

Next-to-leading order evolution of color dipole

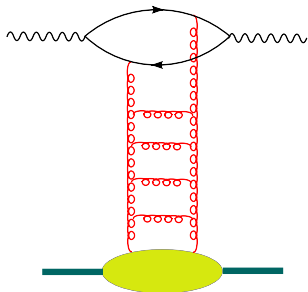
G. A. Chirilli
and
I. Balitsky

JLAB & ODU

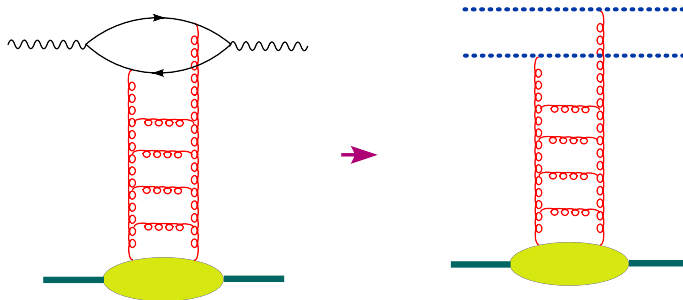
EIC Hampton 2008 May 22, 2008

- DIS from nucleus at high energy and Wilson line.
- Evolution equation.
- Leading order: BK equation.
- Non linear evolution equation in the NLO.
- Quark and gluon loop in the shock-wave.
- NLO kernel.
- $\mathcal{N} = 4$ NLO kernel.
- Conclusions.
- Outlook.

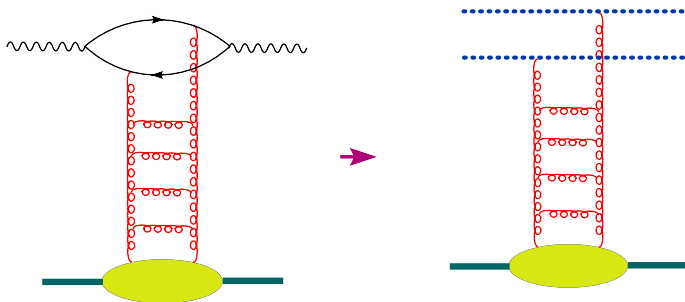
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$$A(s) = \int \frac{d^2 k_{\perp}}{4\pi^2} I^A(k_{\perp}) \langle B | \text{Tr} \{ U(k_{\perp}) U^{\dagger}(-k_{\perp}) \} | B \rangle$$

$$U(x_{\perp}) = P e^{ig \int_{-\infty}^{\infty} du \, n^{\mu} A_{\mu}(un + x_{\perp})}$$

Wilson line

The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{\mathcal{U}}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}^\eta(x_\perp) \hat{U}^{\dagger\eta}(y_\perp)\}$$

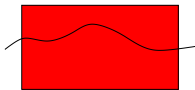
switched between the target's states ($N_c = 3$ for QCD). The gluon parton density is approximately:

$$x_B G(x_B, \mu^2 = Q^2) \simeq \langle p | \hat{\mathcal{U}}^\eta(x_\perp, 0) | p \rangle \Big|_{x_\perp^2 = Q^{-2}}$$

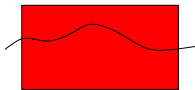
$$\text{where } \eta = \ln \frac{1}{x_B}$$

Propagation in the shock wave: Wilson line

Propagation in the shock wave: Wilson line



Propagation in the shock wave: Wilson line



Boosted Field

Propagation in the shock wave: Wilson line



Propagation in the shock wave: Wilson line

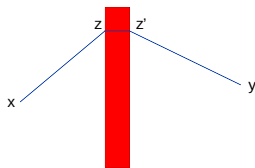


Each path is weighted with the gauge factor $Pe^{ig \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

Propagation in the shock wave: Wilson line



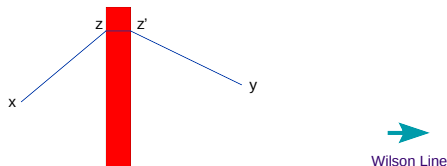
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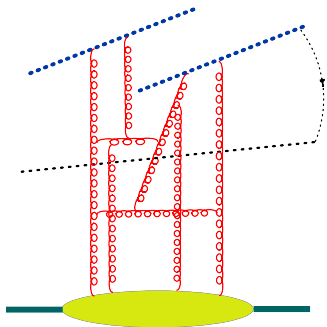
$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = Pe^{ig \int_0^1 du (x-y)^\mu A_\mu (ux + (1-u)y)}$$

To get the evolution equation, consider the dipole with the rapidities up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

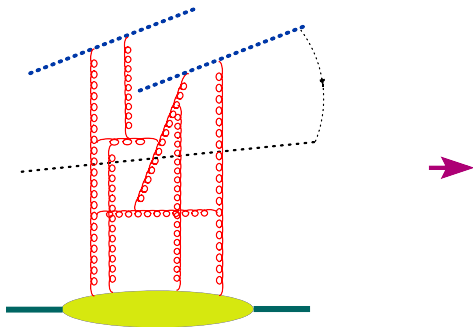
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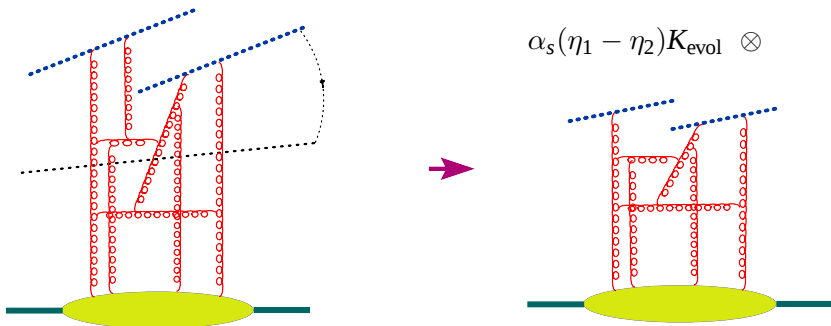
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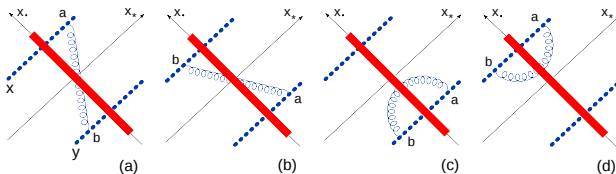


$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$

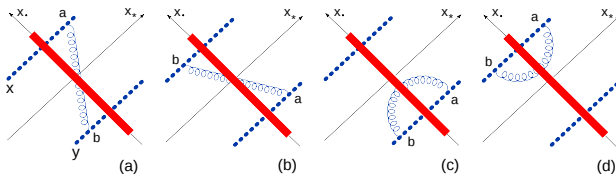
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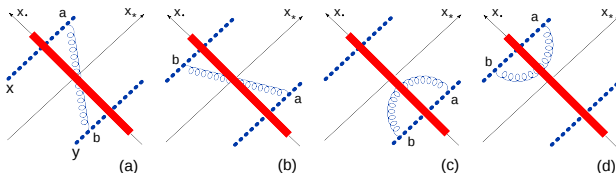
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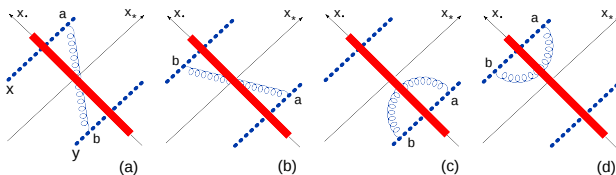
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[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave] $\times$

Leading order: BK equation

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[$z \rightarrow y$: free propagation]

Non linear evolution equation

$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s (\eta_1 - \eta_2) (U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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BK equation: Ian Balitsky (1996), Yu. Kovchegov (1999)

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x - y)^2}{(x - z)^2 (y - z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1$, $\alpha_s \eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1$, $\alpha_s \eta \sim 1$, $\alpha_s^2 A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To get the region of application of the leading order evolution equation.

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- To get the region of application of the leading order evolution equation.
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 - **Theoretical view point:** Whether the coupling constant is determined by the size of the original dipole or by the size of the parent dipole, we have different behavior of the solutions.
 - **Experimental view point:** The cross section is proportional to some power of the coupling constant, so the argument of the coupling constant determined how big or how small the cross section is.

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \int \frac{d^2 z}{2\pi^2} \left(\alpha_s \frac{(x-y)^2}{(x-z)^2 (z-y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_z U_y^\dagger\}] + \\ & \alpha_s^2 \int d^2 z d^2 z' \left(K_4(x, y, z, z') \{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x, y, z, z') \{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\} \right) \end{aligned}$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots$$

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Where $\langle \dots \rangle$ is evaluated in the background of the shock wave

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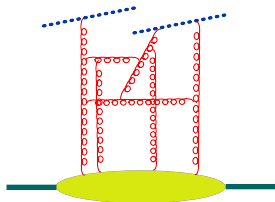
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Subtraction of BK^2 contribution $\implies \left[\frac{1}{u}\right]_+$ prescription

$$\int_0^1 du f(u) \left[\frac{1}{u}\right]_+ \equiv \int_0^1 du \frac{f(u) - f(0)}{u}, \quad \int_0^1 du f(u) \left[\frac{1}{\bar{u}}\right]_+ \equiv \int_0^1 du \frac{f(u) - f(1)}{\bar{u}}$$

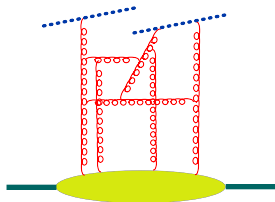
Regularization of the rapidity divergence



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

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Regularization by: slope

$$U^\eta(x_\perp) = \text{Pexp} \left\{ ig \int_{-\infty}^\infty du \, n_\mu A^\mu(un + x_\perp) \right\} \quad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

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$$U_x^\eta = \text{Pexp} \left[ig \int_{-\infty}^{\infty} du \, p_1^\mu A_\mu^\eta(up_1 + x_\perp) \right]$$
$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

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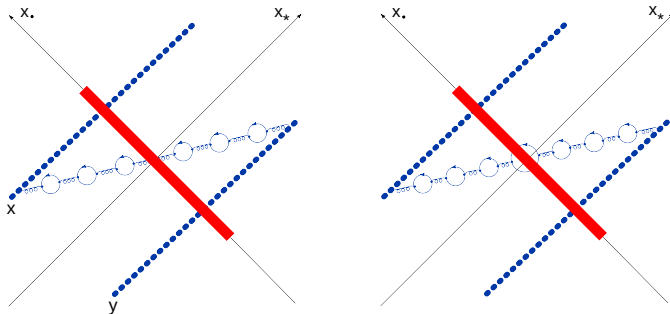
The rigid cut-off leads to (almost) conformal result

Argument of coupling constant

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s(?) N_c}{2\pi^2} \int \frac{dz(\vec{x} - \vec{y})^2}{(\vec{x} - \vec{z})^2 (\vec{z} - \vec{y})^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

I. Balitsky (2006) **Renormalon-based approach: summation of quark bubbles**

$$-\frac{2}{3}n_f \rightarrow b = \frac{11}{3}N_c - \frac{2}{3}n_f$$



Bubble chain sum:

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} &= \frac{\alpha_s((x-y)^2)}{2\pi^2} \int d^2z [\text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}] \\ &\times \left[\frac{(x-y)^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + \dots \end{aligned}$$

I. Balitsky; Yu. Kovchegov and H. Weigert (2006)

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I. Balitsky; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\begin{array}{ll} \frac{\alpha_s((x-y)^2)}{2\pi^2} \frac{(x-y)^2}{X^2 Y^2} & |x-y| \ll |x-z|, |y-z| \\ \frac{\alpha_s(X^2)}{2\pi^2 X^2} & |x-z| \ll |x-y|, |y-z| \\ \frac{\alpha_s(Y^2)}{2\pi^2 Y^2} & |y-z| \ll |x-y|, |x-z| \end{array}$$

Bubble chain sum:

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} &= \frac{\alpha_s((x-y)^2)}{2\pi^2} \int d^2z [\text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}] \\ &\times \left[\frac{(x-y)^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + \dots \end{aligned}$$

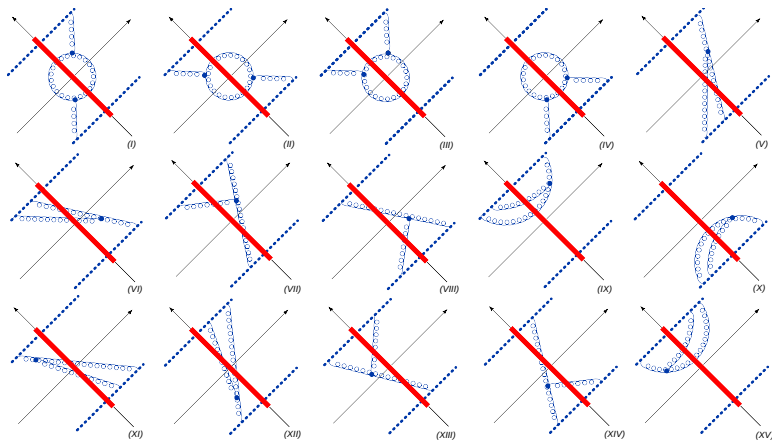
I. Balitsky; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

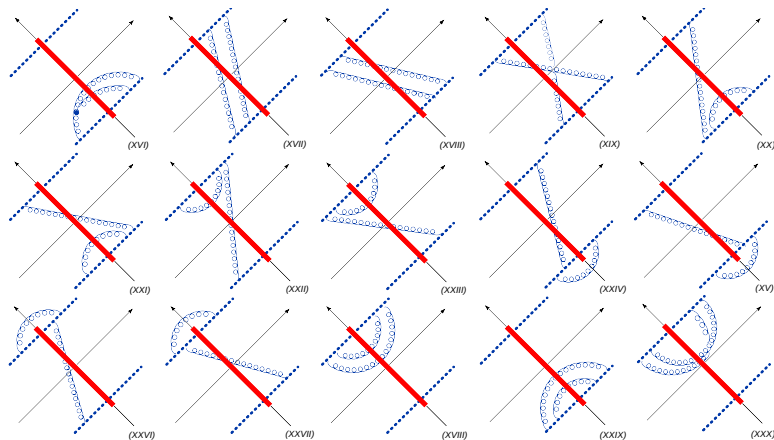
$$\begin{array}{ll} \frac{\alpha_s((x-y)^2)}{2\pi^2} \frac{(x-y)^2}{X^2 Y^2} & |x-y| \ll |x-z|, |y-z| \\ \frac{\alpha_s(X^2)}{2\pi^2 X^2} & |x-z| \ll |x-y|, |y-z| \\ \frac{\alpha_s(Y^2)}{2\pi^2 Y^2} & |y-z| \ll |x-y|, |x-z| \end{array}$$

⇒ the argument of the coupling constant is given by the size of the smallest dipole.

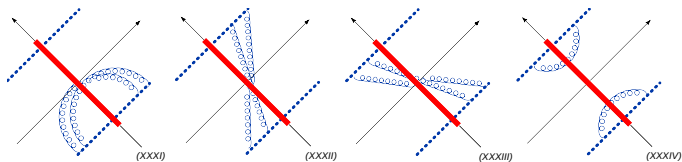
Diagrams with 2 gluons interaction



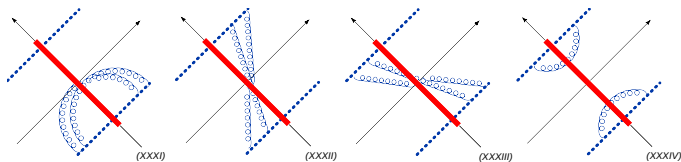
Diagrams with 2 gluons interaction



Diagrams with 2 gluons interaction



Diagrams with 2 gluons interaction

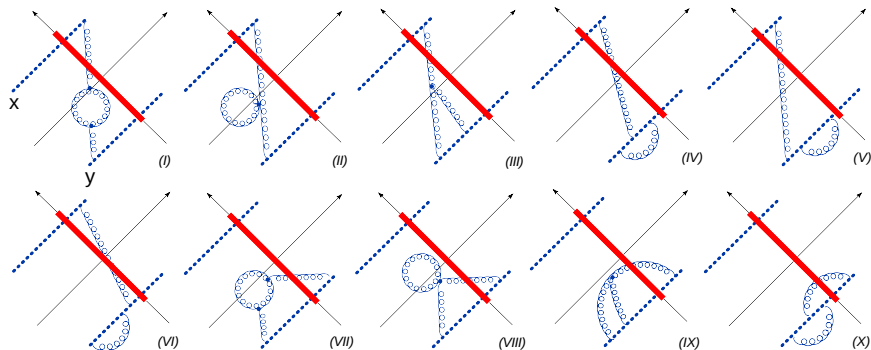


Extracting the UV divergencies

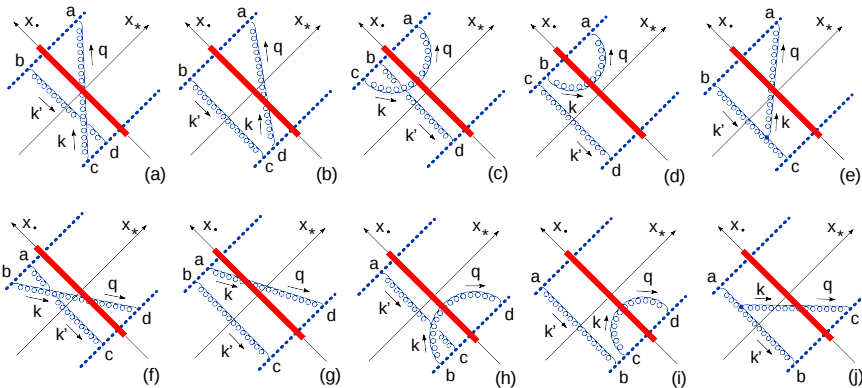
$$\text{Tr}\{t^a U_z t^b U_z^\dagger\} = \text{Tr}\{t^a U_z t^b U_z^\dagger - t^a U_z t^b U_z^\dagger\} + \text{Tr}\{t^a U_z t^b U_z^\dagger\}$$

I. Balitsky 2006; Y. Kovchegov, H. Weigert 2006

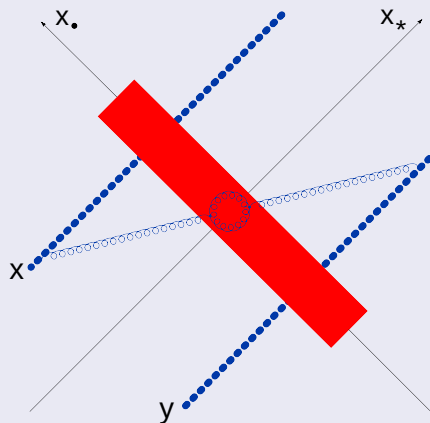
"Running coupling" diagrams



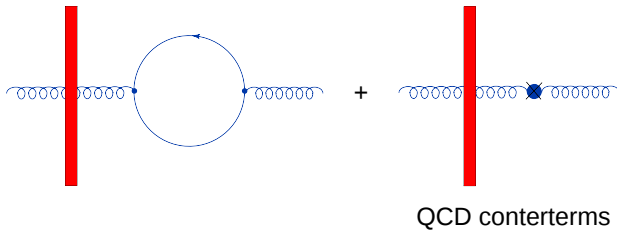
1 $\rightarrow$ 2 dipole transition diagrams



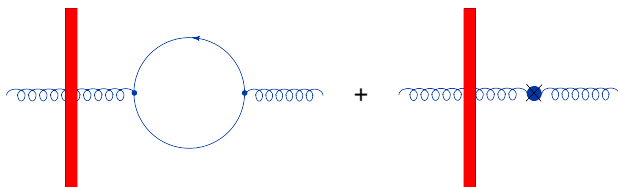
typical diagram: gluon in the shock wave



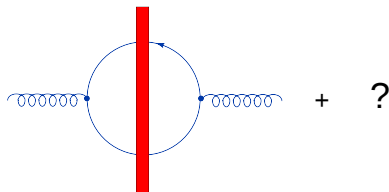
Quark loop in the shock wave



Quark loop in the shock wave



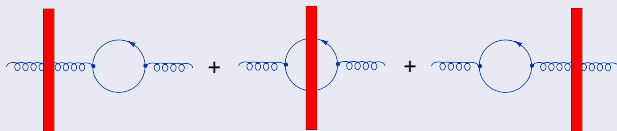
QCD conterterms



If we simply subtract the divergence using $\overline{MS}$, we could be missing some extra finite terms.

Balitsky 2006

A way to get potential extra terms is to consider the exact calculation of the light-cone expansion of $U_x U_y^\dagger$ at $x_\perp \rightarrow y_\perp$ in QCD and compare it with the expansion of the sum of the diagrams:

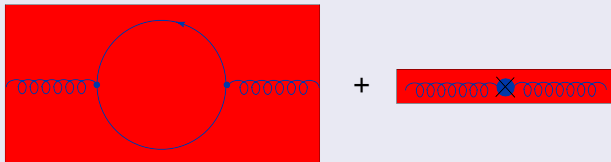


Exact calculation means:

Quark loop in the shock wave

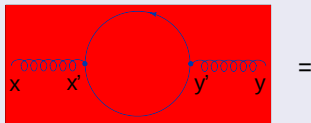
Consider the quark-loop contribution to the gluon propagator in external field

Calculate the light cone expansion of:



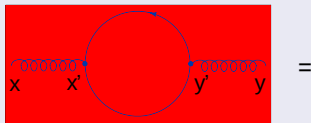
meaning...

Calculate the light cone expansion $(x - y)_\perp \rightarrow 0$ of:



$$\begin{aligned}
 \langle A_\bullet^m(x) A_\bullet^n(y) \rangle_A &= \int dx' dy' \left(x \left| \frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}} \right| x' \right)^{ma} \\
 &\times \text{Tr} \left\{ t^a \gamma_\mu \left(x' \left| \frac{1}{\not{p}} \right| y' \right) t^b \gamma_\nu \left(y' \left| \frac{1}{\not{p}} \right| x' \right) \right\} \left(y' \left| \frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}} \right| y \right)^{bn} \\
 \mathcal{O}_{\bullet\mu} &= \frac{1}{\alpha} D^i G_{i\mu}
 \end{aligned}$$

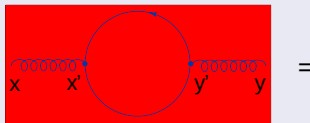
Calculate the light cone expansion $(x - y)_\perp \rightarrow 0$ of:



$$\begin{aligned} \langle A_\bullet^m(x) A_\bullet^n(y) \rangle_A &= \int dx' dy' \left(x \left| \frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}} \right| x' \right)^{ma} \\ &\times \text{Tr} \left\{ t^a \gamma_\mu \left(x' \left| \frac{1}{\not{p}} \right| y' \right) t^b \gamma_\nu \left(y' \left| \frac{1}{\not{p}} \right| x' \right) \right\} \left(y' \left| \frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}} \right| y \right)^{bn} \\ \mathcal{O}_{\bullet\mu} &= \frac{1}{\alpha} D^i G_{i\mu} \end{aligned}$$

then take the limit $d_\perp \rightarrow 2$ (which gives a finite result after adding the counterterms),

Calculate the light cone expansion $(x - y)_\perp \rightarrow 0$ of:



$$\begin{aligned} \langle A_\bullet^m(x) A_\bullet^n(y) \rangle_A &= \int dx' dy' \left(x \left| \frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}} \right| x' \right)^{ma} \\ &\times \text{Tr} \left\{ t^a \gamma_\mu \left(x' \left| \frac{1}{\not{p}} \right| y' \right) t^b \gamma_\nu \left(y' \left| \frac{1}{\not{p}} \right| x' \right) \right\} \left(y' \left| \frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}} \right| y \right)^{bn} \\ \mathcal{O}_{\bullet\mu} &= \frac{1}{\alpha} D^i G_{i\mu} \end{aligned}$$

then take the limit $d_\perp \rightarrow 2$ (which gives a finite result after adding the counterterms),
and finally impose the condition that the external field is very narrow by taking
the limit $\Delta z_* \rightarrow 0$

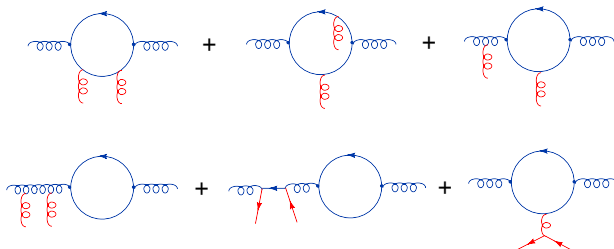
However...

Balitsky (2006) Commutativity of the two limits

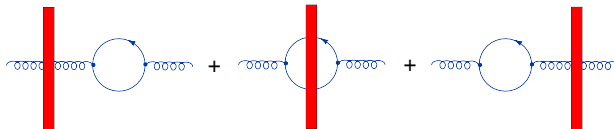
$$\left[d_{\perp} \rightarrow 2, \Delta z_* \rightarrow 0 \right] = 0$$

light-cone expansion

Typical diagrammatic expression of the light-cone expansion of the quark-loop contribution to the gluon propagator in the external field is



Which coincides with the expansion of



⇒ no additional terms at one loop level.

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &- (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\}
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &- (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} =
 \end{aligned}$$

Running coupling part

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &- (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} =
 \end{aligned}$$

Running coupling part + Non-conformal part

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &- (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} =
 \end{aligned}$$

Running coupling part + Non-conformal part + Conformal
 "non-analytic" part

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\}] \right. \\
 &- (z' \rightarrow z) \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} =
 \end{aligned}$$

Running coupling part + Non-conformal part + Conformal
 "non-analytic" part + "conformal-analytic" ($\mathcal{N} = 4$) part

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &- (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} + \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U_y^\dagger\} =
 \end{aligned}$$

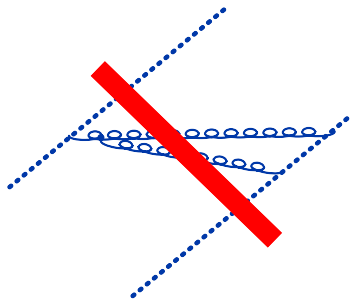
Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

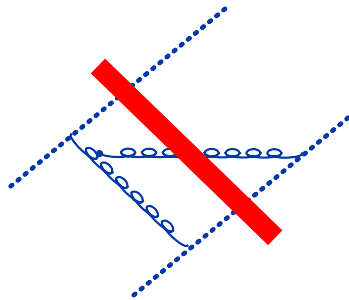
$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &- (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} + \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U_y^\dagger\}
 \end{aligned}$$

However the term $\frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr} U_x U_y^\dagger$ contradicts the requirement $\frac{d}{d\eta} U_x U_y^\dagger = 0$ at $x = y$.

Conformal and non-conformal diagrams



Conformal



Non-Conformal

$$\begin{aligned}\langle K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} &= \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} \\ &\equiv \left[\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle \right]_{+}^{\text{rigid}},\end{aligned}$$

$$\begin{aligned}\langle K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} &= \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} \\ &\equiv \left[\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle \right]_{+}^{\text{rigid}},\end{aligned}$$

But

$$\begin{aligned}\langle K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{slope}} &= \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{slope}} - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{slope}} \\ &\neq \left[\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle \right]_{+}^{\text{slope}}\end{aligned}$$

$$\langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle^{\text{rigid}} - K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}^{\text{slope}} \neq 0$$

The regularization by slope adds an extra term to the K_{NLO} calculated using rigid cut-off regularization. This extra term is not conformal invariant.

I. Balitsky; Yu. Kovchegov and H. Weigert (2006)

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
 &\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
 &+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2z d^2z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_z^\dagger\} \frac{1}{(z-z')^4} \\
 &\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
 \end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
 &\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
 &+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2z d^2z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_z^\dagger\} \frac{1}{(z-z')^4} \\
 &\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
 \end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

Running coupling

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
 &\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
 &+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2z d^2z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_{z'}^\dagger\} \frac{1}{(z-z')^4} \\
 &\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
 \end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

Running coupling + Conformal part

$$\begin{aligned}
& \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \\
&= \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x-y)^2}{X^2 Y^2} \left\{ 1 + \frac{\alpha_s}{4\pi} \left[b \ln(x-y)^2 \mu^2 - b \frac{X^2 - Y^2}{(x-y)^2} \ln \frac{X^2}{Y^2} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} n_f \right. \right. \\
&\quad \left. \left. - 2N_c \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right] \right\} [\text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}] \\
&+ \frac{\alpha_s^2}{16\pi^4} \int d^2z d^2z' \left[\left(-\frac{4}{(z-z')^4} + \left\{ 2 \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4 [X^2 Y'^2 - X'^2 Y^2]} \right. \right. \right. \\
&\quad \left. \left. + \frac{(x-y)^4}{X^2 Y'^2 - X'^2 Y^2} \left[\frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] + \frac{(x-y)^2}{(z-z')^2} \left[\frac{1}{X^2 Y'^2} - \frac{1}{X'^2 Y^2} \right] \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right) \\
&\quad \times [\text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_{z'}^\dagger\} \text{Tr}\{\hat{U}_{z'} \hat{U}_y^\dagger\} - \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger \hat{U}_{z'} \hat{U}_y^\dagger\} - (z' \rightarrow z)] \\
&+ \left\{ \frac{(x-y)^2}{(z-z')^2} \left[\frac{1}{X^2 Y'^2} + \frac{1}{Y^2 X'^2} \right] - \frac{(x-y)^4}{X^2 Y'^2 X'^2 Y^2} \right\} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_{z'}^\dagger\} \text{Tr}\{\hat{U}_{z'} \hat{U}_y^\dagger\} \\
&+ 4n_f \left\{ \frac{4}{(z-z')^4} - 2 \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2(z-z')^2}{(z-z')^4 (X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\} \text{Tr}\{t^a \hat{U}_x t^b \hat{U}_y^\dagger\} [\text{Tr}\{t^a \hat{U}_z t^b \hat{U}_{z'}^\dagger\} - (z' \rightarrow z)] \Big]
\end{aligned}$$

From NLO BK kernl

$$\begin{aligned}
 s \frac{d}{ds} \langle \hat{\mathcal{U}}(n, \gamma) \rangle &= \frac{\alpha_s N_c}{\pi} \left\{ \left[1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right] \chi(n, \gamma) \right. \\
 &\quad \left. + \frac{\alpha_s b}{4\pi} \left[\frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \right. \\
 &\quad \left. + \frac{\alpha_s N_c}{4\pi} \left[-\chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + \textcolor{red}{4}\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{\mathcal{U}}(n, \gamma) \rangle
 \end{aligned}$$

Gluon-loop in the shock wave

From NLO BK kernel

$$\begin{aligned} s \frac{d}{ds} \langle \hat{\mathcal{U}}(n, \gamma) \rangle &= \frac{\alpha_s N_c}{\pi} \left\{ \left[1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right] \chi(n, \gamma) \right. \\ &\quad \left. + \frac{\alpha_s b}{4\pi} \left[\frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \right. \\ &\quad \left. + \frac{\alpha_s N_c}{4\pi} \left[-\chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 4\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{\mathcal{U}}(n, \gamma) \rangle \end{aligned}$$

From NLO BFKL kernel

$$\begin{aligned} s \frac{d}{ds} \langle \hat{\mathcal{U}}(n, \gamma) \rangle &= \frac{\alpha_s N_c}{\pi} \left\{ \left[1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right] \chi(n, \gamma) \right. \\ &\quad \left. + \frac{\alpha_s b}{4\pi} \left[\frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \right. \\ &\quad \left. + \frac{\alpha_s N_c}{4\pi} \left[-\chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 6\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{\mathcal{U}}(n, \gamma) \rangle \end{aligned}$$

Gluon-loop in the shock wave

From NLO BK kernl

$$\begin{aligned} s \frac{d}{ds} \langle \hat{\mathcal{U}}(n, \gamma) \rangle &= \frac{\alpha_s N_c}{\pi} \left\{ \left[1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right] \chi(n, \gamma) \right. \\ &\quad \left. + \frac{\alpha_s b}{4\pi} \left[\frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \right. \\ &\quad \left. + \frac{\alpha_s N_c}{4\pi} \left[-\chi''(n, \gamma) - 2\chi(n, \gamma) \chi'(n, \gamma) + 4\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{\mathcal{U}}(n, \gamma) \rangle \end{aligned}$$

From NLO BFKL kernel

$$\begin{aligned} s \frac{d}{ds} \langle \hat{\mathcal{U}}(n, \gamma) \rangle &= \frac{\alpha_s N_c}{\pi} \left\{ \left[1 - \frac{b\alpha_s}{4\pi} \frac{d}{d\gamma} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right] \chi(n, \gamma) \right. \\ &\quad \left. + \frac{\alpha_s b}{4\pi} \left[\frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \right. \\ &\quad \left. + \frac{\alpha_s N_c}{4\pi} \left[-\chi''(n, \gamma) - 2\chi(n, \gamma) \chi'(n, \gamma) + 6\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{\mathcal{U}}(n, \gamma) \rangle \end{aligned}$$

The coincidence of terms with the nontrivial γ dependence proves that there is no additional $O(\alpha_s)$ correction to the vertex of the gluon-shock wave interaction coming from the small loop inside the shock wave

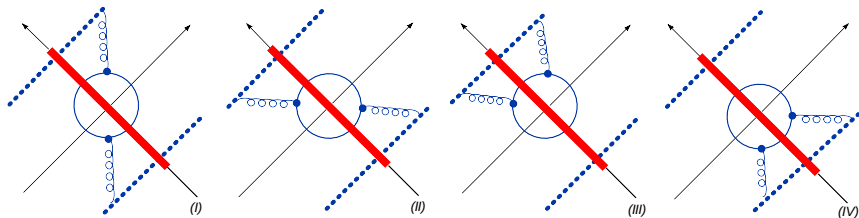
From BK NLO side

The extra term $\frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr} U_x U_y^\dagger$ would contradict the requirement $\frac{d}{d\eta} U_x U_Y^\dagger = 0$ at $x = y$

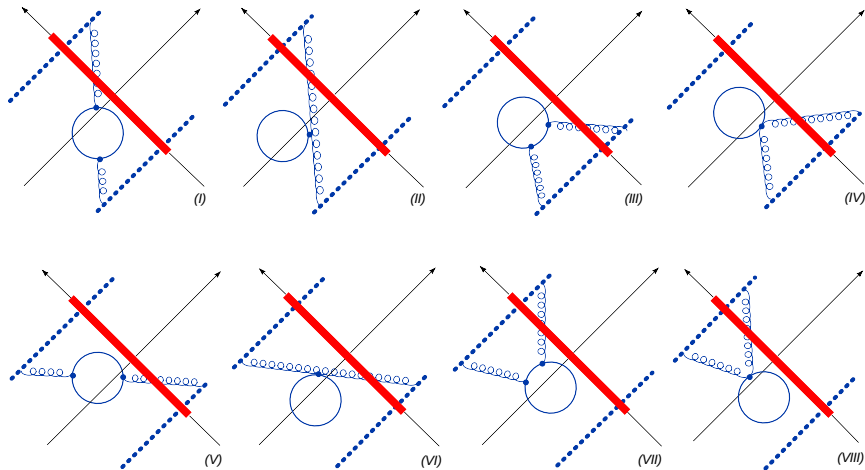
From BFKL NLO side

The coefficient $6\zeta(3)$ agrees with the $j \rightarrow 1$ asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators

$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{12\pi} (1 - \pi^2) \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \\
 &+ \frac{\alpha_s}{16\pi^2} \int d^2z' \left[\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &\quad \left. - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z) \right] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big)
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{12\pi} (1 - \pi^2) \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \\
 &+ \frac{\alpha_s}{16\pi^2} \int d^2z' \left[\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &\quad \left. - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z) \right] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big)
 \end{aligned}$$

Conformal scheme-dependent part

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{12\pi} (1 - \pi^2) \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \\
 &+ \frac{\alpha_s}{16\pi^2} \int d^2z' \left[\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &\quad \left. - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z) \right] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big)
 \end{aligned}$$

Conformal scheme-dependent part + Non-conformal part

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{12\pi} (1 - \pi^2) \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \\
 &+ \frac{\alpha_s}{16\pi^2} \int d^2z' \left[\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &\quad \left. - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z) \right] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big)
 \end{aligned}$$

Conformal scheme-dependent part + Non-conformal part
 + Conformal analitic part

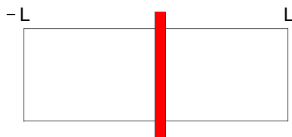
$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{12\pi} (1 - \pi^2) \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \\
 &+ \frac{\alpha_s}{16\pi^2} \int d^2z' \left[\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &\quad \left. - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z) \right] \\
 &\times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Bigg) + \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U_y^\dagger\} =
 \end{aligned}$$

Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL in $\mathcal{N} = 4$
Lipatov and Kotikov, 2004

(Comparing the forward kernel)

Checking for the missing term

We have checked the possible missing term using gauge/scalar links at infinity



Gauge/scalar links

$$\begin{aligned} \text{Tr}\{U_x U_y^\dagger\} &= \lim_{L \rightarrow \infty} \text{Tr}\{[Lp_1 + x_\perp, -Lp_1 + x_\perp][-Lp_1 + x_\perp, -Lp_1 + y_\perp] \\ &\times [-Lp_1 + y_\perp, Lp_1 + y_\perp][Lp_1 + y_\perp, Lp_1 + x_\perp]\} \end{aligned}$$

- The NLO kernel for the evolution of the color dipole has been calculated. It consists of three parts: the running-coupling part proportional to β -function, the conformal part describing $1 \rightarrow 3$ dipoles transition and the non-conformal term.
- The result agrees with the forward NLO BFKL kernel up to a term proportional $\alpha_s^2 \zeta(3)$ times the original dipole.
- For the creation of dipoles in the small- x evolution, the coupling constant is determined by the size of the smallest dipole.
- The NLO evolution kernel depends on the precise definition of the cutoff in the longitudinal momenta.
- With $|\alpha| < \sigma$ cutoff, the NLO-BK and the NLO-BFKL for $\mathcal{N} = 4$ is (almost) conformally invariant in the transverse plane.

- Comparing the forward kernels also in the QCD case
- Leading BK is conformal, what about K_{NLO} in the $\mathcal{N} = 4$?