CHAPTER 7

Stresses: Beams in Bending

7.1 General Remarks

The organization of this chapter mimics that of the last chapter on torsion of circular shafts but the story about stresses in beams is longer, covers more territory, and is a bit more complex. In torsion of a circular shaft, the action was all shear; contiguous cross sections sheared over one another in their rotation about the axis of the shaft. Here, the major stresses induced due to bending are normal stresses of tension or compression. But the state of stress within the beam is more complex for there are shear stresses generated in a beam in addition to the major *normal stresses due to bending* although these are generally of *smaller order* when compared to the latter. Still, in some contexts they must be considered if failure is to be avoided.

Our study of the deflections of a shaft in torsion produced a relationship between the applied torque and the angular rotation of one end of the shaft about its longitudinal axis relative to the other end of the shaft. This had the form of a stiffness equation for a linear spring, or truss member loaded in tension, i.e.,

$$M_T = (GJ/L) \cdot \phi \qquad \text{is like} \qquad F = (AE/L) \cdot \delta$$

Similarly, the rate of rotation of circular cross sections was a constant along the shaft just as the *rate of displacement*, if you like, the extensional strain $\frac{\partial u}{\partial x}$ was constant along the truss member loaded solely at its ends.

We will construct a similar relationship between the moment and the radius of curvature of the beam in bending as a step along the path to fixing the normal stress distribution. We must go further if we wish to determine the transverse displacement and slope of the beam's longitudinal axis. The deflected shape will generally vary as we move along the axis of the beam, and how it varies will depend upon how the loading is distributed over the span Note that we could have considered a *torque per unit length* distributed over the shaft in torsion and made our life more complex – the rate of rotation, the $d\phi/dz$ would then not be constant along the shaft.

In subsequent chapters, we derive and solve a differential equation for the transverse displacement as a function of position along the beam. Our exploration of the behavior of beams will include a look at how they might *buckle*. Buckling is a mode of failure that can occur when member loads are well below the yield or fracture strength. Our prediction of *critical buckling loads* will again come from a study of the deflections of the beam, but now we must consider the possibility of *relatively large deflections*.

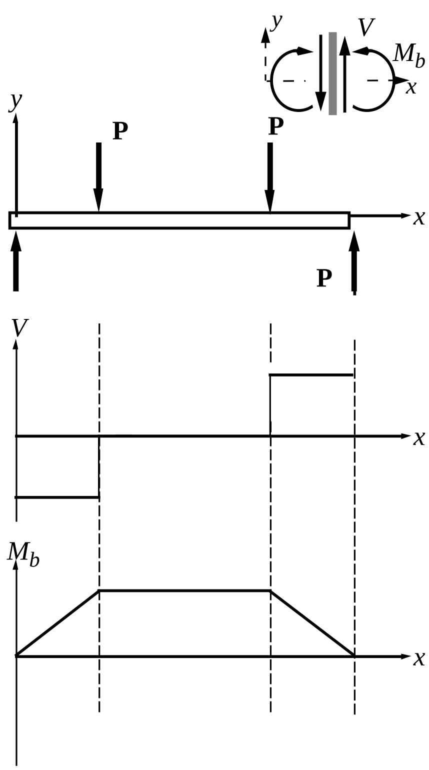
In this chapter we derive the normal stress distribution over the beam's cross-section. To do so, to resolve the indeterminacy we confronted back in chapter 3, we must consider the deformation of the beam.

7.2 Compatibility of Deformation

We consider first the deformations and displacements of a beam in *pure bending*. **Pure bending is said to take place over a finite portion of a span when the bending moment is a constant over that portion**. Alternatively, a portion of a beam is said to be in a state of pure bending when the shear force over that portion is zero. The equivalence of these two statements is embodied in the differential equilibrium relationship

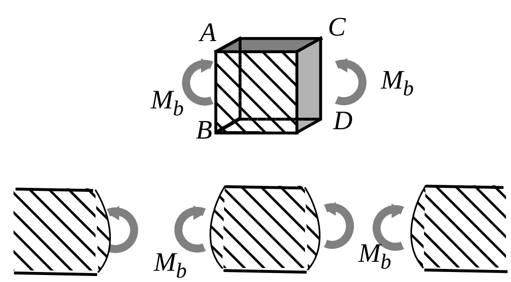
$$\frac{dM_b}{dx} = -V$$

Using symmetry arguments, we will be able to construct a displacement field from which we deduce a compatible state of strain at every point within the beam. The constitutive relations then give us a corresponding stress state. With this in hand we pick up where we left off in section 3.2 and relate the displacement field to the (constant) bending moment requiring that the stress distribution over a cross section be equivalent to the bending moment. This produces a *moment-curvature relationship*, a stiffness relationship which, when we move to the more general case of varying bending moment, can be read as a differential equation for the transverse displacement.



We have already worked up a *pure bending* problem; the *four point bending* of the simply supported beam in an earlier chapter. Over the midspan, $L/4 < x < 3L/4$, the bending moment is constant, the shear force is zero, the beam is in pure bending.

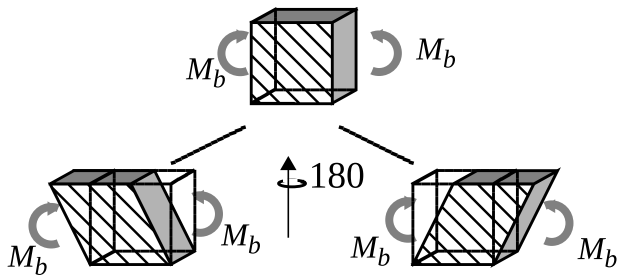
We cut out a section of the beam and consider how it might deform. In this, we take it as given that we have a beam showing a cross section symmetric with respect to the plane defined by z=0 and whose shape does not change as we move along the span. We will claim, on the basis of symmetry that **for a beam in pure bending, plane cross sections remain plane and perpendicular to the longitudinal axis**. For example, postulate that the cross section *CD* on the right does not remain plane but bulges out.



Now run around to the other side of the page and look at the section *AB*. The moment looks the same so section *AB* too must bulge out. Now come back and consider the portion of the beam to the right of section *CD*; its cross section too would be bulged out. But then we could not put the section back without gaps along the beam. This is an incompatible state of deformation.

Any other deformation *out of plane*, for example, if the top half of the section dished in while the bottom half bulged out, can be shown to be incompatible with our requirement that the beam remain all together in one continuous piece. **Plane cross sections must remain plane.**

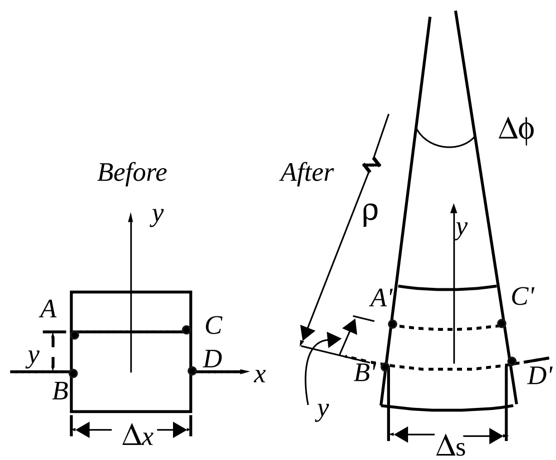
That cross sections remain perpendicular to the longitudinal axis of the beam follows again from symmetry – demanding that the same cause produces the same effect.



The two alternative deformation patterns shown above are equally plausible – there is no reason why one should occur rather than the other. But rotating either about the vertical axis shown by 180 degrees produces a contradiction. Hence they are both impossible. **Plane cross sections remain perpendicular to the longitudinal axis of the beam.**

The deformation pattern of a differential element of a beam in pure bending below is the one that prevails.

Here we show the plane cross sections remaining plane and perpendicular to the longitudinal axis. We show the longitudinal differential elements near the top of the beam in compression, the ones near the bottom in tension – the anticipated effect of a positive bending moment M_b, the kind shown. We expect then that there is some longitudinal axis which is neither compressed nor extended, an axis[1] which experiences no change in length. We call this particular longitudinal axis the *neutral axis*. We have positioned our x,y reference coordinate frame with the x axis coincident with this neutral axis.



We first define a *radius of curvature* of the deformed beam in pure bending.

Because plane cross sections remain plane and perpendicular to the longitudinal axes of the beam, the latter deform into arcs of concentric circles. We let the radius of the circle taken up by the neutral axis be ρ and since the differential element of its length, BD has not changed, that is $BD = B'D'$, we have

$$\rho \cdot \Delta\phi = \Delta s = \Delta x$$

where $\Delta\phi$ is the angle subtended by the arc $B'D'$, Δs is a differential element along the deformed arc, and Δx, the corresponding differential length along the undeformed neutral axis. In the limit, as Δx goes to zero we have, what is strictly a matter of geometry

where ρ is the *radius of curvature* of the neutral axis.

1. We should say "plane", or better yet, "surface" rather than "axis" since the beam has a depth, into the page.

$$\frac{d\phi}{ds} = \rho$$

Now we turn to the extension and contraction of a longitudinal differential line element lying off the neutral axis, say the element *AC*. Its extensional strain is defined by $\varepsilon_x(y) = \lim_{\Delta s \to 0} (A'C' - AC)/AC$

Now *AC*, the length of the differential line element in its undeformed state, is the same as the length *BD*, namely $AC = BD = \Delta x = \Delta s$ while its length in the deformed state is $A'C' = (\rho - y) \cdot \Delta\phi$

where *y* is the vertical distance from the neutral axis.

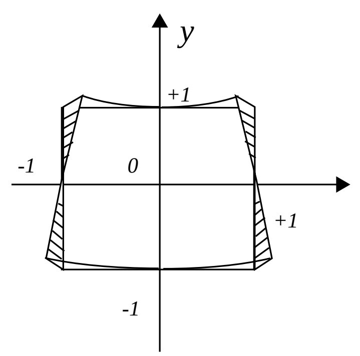
We have then $\varepsilon_x(y) = \lim_{\Delta s \to 0} [(\rho - y)\Delta\phi - \Delta s] = \lim_{\Delta s \to 0} -(y\Delta\phi/\Delta s)$
or, finally, using the fact that $\rho\Delta\phi = \Delta s$ we obtain

$$\varepsilon_x(y) = -y \cdot \left(\frac{d\phi}{ds}\right) = -(y/\rho)$$

We see that the strain varies linearly with *y*; the elements at the top of the beam are in compression, those below the neutral axis in extension. Again, this assumes a positive bending moment M_b. It also assumes that there is no other cause that might engender an extension or contraction of longitudinal elements such as an axial force within the beam. If the latter were present, we would *superimpose* a uniform extension or contraction on each longitudinal element.

The extensional strain of the longitudinal elements of the beam is the most important strain component in pure bending. The shear strain γ_{xy} we have shown to be zero; right angles formed by the intersection of cross sectional planes with longitudinal elements remain right angles. This too is an important result. Symmetry arguments can also be constructed to show that the shear strain component γ_{xz} is zero for our span in pure bending, of uniform cross section which is symmetric with respect to the plane defined by the locus of all points z=0.

In our discussion of strain-displacement relationships, you will find a displacement field defined by $u(x,y) = -\kappa xy$; $v(x,y) = \kappa x^2/2$ which yields a strain state consistent in most respects with the above. In our analysis of pure bending we have not ruled out an extensional strain in the *y* direction which this displacement field does. We repeat below the figure showing the deformed configuration of, what we can now interpret as a short segment of span of the beam.



If the κ is interpreted as the reciprocal of the radius of curvature of the neutral axis, the expression for the extensional strain ε_x derived in an earlier chapter is totally in accord with what we have constructed here. κ is called the *curvature* while, again, ρ is called the *radius of curvature*.

Summarizing, we have, for *pure bending* — the case when the bending moment is constant over the whole or some section of a beam — that plane cross-sections remain plane and perpendicular to the neutral axis, (or surface), that the neutral axis deforms into the arc of a circle of radius of curvature, ρ, and longitudinal elements experience an extensional strain ε where:

$$\frac{d\phi}{ds} = \rho$$

and

$$\varepsilon_x(y) = -(y/\rho)$$

7.3 Constitutive Relations

The stress-strain relations take the form

$$\varepsilon_x = (1/E) \cdot [\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad 0 = \sigma_{xy}/G$$
$$\varepsilon_y = (1/E) \cdot [\sigma_y - \nu(\sigma_x + \sigma_z)] \qquad 0 = \sigma_{xz}/G$$
$$\varepsilon_z = (1/E) \cdot [\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{yz} = \sigma_{yz}/G$$

We now assume that the stress components σ_z and σ_{yz} can be neglected, taken as zero, arguing that for beams whose cross section dimensions are small relative to the length of the span[1], these stresses can not build to any appreciable magnitude if they vanish on the surface of the beam. This is the ordinary *plane stress* assumption.

But we also take σ_y to be insignificant, as zero. This is a bit harder to justify, especially for a beam carrying a distributed load. In the latter case, the stress at the top, load-bearing surface cannot possibly be zero but must be proportional to the load itself. On the other hand, on the surface below, (we assume the load is distributed along the top of the beam), is *stress free* so σ_y, must vanish there. For the moment we make the assumption that it is negligible. When we are through we will compare its possible magnitude to the magnitude of the other stress components which exist within, and vary throughout the beam.

With this, our stress-strain relations reduce to three equations for the normal strain components in terms of the only significant stress component σ_x. The one involving the extension and contraction of the longitudinal fibers may be written

$$\boxed{\sigma_x(x) = E \cdot \varepsilon_x = -y \cdot (E/\rho)}$$

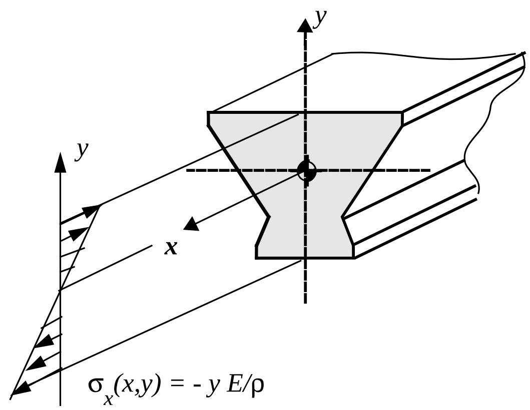
The other two may be taken as machinery to compute the extensional strains in the *y*,*z* directions, once we have found σ_x.

1. Indeed, this may be taken as a geometric attribute of what we allow to be called a beam in the first place.

7.4 The Moment/Curvature Relation

The figure below shows the stress component $\sigma_x(y)$ distributed over the cross-section. It is a linear distribution of the same form as that considered back in an earlier chapter where we toyed with possible stress distributions which would be equivalent to a system of zero resultant force and a couple.

But now we know for sure, for compatible deformation in pure bending, the exact form of how **the normal stress** must vary over the cross section. According to derived expression for the strain, ε_x, σ_x **must be a linear distribution** in *y*.



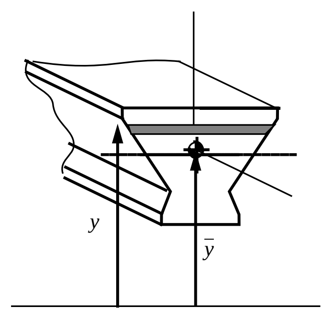
How this *normal stress due to bending* varies with *x*, the position along the *span* of the beam, depends upon how the *curvature*, $1/\rho$, varies as we move along the beam. For the case of pure bending, out analysis of compatible deformations tells us that the curvature is constant so that $\sigma_x(x,y)$ does not vary with *x* and we can write $\sigma_x(x,y) = \sigma_x(y)$, a (linear) function of *y* alone. This is what we would expect since the bending moment is obtained by integration of the stress distribution over the cross section: if the bending moment is constant with *x*, then σ_x should be too. We show this in what follows.

To relate the bending moment to the curvature, and hence to the stress σ_x, we repeat what we did in an earlier exploration of possible stress distributions within beams, first determining the consequences of our requirement that **the resultant force in the axial direction be zero, i.e.,**

$$\int_{Area} \sigma_x \cdot dA = -(E/\rho) \cdot \int_{Area} y \cdot dA = 0 \quad \text{so we must have} \quad \boxed{\int_{Area} y \cdot dA = 0}$$

But what does that tell us? It tells us that **the neutral axis, the longitudinal axis that experiences no extension or contraction, passes through the centroid of the cross section of the beam**. Without this requirement we would be left floating in space, not knowing where to measure *y* from. The centroid of the cross section is indicated on the figure.

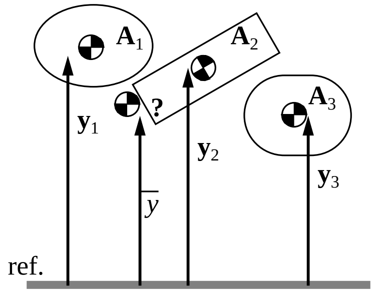
That this is so, that is, the requirement requires that our reference axis pass through the centroid of the cross section, follows from the definition of the location of the centroid, namely



$$\bar{y} \equiv \frac{\int_A y \cdot dA}{A}$$

If y is measured relative to the axis passing thru the centroid, then $\bar{y}$ is zero, our requirement is satisfied.

If our cross section can be viewed as a composite, made up of segments whose centroids are easily determined, then we can use the definition of the centroid of a single area to obtain the location of the centroid of the composite as follows.

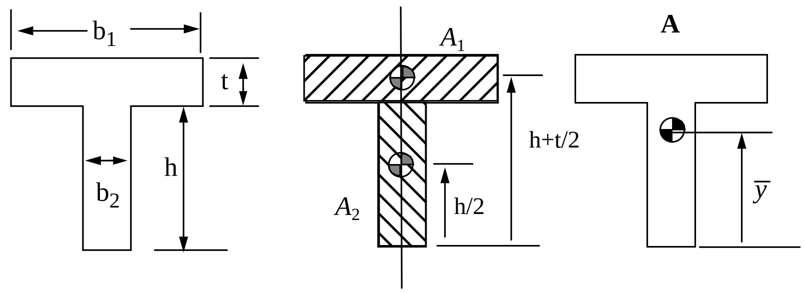


Consider a more general collection of segments whose centroid locations are known relative to some reference: From the definition of the location of the centroid we can write

$$\bar{y} = \frac{\int_1 y \cdot dA_1 + \int_2 y \cdot dA_2 + \int_3 y \cdot dA_3}{A_1 + A_2 + A_3} = \frac{\bar{y}_1 \cdot A_1 + \bar{y}_2 \cdot A_2 + \bar{y}_3 \cdot A_3}{A}$$

where A is the sum of the areas of the segments. We use this in an exercise, to wit:

Exercise 7.1 Determine the location of the neutral axis for the “T" cross-section shown.



We seek the centroid of the cross-section. Now, because the cross-section is symmetric with respect to a vertical plane perpendicular to the page and bisecting the top and the bottom rectangles, the centroid must lie in this plane or, since this plane appears as a line, it must lie along the vertical line, AA’. To find where along this vertical line the centroid is located, we first set a reference axis for measuring vertical distances. This could be chosen anywhere; I choose to set it at the base of the section.

I let $\bar{y}$ be the distance from this reference to the centroid, yet unknown. From the definition of the location of the centroid and its expression at the top of this page in terms of the centroids of the two segments, we have

$$\bar{y} \cdot A = (h/2) \cdot A_2 + (h + t/2) \cdot A_1$$

where

$$A = h \cdot b_2 + t \cdot b_1, \qquad A_2 = h \cdot b_2 \qquad \text{and} \qquad A_1 = t \cdot b_1$$

This is readily solved for $\bar{y}$ given the dimensions of the cross-section; the centroid is indicated on the figure at the far right.

Turning to the equivalence of our distribution to a couple, to the bending moment, we must have

$$-M_b = \int_{Area} y \cdot [\sigma_x dA]$$

where the term within the brackets is a differential element of force due to the stress distributed over the differential element of Area *b*(*y*)*dy*. The negative sign in front of M_b is necessary because, if σ_x were positive at *y* positive, on a positive *x* face, then the differential element of force $\sigma_x\, dA$ would produce a moment about the **negative** z axis, which, according to our convention for bending moment, would be a **negative** bending moment. Now substituting our known linear distribution for the stress σ_x, we obtain

$$M_b = (E/\rho) \cdot \int_{Area} y^2 \cdot dA$$

The integral is again just a function of the geometry of the cross section. It is often called **the moment of inertia about the *z* axis**. We will label it I[1]. That is

$$\boxed{I = \int_{Area} y^2 \cdot dA}$$

Our *Moment/Curvature relationship* is then:

$$M_b = \frac{(EI)}{\rho}$$

where the curvature is also defined by

$$\kappa = \frac{1}{\rho} = \frac{d\phi}{ds}$$

Here is a most significant result, very much of the same form of the stiffness relation between the torque applied to a shaft and the rate of twist — but with a quite different ϕ

$$M_T = (GJ) \cdot \frac{d\phi}{dz}$$

and of the same form of the stiffness relation for a rod in tension

$$F = (AE) \cdot \frac{du}{dx}$$

This moment-curvature relationship tells us the radius of curvature of an initially straight, uniform beam of symmetric cross-section, when a bending moment M_b is applied. And, in a fashion analogous to our work a circular shaft in torsion, we can go back and construct, using the moment curvature relation, an expression for the normal stress in terms of the applied bending moment. We obtain.

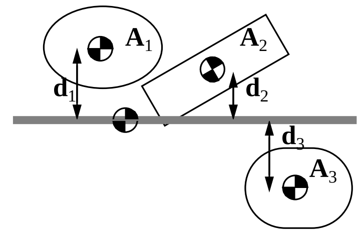
$$\boxed{\sigma_x(y) = -\frac{M_b(x) \cdot y}{I}}$$

Note again the similarity of form with the result obtain for the shaft in torsion, $\tau = M_T r/J$, (but note here the negative sign in front) and observe that the maximum stress is going to occur either at the top or bottom of the beam, whichever is further off the neutral axis (just as the maximum shear stress in torsion occurs at the outermost radius of the shaft).

Here we have revised Galileo. We have answered the question he originally posed. While we have done so strictly only for the case of "pure bending", this is no serious limitation. In fact, we take the above relationships to be accurate enough for the design and analysis of most beam structures even when the loading is not pure bending, even when there is a shear force present.

1. Often subscripts are added to I, e.g., I_{yy} or I_z; both are equally acceptable and/or confusing. The first indicates the integral is over y and y^2 appears in the integrand; the second indicates that the moment of inertia is "about the z axis", as if the plane area were rotating (and had inertia).

It remains to develop some machinery for the calculation of the moment of inertia, *I*, when the section can be viewed as a composite of segments whose "local" moments of inertia are known. That is, we need *a parallel axis theorem* for evaluating the moment of inertia of a cross-section.



We use the same composite section as above and seek the total moment of inertia of all segments with respect to the centroid of the composite. We first write *I* as the sum of the *I*'s

$$I = \int_{Area} y^2 \cdot dA = \int_{A_1} y^2 \cdot dA_1 + \int_{A_2} y^2 \cdot dA_2 + \int_{A_3} y^2 \cdot dA_3$$

then, for each segment, express *y* in terms of *d* and a local variable of integration, η

That is, we let $y = d_1 + \eta_1$ etc. and so obtain;

$$I = \int_{A_1} (d_1 + \eta_1)^2 \cdot dA_1 + \text{etc.} = \int_{A_1} (d_1^2 + 2d_1\eta_1 + \eta_1^2) \cdot dA_1 = d_1^2 A_1 + \int_{A_1} 2d_1\eta_1 \cdot dA_1 + \int_{A_1} \eta_1^2 \cdot dA_1 + \text{etc.}$$

Now the middle term in the sum on the right vanishes since $\int_{A_1} 2d_1\eta_1 \cdot dA_1 = 2d_1 \int_{A_1} \eta_1 \cdot dA_1$ and η_1 is measured from the local centroid. Furthermore, the last term in the sum on the right is just the local moment of inertia. The end result is the parallel axis theorem (employed three times as indicated by the "etc".)

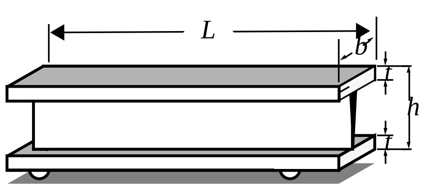
$$I = d_1^2 A_1 + I_1 \quad + \text{ etc.}$$

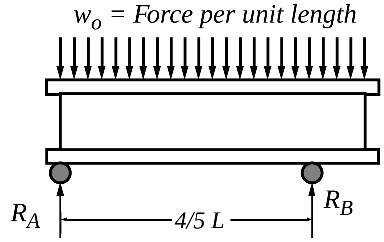
The bottom line is this: Knowing the local moment of inertia (with respect to the centroid of a segment) we can find the moment of inertia with respect to any axis parallel to that passing through the centroid of the segment by adding a term equal to the product of the area and the square of the distance from the centroid to the arbitrarily located, parallel axis. Note that the moment of inertia is a minimum when taken with respect to the centroid of the segment.

An example:

Exercise7.2 – The uniformly loaded "*I*" *beam* is simply supported at its left end and at a distance *L*/5 in from its right end.

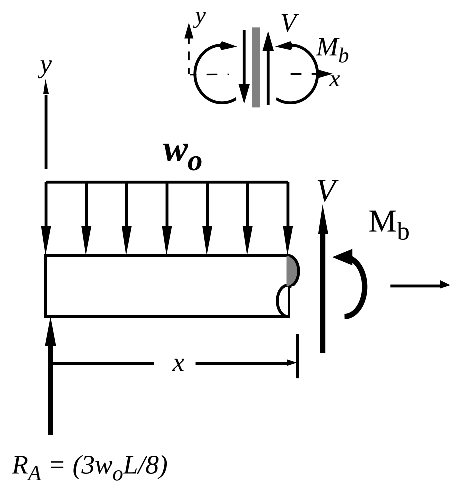
Construct the shear-force and bending-moment diagrams, noting in particular the location of the maximum bending moment. Then develop an *estimate* of the maximum stress due to bending.





An isolation of the entire span and requiring equilibrium gives the two vertical reactions,

$$R_A = (3/8)w_oL \qquad \text{and} \qquad R_B = (5/8)w_oL$$



A section of the beam cut between the two supports will enable the evaluation of the shear force and bending moment within this region. We have indicated our positive sign convention as the usual. Force equilibrium gives the shear force

$$V(x) = -(3/8)w_oL + w_ox$$

which, we note, passes through zero at $x = (3/8)L$ and, as we approach the right support, approaches the value $(17/40)w_oL$. At this point, the shear force suffers a discontinuity, a jump equal in magnitude to the reaction at *B*. Its value just to the right of the right support is then $-(1/5)w_oL$. Finally, it moves to zero at the end of the beam at the same rate, w_0, as required by the differential equilibrium relation

$$\frac{dV}{dx} = w_o$$

We could, if we wish, at this point use the same free body diagram above to obtain an expression for the bending moment distribution. I will not do this. Rather I will construct the bending moment distribution using insights gained from evaluating M_b at certain critical points and reading out the implications of the other differential, equilibrium relationship,

$$\frac{dM_b}{dx} = -V(x)$$

One interesting point is at the left end of the beam. Here the bending moment must be zero since the roller will not support a couple. Another critical point is at $x=(3/8)L$ where the shear force passes through zero. Here the **slope** of the bending moment must vanish. We can also infer from this differential, equilibrium relationship that the slope of the bending moment distribution is positive to the left of this point, negative to the right. Hence at this point the bending moment shows a local maximum. We *cannot* claim that it is the maximum over the whole span until we check all boundary points and points of shear discontinuity.

Furthermore, since the shear force is a linear function of *x*, the bending moment must be quadratic in this region, symmetric about $x=(3/8)L$. Now since the distance from the locus of the local maximum to the roller support at the right is greater than the distance to the left end, the bending moment will diminish to less than zero at the right support. We can evaluate its magnitude by constructing an isolation that includes the portion of the beam to the **right** of the support.

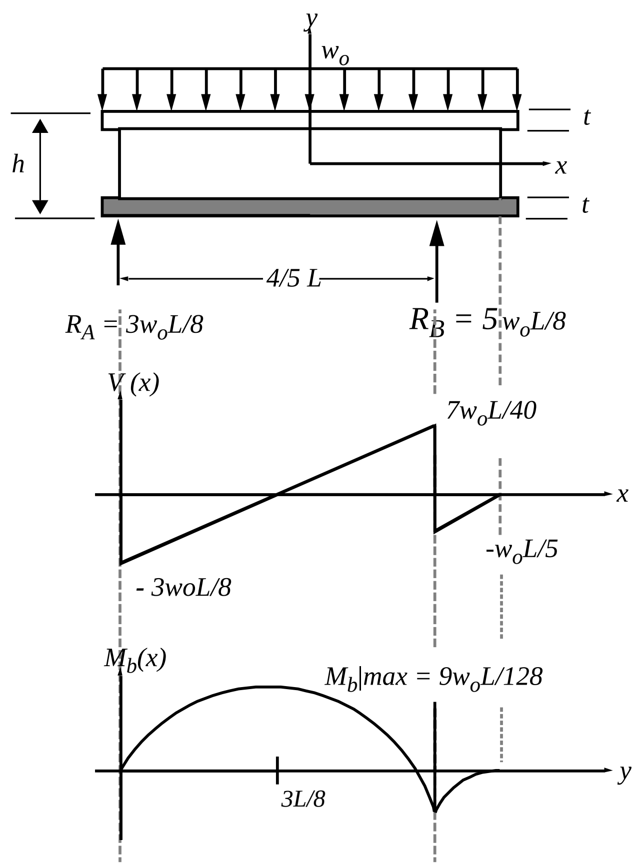
We find, in this way that, at $x=4L/5$, $M_b = -w_oL^2/50$.

At the right support there is a discontinuity in the slope of the bending moment equal to the discontinuity in the value of the shear force. The jump is just equal to the reaction force R_B. In fact the slope of the bending moment must switch from negative to positive at this point because the shear force has changed sign. The character of the bending moment distribution from the right support point out to the right end of the beam is fully revealed noting that, first, the bending moment must go to zero at the right end, and, second, that since the shear force goes to zero there, so must the slope of the bending moment.

All of this enables sketching the shear force and bending moment distributions shown. We can now state definitively that the maximum bending moment occurs

at $x = 3L/8$. Its value is

$$M_b|_{max} = (9/128)w_oL$$



The maximum stress due to this maximum bending moment is obtained from

$$\sigma_x|_{max} = -\frac{y \cdot (M_b|_{max})}{I}$$

It will occur at the top and bottom of the beam where $y = \pm h/2$, measured from the neutral, centroidal axis, attains its maximum magnitude. At the top, the stress will be compressive while at the bottom it will be tensile since the maximum bending moment is a positive quantity.

We must still evaluate the moment of inertia *I*. Here we will estimate this quantity assuming that $t < h$, *and/or b*, that is, the *web of the I beam* is thin, or has negligible area relative to the *flanges* at the top and bottom. Our estimate is then

$$I \approx 2 \cdot [(h/2)^2] \cdot [t \cdot b]$$

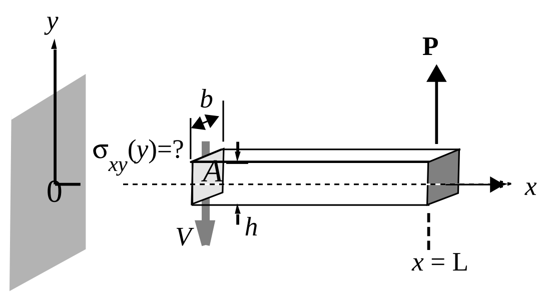
The last bracketed factor is the area of one flange. The first bracketed factor is the square of the distance from the y origin on the neutral axis to the centroid of the flange, or an estimate thereof. The factor of two out front is there because the two flanges contribute equally to the moment of inertia. How good an estimate this is remains to be tested. With all of this, our estimate of the maximum stress due to bending is

$$|\sigma_x|_{max} \approx (9/128) \cdot \frac{w_oL^2}{(thb)}$$

7.5 Shear Stresses in Beams

In this last exercise we went right ahead and used an equation for the normal stress due to bending constructed on the assumption of a particular kind of loading, namely, *pure bending*, a loading which produces no shear force within the beam. Clearly we are not justified in this assumption when a distributed load acts over the span. No problem. The effect of the shear force on the normal stress distribution we have obtained is negligible. Furthermore, the effect of a shear force on the deflection of the beam is also small. All of this can be shown to be accurate enough for most engineering work, at least for a true beam, that is when its length is much greater than any of its cross-section's dimensions.We will first show that the shear stresses due to a shear force are small with respect to the normal stresses due to bending.

Reconsider Galileo's end loaded, cantilever beam. At any section, *x*, a shear force, equal to the end load, which we now call P, acts in accord with the requirement of static equilibrium. I have shown the end load as acting up. The shear force is then a positive quantity according to our convention.



We postulate that this shear force is distributed over the plane cross section at *x* in the form of a shear stress σ_{xy}. Of course there is a normal stress σ_x distributed over this section too, with a resultant moment equal to the bending moment at the section. But we do not show that on our picture just yet.

What can we say about the shear stress distribution $\sigma_{xy}(y)$? For starters we can claim that it is only a function of *y*, not of z (nor of x with V(x) constant) as we have indicated. The worth of this claim depends upon the shape of the contour of the cross-section as we shall see. For a rectangular cross section it's a reasonable claim. We can also claim with more assurance that the shear stress must vanish at the top and bottom of the beam, at $y = \pm h/2$, because we know from chapter 3 that a differential element must not experience any resultant moment. Thus $\sigma_{xy} = \sigma_{yx}$ and at the top and bottom surfaces so σ_{yx} must vanish.

We expect then the shear stress to grow continuously to some finite value at some point in the interior. Whatever its distribution we expect that it will be relatively smooth, not jumping up and down as it varies with *y*. Its maximum value, in this case, ought not to be too different from its **mean** value defined by

$$\sigma_{xy}\big|_{mean} \approx \frac{V}{A} = \frac{P}{(bh)}$$

Now compare this with the maximum of the normal stress due to bending. Recalling that the maximum bending moment is *PL* at *x=0*, at the wall, and using our equation for pure bending, we find that

$$\sigma_x\big|_{max} = \frac{M_b \cdot y}{I} = \frac{PL(h/2)}{I}$$

I now evaluate the moment of inertia of the cross-section, I have $I = \int_{Area} y^2 \cdot dA = \int_{-h/2}^{h/2} y^2 \cdot dA$

This yields, with careful attention to the limits of integration,

$$I = \frac{bh^3}{12}$$

one of the few equations worth memorizing in this course.[1] The maximum normal stress due to bending is

$$\sigma_x\big|_{max} = 6\frac{PL}{(bh^2)} \text{ then}$$

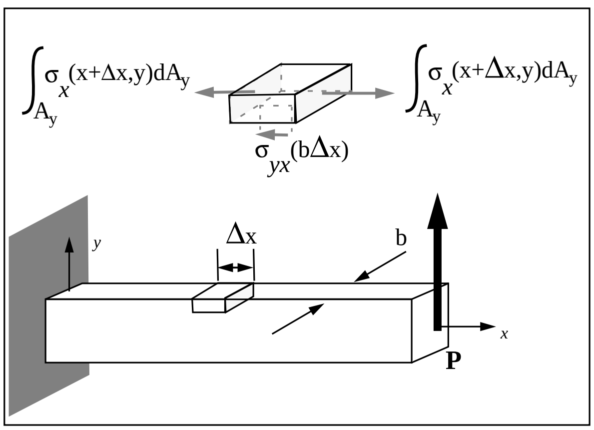
We observe:

- The units check; the right hand side has dimensions of stress, F/L^2. This is true also for our expression for the average shear stress.
- The ratio of the maximum shear stress to the maximum normal stress due to bending is *on the order of*

$$\sigma_{xy}\big|_{max} \big/ \sigma_x\big|_{max} = \text{Order } (h/L)$$

which if the beam is truly a beam, is on the order of 0.1 or 0.01 — as Galileo anticipated!

- While the shear stress is small relative to the normal stress due to bending, it does not necessarily follow that we can neglect it even when the ratio of a dimension of the cross section to the length is small. In particular, in built up, or *composite* beams excessive shear can be a cause for failure.



We next develop a more accurate, more detailed, picture of the shear stress distribution making use of an ingeneous free-body diagram. Look left.

We show the forces acting on a differential element of the cantilever, of length Δx cut from the beam at some y station which is arbitrary. (We do not show the shear stress σ_{xy} acting on the two "x faces" of the element as these will not enter into our analysis of force equilibrium in the x direction.

For force equilibrium in the x direction, we must have

$$\int_{A_y} \sigma_x(x + \Delta x, y)dA - \int_{A_y} \sigma_x(x, y)dA = \sigma_{yx}(y)b\Delta x$$

1. Most practitioners say this as "*bee* h cubed over twelve". Like "sigma is equal to *em y* over *eye*" it has a certain ring to it.

This can be written

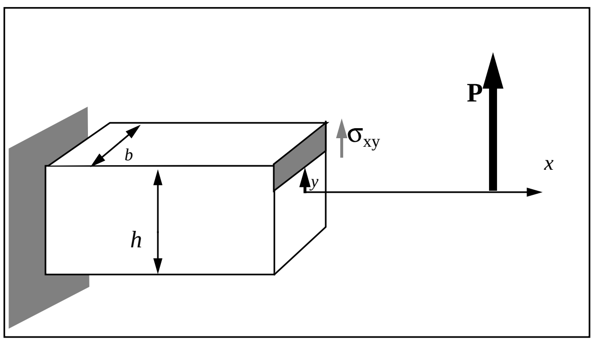
$$\int_{A_y} \frac{\sigma_x(x+\Delta x, y) - \sigma_x(x, y)}{\Delta x} \cdot dA = b\sigma_{yx}(y)$$

Which, as Δx approaches zero, yields $\int_{A_y} \frac{\partial}{\partial x}\sigma_x(x, y) \cdot dA = b\sigma_{yx}(y)$ Now, our engineering beam theory says

$$\sigma_x(x, y) = -\frac{M_b(x) \cdot y}{I} \qquad \text{and we have from before} \qquad \frac{d}{dx}M_b(x) = -V$$

so our equilibrium of forces in the direction of the longitudinal axis of the beam, on an oddly chosen, section of the beam (of length Δx and running from y up to the top of the beam). gives us the following expression for the shear stress σ_{yx} and thus σ_{xy} namely:

$$\sigma_{yx}(y) = \sigma_{xy}(y) = \frac{V}{bI} \cdot \int_{A_y} y \cdot dA$$



For a rectangular section, the element of area can be written

$$dA = b \cdot d\eta$$

where we introduce eta as our "y" variable of integration so that we do not confuse it with the "y" that appears in the lower limit of integration.

We have then, noting that the b's cancel:

$$\sigma_{xy}(y) = \frac{V}{I}\int_y^{h/2} \eta \cdot d\eta$$

which, when integrated gives $\sigma_{xy}(y) = \frac{V}{2I} \cdot \left[\left(\frac{h}{2}\right)^2 - y^2\right]$ which is a parabolic distribution with maximum at y=0. The maximum value is , once putting $I = bh^3/12$, $\sigma_{xy}(y) = \frac{3}{2} \cdot \frac{P}{bh}$ where we have assumed an end-loaded cantilever as in the figure.

This is to be compared with the average value obtained in our order of magnitude analysis. The order of magnitude remains essentially less than the maximum normal stress due to bending by a factor of (h/L).

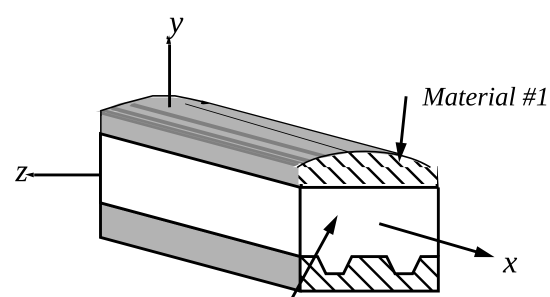
7.6 Stresses within a Composite Beam

A *composite* beam is composed of two or more elemental structural forms, or different materials, bonded, knitted, or otherwise joined together. *Composite materials or forms* include such heavy handed stuff as concrete (one material) reinforced with steel bars (another material); high-tech developments such as tubes built up of graphite fibers embedded in an epoxy matrix; sports structures like *laminated* skis, the poles for vaulting, even a golf ball can be viewed as a *filament wound* structure encased within another material. *Honeycomb* is another example of a composite – a *core material*, generally lightweight and relatively flimsy, maintains the distance between two *face sheets*, which are relatively sturdy with respect to *in-plane* extension and contraction.

To determine the moment/curvature relation, the normal stresses due to bending, and the shear stresses within a composite beam, we proceed through the *pure bending* analysis all over again, making careful note of when we must alter our constructions due to the *inhomogeneity* of the material.

Compatibility of Deformation

Our analysis of deformation of a beam in pure bending included no reference to the material properties or how they varied throughout the beam. We did insist that the cross-section be symmetric with respect to the z=0 plane and that the beam be uniform, that is, no variation of geometry or properties as we move in the longitudinal direction. A composite structure of the kind shown below would satisfy these conditions.



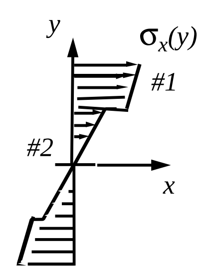
Constitutive Relations

We have two materials so we must necessarily contend with two sets of material properties. We still retain the assumptions regarding the smallness of the stress components σ_y, σ_z and τ_{yz} in writing out the relations for each material.

For material #1 we have $\sigma_x = -E_1 \cdot (y/\rho)$ while for #2 $\sigma_x = -E_2 \cdot (y/\rho)$

Equilibrium

Equivalence of this normal stress distribution sketched below to zero resultant force and a couple equal to the bending moment at any station along the span proceeds as follows:

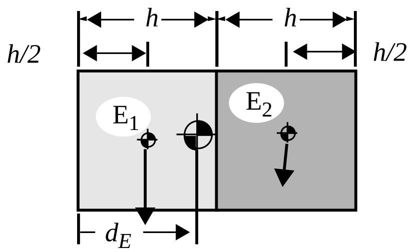


For zero resultant force we must have

$$\int_{Area_1} \sigma_x \cdot dA_1 + \int_{Area_2} \sigma_x \cdot dA_2 = 0$$

Upon substituting our strain-compatible variation of stress as a function of y into this we obtain, noting that the radius of curvature, ρ is a common factor,

$$E_1 \int_{Area_1} y \cdot dA_1 + E_2 \int_{Area_2} y \cdot dA_2 = 0$$



What does this mean? Think of it as a machine for computing the location of the unstrained, neutral axis, $y = 0$. However, in this case it is located, not at the centroid of the cross-sectional area, but at the *centroid of area weighted by the elastic moduli*. The meaning of this is best exposed via a short thought experiment. Turn the composite section over on its side. For ease of visualization of the special effect I want to induce I consider a composite cross section of two rectangular subsections **of equal area** as shown below. Now think of the elastic modulus as a weight density, and assume $E_1 > E_2$, say $E_1 = 4\,E_2$.

This last equation is then synonymous with the requirement that the location of the neutral axis is at the center of gravity of the elastic-modulus-as-weight-density configuration shown.[1] Taking moments about the left end of the tipped over cross section we must have

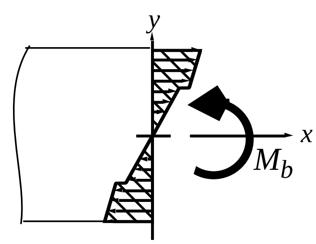
$$[AE_1 + AE_2] \cdot d_E = (AE_1) \cdot (h/2) + (AE_2) \cdot (3h/2)$$

With our assumed relative values for elastic modulus this gives, for the location of the *E-weighted centroid* $d_E = (7/10) \cdot h$

Note that if the elastic moduli were the same the centroid would be at *h*, at the mid point. On the other hand, if the elastic modulus of material #2 were greater than that of material #1 the centroid would shift to the right of the interface between the two.

Now that we have a way to locate our neutral axis, we can proceed to develop a moment curvature relationship for the composite beam in pure bending. We require for equivalence

$$-M_b = \int_{Area} y \cdot \sigma_x dA$$



as before, but now, when we replace σ_x with its variation with y we must distinguish between integrations over the two material, cross-sectional areas. We have then, breaking up the area integrals into A_1 over one material's cross section and A_2, the other material's cross section[2]

$$M_b = (E_1/\rho) \cdot \int_{A_1} y^2 \cdot dA_1 + (E_2/\rho) \cdot \int_{A_2} y^2 \cdot dA_2$$

The integrals again are just functions of the geometry. I designate them I_1 and I_2 respectively and write

$$M_b/[E_1 I_1 + E_2 I_2] = 1/\rho \qquad \text{or} \qquad M_b/(\overline{EI}) = 1/\rho$$

1. I have assumed in this sketch that material #1 is *stiffer*, its elastic modulus E_1 is greater than material #2 with elastic modulus E_2.

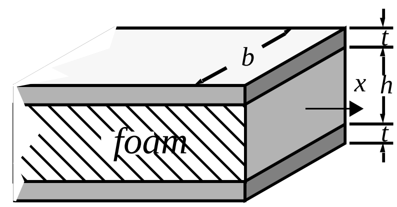
2. Note that we can have the area of either one or both of the materials distributed in any manner over the cross section in several non-contiguous pieces. Steel reinforced concrete is a good example of this situation. We still, however, insist upon symmetry of the cross section with respect to the *x-y* plane.

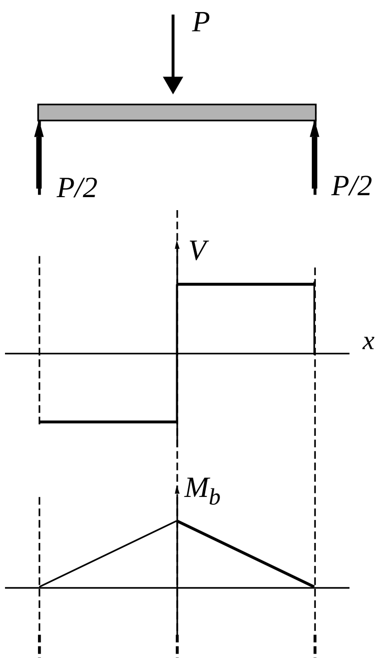
Here then is our moment curvature relationship for pure bending of a composite beam. It looks just like our result for a homogeneous beam but note

- Plane cross sections remain plane and perpendicular to the longitudinal axis of the beam. Compatibility of Deformation requires this as before.
- The neutral axis is located not at the centroid of area but at the centroid of the E-weighted area of the cross section. In computing the moments of inertia I_1, I_2 the integrations must use y measured from this point.
- The stress distribution is linear within each material but there exists a discontinuity at the interface of different materials. The exercise below illustrates this result. Where the maximum normal stress appears within the cross section depends upon the relative stiffnesses of the materials **as well as** upon the geometry of the cross section.

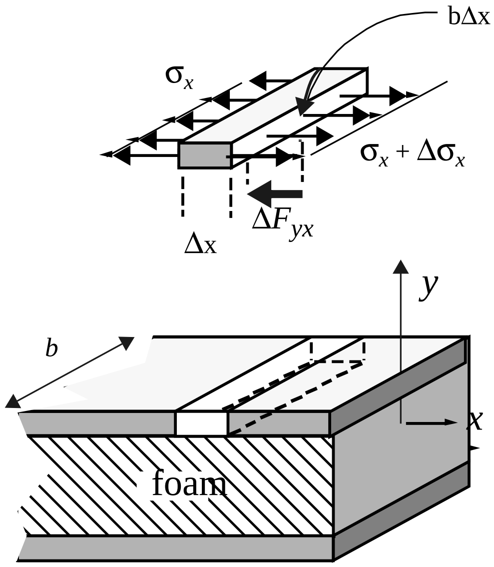
We will apply the results above to loadings other than pure bending, just as we did with the homogeneous beam. We again make the claim that the effect of shear upon the magnitude of the normal stresses and upon the deflected shape is small although here we are skating on thinner ice – still safe for the most part but thinner. And we will again work up a method for estimating the shear stresses themselves. The following exercise illustrates:

Exercise 7.3 – A composite beam is made of a solid polyurethane core and aluminum face sheets. The modulus of elasticity, E for the polyurethane is 1/30 that of aluminum. The beam, of the usual length L, is simply supported at its ends and carries a concentrated load P at midspan. If the ratio of the thickness of the aluminum face sheets to the thickness of the core is $t/h = 1/20$ develop an *estimate* for the maximum shear stress acting at the interface of the two materials.





We first sketch the shear force and bending moment diagram, noting that the maximum bending moment occurs at mid span while the maximum shear force occurs at the ends.



Watch this next totally unmotivated step. I am going to move to estimate the shear stress at the interface of the aluminum and the core. I show an isolation of a differential element of the aluminum face sheet alone. I show the normal stress due to bending and how it varies both over the thickness of the aluminum and as we move from x to $x+\Delta x$. I also show a differential element of a shear force ΔF_{yx} acting on the underside of the differential element of the aluminum face sheet. I **do not** show the shear stresses acting of the x faces; their resultant on the x face is in equilibrium with their resultant on the face $x + \Delta x$.

Equilibrium in the x direction will be satisfied if

$$\Delta F_{yx} = \int_{A_{al}} \Delta\sigma_x dA_{al}$$

where A_{al} is the cross-sectional area of the aluminum face sheet. Addressing the left hand side, we set

$$\Delta F_{yx} = \sigma_{yx} b \Delta x$$

where σ_{yx} is the shear stress at the interface, the quantity we seek to estimate. Addressing the right hand side, we develop an expression for $\Delta \sigma_x$ using our pure bending result. From the moment curvature relationship for a composite cross section we can write

$$M_b / (\overline{EI}) = 1/\rho$$

The stress distribution within the aluminum face sheet[1]

$$\sigma_x = -E_{al}\, y/\rho \qquad \textit{which is then} \quad \sigma_x = -E_{al}\, y\, M_b / (\overline{EI})$$

Taking a differential view, as we move a small distance Δx and noting that the only thing that varies with x is M_b, the bending moment, we have

$$\Delta\sigma = -[E_{al}\, y\, M_b / (\overline{EI})]]\Delta M_b(x)$$

1. Note the similarity to our results for the torsion of a composite shaft.

But the change in the bending moment is related to the shear force through differential equation of equilibrium which can be written $\Delta M_b = - V(x)\Delta x$. Putting this all together we can write

$$\sigma_{yx} \cdot b\Delta x = \left\{ \int_{A_{al}} E_{al} \cdot V(x) \cdot \frac{y}{(\overline{\overline{EI}})} dA_{al} \right\} \cdot \Delta x \qquad \text{or, in the limit} \qquad \boxed{\sigma_{yx} = \frac{E_{al}(V/b)}{\overline{\overline{EI}}} \int_{A_{al}} y \cdot dA_{al}}$$

This provides an estimate for the shear stress at the interface.

Observe:

- This expression needs elaboration. It is essential that you read the phrase $\int_{Aal} ydA$ correctly. First, y is to be measured from the E-weighted centroid of the cross section (which in this particular problem is at the center of the cross section because the aluminum face sheets are symmetrically disposed at the top and the bottom of the cross section and they are of equal area). Second, the integration is to be performed over the aluminum cross section only. More specifically, from the coordinate $y = h/2$, where one is estimating the shear stress up to the top of the beam, $y = h/2 + t$. This *first moment of area* may be approximated by

$$\int_{A_{al}} y \cdot dA_{al} = bt(h/2)$$

- The shear stress is dependent upon the change in the normal stress component σ_x with respect to x. This resonates with our derivation, back in chapter 3, of the differential equations which ensure equilibrium of a differential element.

- The *equivalent* $\overline{\overline{EI}}$ can be evaluated noting the relative magnitudes of the elastic moduli and approximating the moment of inertia of the face sheets as $I_{al} = 2(bt)(h/2)^2$ while for the foam we have $I_f = bh^3/12$. This gives

$$\overline{\overline{EI}} = (5/9)(E_{al} \cdot bth^2)$$

Note the consistent units; FL^2 on both sides of the equation. The foam contributes 1/9th to the *equivalent bending stiffness*.

- The magnitude of the shear stress at the interface is then found to be, with V taken as $P/2$ and the first moment of area estimated above,

$$\sigma_{yx}\Big|_{interface} = (9/10) \cdot \frac{P}{bh}$$

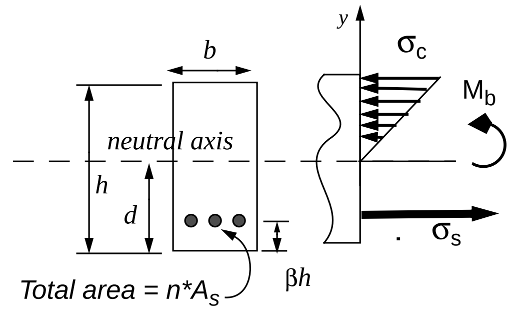
- The maximum normal stress due to bending will occur in the aluminum[1]. Its value is approximately

$$\sigma_x\Big|_{max} = (9/2) \cdot \frac{P}{bh} \cdot \frac{L}{h}$$

which, we note again is on the order of L/h times the shear stress at the interface.

1. The aluminum is stiffer; for comparable extensions, as compatibility of deformation requires, the aluminum will then carry a greater load. But note, the foam may fail at a much lower stress. A separation due to shear at the interface is a possibility too.

As a further example, we consider a steel-reinforced concrete beam which, for simplicity, we take as a rectangular section.



We assume that the beam will be loaded with a positive bending moment so that the bottom of the beam will be in tension and the top in compression.

We reinforce the bottom with steel rods. They will carry the tensile load. We further assume that the concrete is unable to support any tensile load. So the concrete is only effective in compression, over the area of the cross section above the neutral axis.

In proceeding, we identify the steel material with material #1 and the concrete with material #2 in our general derivation. We will write

$$E_1 = E_s = 30e06 \text{ psi} \qquad \text{and} \qquad E_2 = E_s = 3.6e06 \text{ psi}$$

The requirement that the resultant force, due to the tensile stress in the steel and the compressive stress in the concrete, vanish then may be written

$$\int_{A_s} \sigma_s \cdot dA_s + \int_{A_c} \sigma_c \cdot dA_c = 0 \quad \text{or} \qquad \int_{A_s} -E_s \cdot (y/\rho) \cdot dA_s + \int_{A_c} -E_c \cdot (y/\rho) \cdot dA_c = 0 \quad \text{which}$$

since the radius of curvature of the neutral axis is a constant relative to the integration over the area, can be written:

$$E_s \int_{A_s} y \cdot dA_s + E_c \int_{A_c} y \cdot dA_c = 0$$

The first integral, assuming that all the steel is concentrated at a distance (d - βh) *below* the neutral axis, is just

$$-E_s(d - \beta h)nA_s$$

where the number of reinforcing rods, each of area A_s, is n. The negative sign reflects the fact that the steel lies below the neutral axis.

The second integral is just the product of the distance to the centroid of the area under compression, (h-d)/2 , the area b(h-d), and the elastic modulus.

$$E_c \cdot \frac{(h-d)}{2} \cdot b(h-d)$$

The zero resultant force requirement then yields a quadratic equation for d, or d/h, putting it in nondimensional form. In fact

$$\left(\frac{d}{h}\right)^2 - (2+\lambda) \cdot \left(\frac{d}{h}\right) + (1 + \beta\lambda) = 0 \qquad \text{where we have defined} \qquad \lambda = \frac{2E_s n A_s}{E_c b h}$$

This gives
$$\frac{d}{h} = \frac{1}{2} \cdot [(2+\lambda) \pm \sqrt{(2+\lambda)^2 - 4(1+\beta\lambda)}]$$

There remains the task of determining the stresses in the steel and concrete. For this we need to obtain and expression for the equivalent bending stiffness, $\overline{EI}$.

The contribution of the steel rods is easily obtained, again assuming all the area is concentrated at the distance (d-bh) below the neutral axis. Then

$$I_s = (d - \beta h)^2 n A_s$$

The contribution of the concrete on the other hand, using the transfer theorem for moment of inertia, includes the "local" moment of inertia as well as the transfer term.

$$I_c = b(h-d) \cdot \left(\frac{h-d}{2}\right)^2 + \frac{b \cdot (h-d)^3}{12}$$

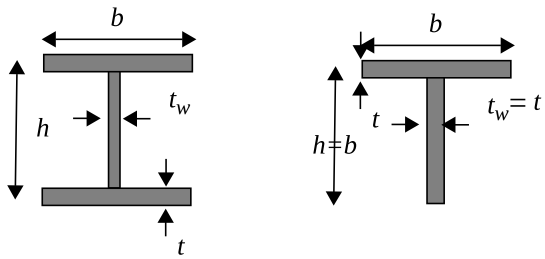
Then $\overline{EI} = E_s \cdot I_s + E_c \cdot I_c$ and the stress are determined accordingly, for the steel, by

$$\sigma_s\big|_{tension} = M_b \cdot E_s \cdot \frac{(d - \beta h)}{\overline{EI}}$$ while for the concrete $$\sigma_c\big|_{compression} = M_b \cdot E_c \cdot \frac{y}{\overline{EI}}$$

7.7 Problems - Stresses in Beams

7.1 In some of our work we have approximated the moment of inertia of the cross-section effective in bending by

$$I \sim 2\,(h/2)^2(bt)$$



It t/h ~ 0.01, or 0.1, estimate the error made by comparing the number obtained from this approximate relationship with the exact value obtained from an integration.

7.2 For a beam with a *T* section, as shown above right, Locate the centroid of the section.

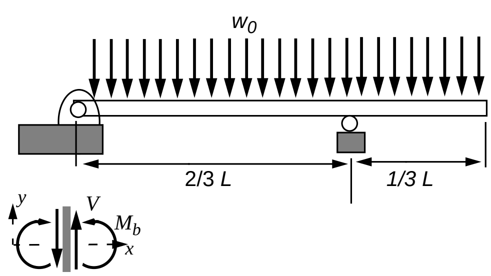
i) Construct an expression for the moment of inertia about the centroid.

ii) Locate where the maximum tensile stress occurs and express its magnitude in terms of the bending moment and the geometry of the section. Do the same for the maximum compressive stress. In this assume the bending moment puts the top of the beam in compression.

iii) If you take *b* equal to the *h* of the *I* beam, so that the cross-sectional areas are about the same, compare the maximum tensile and compressive stresses within the two sections.

7.3 A steel wire, with a radius of *0.0625 in*, with a yield strength of $120x10^3$ *psi*, is wound around a circular cylinder of radius $R = 20$ *in*. for storage. What if your boss, seeking to save money on storage costs, suggests reducing the radius of the cylinder to $R = 12in$. How do you respond?

7.4 A beam is pinned at its left end and supported by a roller at 2/3 the length as shown. The beam carries a uniformly distributed load, w_0 , <F/L>



i) Where does the maximum normal stress due to bending occur.

ii) If the beam has an I cross section with

flange width = .5”

section depth = 1.0 “

and t_w = t = 0.121 “

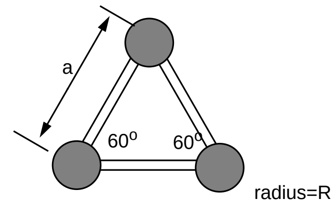
and the length of the beam is 36” and the distributed load is 2 lb/inch, determine the value of the maximum normal stress.

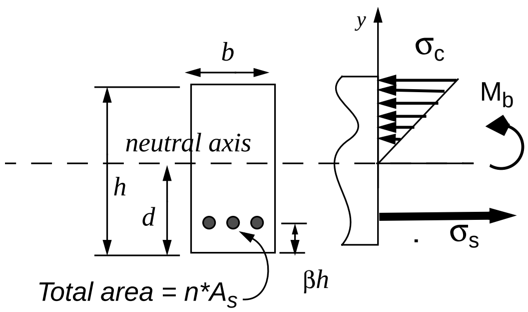
iii) What if the cross section is rectangular of the same height and area? What is the value of the maximum normal stress due to bending?

7.5 The cross-section of a beam made of three circular rods connected by three thin "shear webs" is shown.

i) Where is the centroid?

ii) What is the moment of inertia of the cross-section?





7.6 A steel reinforced beam is to be made such that the steel and the concrete fail simultaneously. If

E_s = 30 e06 psi steel

E_c= 3.6 e06 psi concrete

how must β be related to d/h for this to be the case?

Defining $$\lambda = \frac{2 \cdot E_s \cdot nA_s}{E_c \cdot bh}$$

find d/h and β values for a range of "realistic" values for the area ratio, (nAs/bh), hence for a range of values for Λ.

Make a sketch of one possible composite cross-section showing the location of the reinforcing rod. Take the diameter of the rod as 0.5 inches.