Two-Period Planning for Hybrid Seed Corn: An Alternative Solution Procedure

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Reference:


Introduction

This is an attempt to fathom the approach of Jones, et al to the problem stated in the title. We assume that the reader has this reference at hand. We also wish to avoid the need for a solver such as What’s Best! as used in that study.

Throughout we rely on the results from Jones, et al that the marginal profit functions are well behaved and that local optima are global. For the most part, we adopt the notation used in the Jones study. However, we will assume that the probability distributions for crop yield and demand are statistically independent, normal distributions with the same means and standard deviations used by Jones in his Managerial Insights section. In that section, he examined 9 combinations of variability for yield and demand: high, medium and zero, using symmetrical, discrete distributions. We will confine our attention to only 2 of these combinations: yield and demand at high variability; yield and demand at medium variability.

Some Notation

\( i = 1, 2 \) index for planning periods

\( K_i = $900 \), cost per acre

\( c_i = $10 \), variable cost per bushel

\( \pi = $27.50 \), shortage cost per bushel

\( v = $23.50 \), salvage value per bushel

\( p = $60 \), price per bushel

\( y_i \) = yield in bushels per acre for period \( i \), a random variable

\( D \) = demand in bushels, a random variable

\( \mu_{yi} = 40 \) bushels per acre, mean yield
\[ \mu_D = 210 \text{ (1000s of) bushels, mean demand} \]
\[ \sigma_{yi} = 7.4833, 4.4721 \text{ high and medium values of standard deviation for yield} \]
\[ \sigma_D = 7.4833, 4.4721 \text{ high and medium values of standard deviation for demand} \]
\[ k_{yi} = \text{coefficient of variation for yields} \]
\[ k_D = \text{coefficient of variation for demand} \]
\[ Q_i = \text{decision variables for acreage planted in each period} \]
\[ w_i = \text{bushels on-hand beginning period } i, \ w_1 \text{ is specified and } w_2 = Q_1 y_1 \]
\[ V_i = \text{marginal profit function for period } i \]

Additional notation will be introduced as needed. For convenience, we assume as did Jones that \( w_1 = 0 \) and that yields and various costs remain the same from period 1 to period 2.

**A General Description of the Method**

The objective of the analysis is to find the optimal acreages \( Q_1 \) and \( Q_2 \) for two sequential planning periods with unknown crop carry-over \( w_2 \) at the start of period 2. The process, in general terms follows.

1. Equation (1) of Jones is the marginal profit function \( V_2(Q_2, w_2) \) for the two-period planning problem. If we set the derivative of this function equal to zero, we can then search for the optimal \( Q_2 \) for each \( w_2 \) over a range of specified values. The result permits construction of the function \( Q_2^*(w_2) \), so that the marginal profit function becomes \( V_2^*(w_2) \). The search process involves numerical integration of the normal density function and its cumulative and the use of the Goal Seek utility in Excel.

2. Equation (2) of Jones is the marginal profit function \( V_1(Q_1) \) for the first period, assuming an optimal choice of acreage for period 2, e.g. \( V_1(Q_1) \) depends on \( V_2^*(w_2) \). We next differentiate equation (2) with respect to \( Q_1 \), bearing in mind that \( w_2=Q_1 y_1 \), so that we must employ the chain rule: \( \frac{dV_2^*(w_2)}{dQ_1} = \left( \frac{dV_2^*(w_2)}{dw_2} \right) x \left( \frac{dw_2}{dQ_1} \right) \).

3. In order to determine \( \frac{dV_2^*(w_2)}{dw_2} \), we must return to the modified form of equation (1) as determined in step 1. above. After differentiation and carrying out numerical integration for a specified range of \( w_2 \), we can construct an explicit function for the derivative of \( V_2^* \).

4. We now return to the differentiated form of equation (2) and carry out a search for the optimal value of \( Q_1 \). Once again, this will require numerical integration and the use of Goal Seek.
In the next section, we will examine the details of this alternative approach and present results for the optimal choices for acreage in each period. As we shall see, they compare quite closely to those obtained by Jones.

**Analysis and Results**

**Step 1.**

Setting the derivative of \( V_2(Q_2, w_2) \) with respect to \( Q_2 \) equal to zero gives:

\[
\int_{-\infty}^{\infty} \left[ \frac{F(k) - k y f_x(k)}{f(z_D) dz_D} \right] \frac{(K_2/\mu y_2 + c_2 - v)}{(p + \pi - v)} = 0
\]

where

\[
f_x(.) \text{ is the standard normal density function,}
\]

\[
F(.) \text{ is the normal cumulative,}
\]

\[
z_D \text{ is the standard normal variate for demand (subscript for clarity only)}
\]

and

\[
k = \left[ \frac{(1 + k D z_D - w_2/\mu_D)}{Q_2 \mu y_2/\mu_D} - 1 \right] / k y_2
\]

In equation (1) \( w_2/\mu_D \) represents the proportion of mean demand met by first period ending crop inventory while \( Q_2 \mu y_2/\mu_D \) is the proportion of mean demand met by second period output.

We will require numerical integration to search equation (1) for \( Q_2 \) and for subsequent analysis. In all cases we used Simpson’s Rule with the standard normal variate in the range \([-5.0, +5.0]\) in steps of size 0.2. The value of \( w_2/\mu_D \) was varied in the range \([0, 1]\) in steps of 0.1 and an additional point was evaluated at 0.95. We found that Goal Seek was unable to find a solution for values of \( w_2/\mu_D > 1.02 \).

Figure 1 below, shows the results of the search for optimal \( Q_2 \) for the case of high variability. The linear nature of the relationship is surprising, but was observed at many different combinations of the coefficients of variation for yield and demand. Then we can write

\[
Q_2^* = a_0 + a_1 (w_2/\mu_D)
\]

Table 1 below lists the coefficients of equation (3) for the high and medium variability cases.
Table 1: Coefficients of equation (3).

<table>
<thead>
<tr>
<th></th>
<th>$a_0$</th>
<th>$a_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Var.</td>
<td>6.2621</td>
<td>-6.1240</td>
</tr>
<tr>
<td>Med Var.</td>
<td>5.8742</td>
<td>-5.7915</td>
</tr>
</tbody>
</table>

Note that the coefficient $a_0$ in equation (3) is exactly the optimal value for acreage for a single period planning situation, since $w_2 = 0$. We are ready to take on step 2 of the procedure outlined earlier.
Step 2.

Our next task is to consider Jones’ equation (2) for the marginal profit function in period 1, differentiating that expression with respect to \( Q_1 \) and equating the result to zero. That result, after rearranging, follows.

\[
\int_{-\infty}^{\infty} (1 + k y_1 z_1 y_1) \left[ \frac{dV_2^*(w_2)}{dw_2} \right] f(z_{y_1})dz_{y_1} = K_1 / \mu_{y_1} + c_1 \tag{4}
\]

After we establish an algebraic form for the term in brackets in equation (4) as a function of \( w_2 \) (in the next step), we then make use of the fact that \( w_2 = Q_1 y_1 \). This will permit a Goal Seek search for the \( Q_1^* \) that satisfies equation (4).

Step 3.

First define two new constants:

\[
\hat{a}_i = a_i (\mu_{y_2} / \mu_D) \quad i = 0, 1
\]

Differentiation of equation (1) with respect to \( w_2 \) after substitution of equation (3) gives

\[
\frac{dV_2^*(w_2)}{dw_2} = b_0 + b_1 \int_{-\infty}^{\infty} (1 + \hat{a}_1) F(k) - \hat{a}_1 k y_2 f_x(k) \right] f(z_D)dz_D \tag{5}
\]

where

\[
b_0 = -K_2 \hat{a}_1 / \mu_{y_2} - c_2 \hat{a}_1 + v(1 + \hat{a}_1) \quad , \quad b_1 = p + \pi - v
\]

and

\[
k = \left[ \frac{1 + k D z_D - w_2 / \mu_D}{\hat{a}_0 + \hat{a}_1 w_2 / \mu_D} - 1 \right] / k y_2 \tag{6}
\]

The procedure here is to evaluate equation (5) for a range of values of \( w_2 / \mu_D \), plot the results and fit a function of \( w_2 / \mu_D \) to the derivative. Figure 2 illustrates such a plot for the high variability case. For the medium variability case, the curve is very similar. However, at very small values for the coefficients
of variation (i.e. <.01), the plot becomes linear.

![Figure 2: High Variability Case](image)

A function of the form

$$\frac{dV}{dw}\bigg|_{w_2} = d_0 + d_1, \exp\left(\frac{w_2}{\mu_D}\right)^6$$

provides an excellent, yet simple, fit for both high and medium variability cases with $R^2 > .996$. Table 2 below lists the coefficients for both cases.

<table>
<thead>
<tr>
<th></th>
<th>$d_0$</th>
<th>$d_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>High Var.</strong></td>
<td>37.691</td>
<td>-2.0287</td>
</tr>
<tr>
<td><strong>Med Var.</strong></td>
<td>35.462</td>
<td>-1.2214</td>
</tr>
</tbody>
</table>

**Table 2: Coefficients for equation (7)**

Now we turn our attention to determination of the optimal value of the first period acreage, $Q_1$.

**Step 4.**

Return to equation (4) and substitute equation (7). Next substitute $Q_1y_1$ for $w_2$ and $(\mu_{y_1} + \sigma_{y_1}z_{y_1})$ for $y_1$. 

6
Then Goal Seek can be used to find the optimal $Q_1$. This result can then be used to compute $w_2 = Q_1\mu_1$. Equation (3) will provide the optimal $Q_2$. Total supply is found from $w_2 + Q_2\mu_2$. These results are shown in Table 3 below along with a comparison to the Jones results.

<table>
<thead>
<tr>
<th></th>
<th>High Variability</th>
<th></th>
<th>Medium Variability</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present Study</td>
<td>Jones Study</td>
<td>Present Study</td>
<td>Jones Study</td>
</tr>
<tr>
<td><strong>Single Period Plan</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_1$</td>
<td>6.26</td>
<td>6.0</td>
<td>5.89</td>
<td>6.0</td>
</tr>
<tr>
<td>Total Supply</td>
<td>250.40</td>
<td>246.0</td>
<td>234.80</td>
<td>240.0</td>
</tr>
<tr>
<td><strong>Two Period Plan</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_1$</td>
<td>4.15</td>
<td>4.5</td>
<td>4.77</td>
<td>4.8</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>1.42</td>
<td>1.1</td>
<td>0.61</td>
<td>0.6</td>
</tr>
<tr>
<td>Total Supply</td>
<td>222.84</td>
<td>224.0</td>
<td>215.29</td>
<td>217.0</td>
</tr>
</tbody>
</table>

**Table 3: Comparison of some key results**

Differences in the results of the two approaches as revealed in the table above may be due to any of several factors:

1. The present study used normal distributions for both yield and demand random variables while the Jones analysis used discrete, symmetrical distributions.
2. Numerical integration does not provide exact answers.
3. The Goal Seek utility in Excel does not provide exact answers.
4. The values for high and medium coefficients of variation of demand stated in the Jones study (0.05 and 0.03) do not correspond to the values we computed using the discrete probability distribution of demand given in his Table 4. We obtained values of 0.0356 and 0.0213 respectively and used these throughout our analysis.