$$

$$X2(0,1) = 0$$

$$OUT(0,1) = \overset{0}{OUT(-1,0)} + \frac{.05}{3}\left(\overset{0}{X2(-1,0)} - \overset{0}{OUT(-1,0)}\right)$$

$$\underline{OUT(0,1) = 0}$$

$K=1$
$(t=.05)$

$$X1(1,2) = \overset{0}{X1(0,1)} + \frac{.05}{3}\left(\overset{F_o}{FIN(0,1)} - \overset{0}{X1(0,1)}\right)$$

$$X1(1,2) = .0167\,F_o$$

$$X2(1,2) = \overset{0}{X2(0,1)} + \frac{.05}{3}\left(\overset{0}{X1(0,1)} - \overset{0}{X2(0,1)}\right)$$

$$X2(1,2) = 0$$

$$OUT(1,2) = \overset{0}{OUT(0,1)} + \frac{.05}{3}\left(\overset{0}{X2(0,1)} - \overset{0}{OUT(0,1)}\right)$$

$$OUT(1,2) = 0$$

$K=2$
$t=.10$

$$X1(2,3) = \overset{.0167\,F_0}{X1(1,2)} + \frac{.05}{3}\left(\overset{F_0}{FIN(1,2)} - \overset{.0167F_0}{X1(1,2)}\right)$$

$$X1(2,3) = .03309\ F_0$$

$$X2(2,3) = \overset{0}{X2(1,2)} + \frac{.05}{3}\left(\overset{.0167F_0}{X1(1,2)} - \overset{0}{X2(1,2)}\right)$$

$$X2(2,3) = .0002783\ F_0$$

$$OUT(2,3) = \overset{0}{OUT(1,2)} + \frac{.05}{3}\left(\overset{0}{X2(1,2)} - \overset{0}{OUT(1,2)}\right)$$

$$OUT(2,3) = 0$$

$K=3$
$t=.15$

$$X1(3,4) = \overset{.0331\,F_0}{X1(2,3)} + \frac{.05}{3}\left(F_0 - \overset{.03309F_0}{X1(2,3)}\right)$$

$$X1(3,4) = .0492\ F_0$$

$$X2(3,4) = \overset{.000278F_0}{X2(2,3)} + \frac{.05}{3}\left(\overset{.0331\,F_0}{X1(2,3)} - \overset{.000278F_0}{X2(2,3)}\right)$$

$$X2(3,4) = .000825\,F_0$$

$$OUT(3,4) = \overset{0}{OUT(2,3)} + \frac{.05}{3}\left(\overset{.0002783\,F_0}{X2(2,3)} - \overset{0}{OUT(2,3)}\right)$$

$$OUT(3,4) = .000004638\ F_0$$

$\underline{K=4}$
$t=.20$

$$X1(4,5) = \overset{.0492\,F_o}{X1(3,4)} + \frac{.05}{3}\left(F_o - \overset{.0492\,F_o}{X1(3,4)}\right)$$

$$X1(4,5) = .0650\,F_o$$

$$X2(4,5) = \overset{.000825\,F_o}{X2(3,4)} + \frac{.05}{3}\left(\overset{.0492\,F_o}{X1(3,4)} - \overset{.000825\,F_o}{X2(3,4)}\right)$$

$$X2(4,5) = .00163\,F_o$$

$$OUT(4,5) = \overset{.000004638\,F_o}{OUT(3,4)} + \frac{.05}{3}\left(\overset{.000825\,F_o}{X2(3,4)} - \overset{.000004638\,F_o}{OUT(3,4)}\right)$$

$$\underline{OUT(4,5) = .00001827\,F_o}$$

$\underline{K=5}$
$t=.25$

$$X1(5,6) = \overset{.065\,F_o}{X1(4,5)} + \frac{.05}{3}\left(F_o - \overset{.065\,F_o}{X1(4,5)}\right)$$

$$X1(5,6) = .0806\,F_o$$

$$X2(5,6) = \overset{.00163\,F_o}{X2(4,5)} + \frac{.05}{3}\left(\overset{.065\,F_o}{X1(4,5)} - \overset{.00163\,F_o}{X2(4,5)}\right)$$

$$X2(5,6) = .002686\,F_o$$

$$OUT(5,6) = \overset{.00001827}{OUT(4,5)} + \frac{.05}{3}\left(\overset{.00163\,F_o}{X2(4,5)} - \overset{.00001827\,F_o}{OUT(4,5)}\right)$$

$$\underline{OUT(5,6) = .00004516\,F_o}$$

$\underline{K=6}$
$t=.30$

$$X1(6,7) = \overset{.0806F_o}{X1(5,6)} + \frac{.05}{3}\left(F_o - \overset{.0806F_o}{X1(5,6)}\right)$$

$$X1(6,7) = .0959\,F_o$$

$$X2(6,7) = \overset{.002686F_o}{X2(5,6)} + \frac{.05}{3}\left(\overset{.0806F_o}{X1(5,6)} - \overset{.002686F_o}{X2(5,6)}\right)$$

$$X2(6,7) = .00398\,F_o$$

$$OUT(6,7) = \overset{.00004516F_o}{OUT(5,6)} + \frac{.05}{3}\left(\overset{.002686F_o}{X2(5,6)} - \overset{.00004516F_o}{OUT(5,6)}\right)$$

$$\underline{OUT(6,7) = .0000891\,F_o} \qquad \text{(no match)}$$

Lets do 1 more

$\underline{K=7}$
$t=.35$

$$X1(7,8) = \overset{.0959F_o}{X1(6,7)} + \frac{.05}{3}\left(F_o - \overset{.0959F_o}{X1(6,7)}\right)$$

$$X1(7,8) = .1110F_o$$

$$X2(7,8) = \overset{.00398F_o}{X2(6,7)} + \frac{.05}{3}\left(\overset{.0959F_o}{X1(6,7)} - \overset{.00398F_o}{X2(6,7)}\right)$$

$$X2(7,8) = .00551\,F_o$$

$$OUT(7,8) = \overset{.0000891}{OUT(6,7)} + \frac{.05}{3}\left(\overset{.00398F_o}{X2(6,7)} - \overset{.0000891}{OUT(6,7)}\right)$$

$$\underline{OUT(7,8) = .000154\,F_o}$$

OK. this matches the analytical result. However, we seem to be one time step out of phase.

<u>More analytical</u>

$$t=.25\,, \quad t/\hat{\tau} = .25/3 = .0833$$

$$OUT(t=.25) = .0000906F_o \qquad \text{matches } K=6$$

Perhaps, if we had taken

$$\boxed{FIN(-1,0) = F_0}$$

then every Finite Difference result would have occurred one step earlier. We would then have a complete match.

Note : Finite Difference Model inputs occur during $[K-1, K]$ interval at beginning of run.

Now we are ready to explore the Finite Difference solution process for the Forrester Model.