General Relativity from Special Relativity Using Tetrads

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February 23, 2005

Outline

The Equivalence Principle

In Words

Imagine you were in a giant elevator in space which was being towed by a rocket that accelerated at precisely 9.8 m/s. Could you tell that you weren't on earth sitting in a stuck elevator? Alternately, can you tell the difference between falling toward earth in a broken elevator and sitting in empty space?

The Equivalence Principle It is always possible to make a local coordinate transformation such that the laws of physics take their form in an inertial coordinate system at any point, **even in the presence of gravity**.

The Equivalence Principle II
In Words

In a non-uniform gravitational field, this cannot hold globally! (Think of two objects which start on opposite sides of the earth, and fall to its center. They cannot share the same inertial frame!) Another way to phrase the equivalence principle is in terms of spatially constant fields; locally, every field is spatially constant.

- We can no longer talk about “the gravitational field”; we can only talk about relative accelerations. (Since these will occur even after a coordinate transformation to a locally inertial frame.)

Tetrads

A mathematical statement of Equivalence:

The Equivalence Principle We can define four smooth 1-form fields, e^A_μ, which are projectors onto the local Lorentz frame axes of observers in an inertial coordinate system at every point.

Gravity will cause the Lorentz axes at different points to "mis-align" (as we shall see later).

Tetrads Are Great!

Tetrads are great because they let us use the Special Relativity we already know:

- Want $|v|$? Work in a Lorentz frame:

$$|v|^2 = |e^A_\mu v^\mu|^2 = \eta_{AB} e^A_\mu e^B_\nu v^\mu v^\nu$$

.

- They define a natural mapping (experimentally verified) between vectors and forms:

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu \qquad g^{\mu\nu} \to g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$$

From now on I will be somewhat caviler about which space I'm contracting in: $t^\mu u_\mu = t^A u_A$.

This is Really Nifty

We can now derive the geodesic equation for a freely falling particle. The particle has a path with parameterization σ; the coordinates of points on this path are $\xi^\mu(\sigma)$. The tangent vector at each point on the path is $D\xi^\mu(\sigma) \equiv \dot{\xi}^\mu(\sigma)$. Each inertial observer sees the particle move on the path which maximizes its proper time (this is known from SR). We therefore have the following Lagrangian:

$$L = \int d\sigma \sqrt{\eta_{AB} e^A_\mu e^B_\nu \dot{\xi}^\mu(\sigma) \dot{\xi}^\nu(\sigma)};$$

extremizing will give the geodesic equation (we'll define Γ later)

$$D^2 \xi^\mu(\sigma) + \Gamma^\mu_{\alpha\beta} D\xi^\alpha(\sigma) D\xi^\beta(\sigma) = 0.$$

Local Transformation Invariance

- We can always have our freely falling observer (inertial frame) make a Lorentz transformation; this can be local. It shouldn't change anything physical.
- We can always make a local coordinate transformation (i.e. change charts); it shouldn't make any physical difference.

Invariance under these two types of transformations will require us to re-consider derivatives.

What we Want from D_μ

1. Linearity.
2. Leibnitz rule: $D_\mu(ab) = aD_\mu b + bD_\mu a$.
3. Behave like ∂_μ on scalars: $D_\mu D_\nu \phi = D_\nu D_\mu \phi$. (Called "torsion free").
4. Respect contractions: $D_\mu v^A$ gives Lorentz vector components; $D_\mu v^\nu$ gives manifold vector components.
5. Want $t^\mu D_\mu a$ to measure the rate of change of a along curve with tangent t^μ.

∂_μ gives 1–3, but some cleanup needed for 4; 5 happens automatically.

Local Lorentz Invariance

Allow local Lorentz transformations: vector components v^A go to $\tilde{v}^A = \Lambda^A_B(x) v^B$. It shouldn't matter whether we take derivatives before or after Lorentz transformation:

$$\tilde{D}_\mu \Lambda^A_B(x) v^B = \Lambda^A_B D_\mu v^B.$$

Clear that $D_\mu = \partial_\mu$ won't work; the most general "fix up" we can make consistent with linearity is

$$D_\mu v^A = \partial_\mu v^A + \omega_{\mu B}^A v^B.$$

ω Transformations

To allow

$$\tilde{D}_\mu \Lambda^A_B(x) v^B = \Lambda^A_B D_\mu v^B.$$

require

$$v^C \partial_\mu \Lambda^A_C(x) + \tilde{\omega}^A_{\mu B} \Lambda^B_C v^C = \Lambda^A_B \omega^B_{\mu C} v^C.$$

Abstracting off of v^C (components are arbitrary), we have

$$\tilde{\omega}^A_{\mu B} = \Lambda^A_C \omega^C_{\mu D} \Lambda^{(-1)D}_B - \Lambda^{(-1)C}_B \partial_\mu \Lambda^A_C(x).$$

Conclusion: ω is not a Lorentz tensor. Field theorists would call ω the gauge connection; we call it the spin connection or the Cartan forms.

Local Coordinate Invariance

If we have a chart for a manifold x^μ, and we want to transform vectors which have components with respect to that chart v^μ to a second chart $\tilde{x}^\mu(x)$ (several trips back and forth to the manifold here):

$$\tilde{v}^\mu = (\partial_\nu \tilde{x}^\mu) v^\nu \equiv R^\mu_\nu v^\nu.$$

In order that our derivative respect local coordinate transformations, we must have

$$R^\alpha_\nu D_\mu v^\nu = \tilde{D}_\mu R^\alpha_\nu v^\nu;$$

again our fix up is

$$D_\mu v^\nu = \partial_\mu v^\nu - \Gamma^\nu_{\mu\alpha} v^\alpha.$$

Γ transformations

In order that

$$R_\nu^\alpha D_\mu v^\nu = \tilde{D}_\mu R_\nu^\alpha v^\nu,$$

we must have

$$R_\alpha^\beta R_\beta^{(-1)\gamma} \left[\partial_\gamma R_\delta^\mu(x)\right] v^\delta + R_\alpha^\beta \tilde{\Gamma}_{\beta\gamma}^\mu R_\delta^\gamma v^\delta = R_\nu^\mu \Gamma_{\alpha\delta}^\nu v^\delta.$$

And so on.... This isn't very illuminating, so I won't continue, but you get the idea.

Which Γ, ω?

So far, everything has been computing the change in Γ or ω when the coordinates change; how do we choose the right Γ and ω in a particular coordinate system?

Use Special Relativity! If we have a path with tangent vector components t^μ (or t^A, if you prefer), then the rate of change of a vector field with components v^A along the path is:

$$t^\mu D_\mu v^A.$$

If

$$t^\mu D_\mu v^A = 0,$$

then we say that the vector field v is parallel-transported along the curve. This is, at this point, a completely SR notion!

Parallel Transport -> Metric Compatibility

If we have two vector fields, v^A and w^A, each of which is parallel-transported along a curve with tangent vector t^μ, then our Special Relativity physics says that the inner product of these vector fields is constant along the path:

$$t^\mu D_\mu v^A w_A = t^\mu \partial_\mu v^A w_A = 0.$$

But, by the Leibniz rule and the assumption of parallel transport, we have

$$t^\mu D_\mu v^A w_A = t^\mu D_\mu \eta_{AB} e^A_\alpha e^B_\nu v^\alpha w^\nu = t^\mu v^\alpha w^\nu D_\mu \eta_{AB} e^A_\alpha e^B_\nu = 0.$$

Since all the vector components are arbitrary, this implies

$$D_\mu g_{\alpha\nu} = 0.$$

Γ

Through some index manipulation, we can show that this gives the usual expression for Γ:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{g^{\alpha\lambda}}{2}\left[\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right].$$

ω

To determine ω in terms of e, we require that parallel transport along a curve must be parallel transport in either the Lorentz frame or the coordinate frame:

$$t^\mu D_\mu v^A = 0 \implies t^\mu D_\mu v^\nu = 0,$$

which leads to a similar condition on ω:

$$D_\mu e_\nu^A = 0 = \partial_\mu e_\nu^A - \Gamma^\alpha_{\mu\nu} e_\alpha^A + \omega_{\mu B}^{A} e_\nu^B.$$

This gives

$$\omega_{\mu C}^{A} = -e_C^\nu \partial_\mu e_\nu^A + \Gamma^\alpha_{\mu\nu} e_\alpha^A e_C^\nu.$$

There is a way to eliminate the Γs in this definition which makes things look cleaner, but is not conceptually different.

Next Time?

- ω should be anti-symmetric in its Lorentz indices (because of $SO(3,1)$).
- Digression into field theory for a moment.
- Field strength tensor for ω is R!