

Hedging Demands under Incomplete Information

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Abstract

I present a model of consumption and portfolio choice under market incompleteness and imperfect information regarding the investment opportunity set. I solve analytically the consumption and portfolio choice problem for an investor learning about the true state of the economy. When prices are the only observations, the previously unspanned state variables are spanned by the market securities under the optimal inference/learning process. The market is observationally complete for the imperfectly informed investor. I show how learning affects both the covariance and the duration component of the hedging portfolio. I apply the model to the case where the Sharpe ratio is mean reverting. For the parameters presented in Wachter (2002), I show a reduction in hedging demands due to imperfect information. I solve in closed-form for the model implied R^2 for the return forecast regression. I discuss the relationship between the reduction in hedging demands and the reduction in the model implied R^2 for the return forecast regression.

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1 Introduction

This paper studies consumption and portfolio choice when markets are incomplete and investors cannot observe variables which determine the investment opportunity set. I establish conditions under which the investor's optimization problem under incomplete markets can be transformed into a complete markets problem. Investors use prices as noisy signals to infer the value of the unobservable state variables. The estimation process allows the agent to project the dynamics of the unobserved variables into the space of the securities. From the investors point of view markets are complete since the inferred processes for the state variables are spanned by the market securities. This allows me to apply martingale methods presented in Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) and solve the consumption and portfolio choice problem analytically.

I apply the model to the case where excess expected returns on the risky asset are mean reverting and unobservable. This setup is motivated by the standard assumption that excess returns to risky assets are a function of the volatility and the market price of risk. Several empirical studies, particularly Merton (1980), have shown volatility is easily estimated. Under this assumption we should care about parameter uncertainty for the market price of risk. Hence the problem boils down to uncertainty regarding the current Sharpe ratio. In this case, when the investors' inference does not reduce the estimation error for the unobservable variable, closed-form solutions are obtained. My results are novel in two dimensions. First, I show how parameter uncertainty, a reasonable assumption to make given the empirical evidence, can help us simplify the consumption and portfolio choice problem. Second, I can analytically show the role of imperfect information in the duration and covariance component of hedging demands. I calibrate to the implied model of mean reverting returns in Wachter (2002) based on the analysis in Barberis (2000). I find imperfect information reduces the hedging demand duration, the sensitivity of hedging demand to changes in the state variable, as well as the covariance component of hedging demand. The reduction in both components is due to the variance of the estimation error. For the calibration, the variance of the estimation error has a different sign than the covariance between the shocks to the state variable and the risky asset. Therefore, the variance of estimation error has a tempering effect in the hedging demand of the investor. I relate the changes in hedging demand to the model implied R^2 when the investor accounts for incomplete information. I find imperfect information reduces the model implied R^2 for future returns at any horizon. I find the reduction in the model implied

R^2 is also linked to the variance of the estimation error.

Evidence of predictability in asset markets has revived the consumption and portfolio choice literature. Recently, economists have focused on quantifying hedging demands due to changes in the investment opportunity set. Merton (1971) derives the existence of a hedging portfolio that accounts for changes in variables determining the attractiveness of future investment opportunities. At the time, the empirical evidence was unable to reject the hypothesis that asset prices followed a random walk. Without time varying returns, it followed naturally that portfolio choice should be entirely myopic and thus there would be no hedging component to the optimal asset allocation policy. Poterba and Summers (1988), Campbell and Shiller (1988) and Fama and French (1989) find evidence of predictability in the time series of asset prices. Lewellen (2001a) shows mean reversion in stock return may be even stronger than previously perceived. He shows that mean reverting component comprises more than 25% of stock returns. With abundant evidence of time-varying expected returns, Kim and Omberg (1996) study the role return predictability on the optimal asset allocation problem, finding closed form solutions for the hedging demands. Recently, Brennan (1998), Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2002), Chacko and Viceira (2001), Liu (2001), and Wachter (2002) extend this work in a variety of directions.

All of the papers mention above assume the current value in the estimate of expected returns is observable. Given the amount of evidence regarding predictability in asset prices and the difficulties associated with determining such predictability, any reasonable normative model of portfolio choice must hence acknowledge a role for parameter uncertainty and incomplete information. Bawa and Klein (1976) and Bawa, Brown, and Klein (1979) study the role of uncertainty in asset allocation. Kandel and Stambaugh (1996) extend the theory to consider uncertainty about the predictability in asset prices. They find that the predictive relation between returns and the dividend to price ratio, although statistically weak, is economically significant even in the presence of estimation risk. In other words, investors should account for predictability in the portfolio decision, hence it would be suboptimal for the investor to invest under the assumption of a random walk process for asset prices and ignore the role predictability should play in asset allocation even when the evidence of predictability is statistically weak.

The works of Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) lay the foundation of the portfolio choice problem under incomplete information. They show that the op-

timization problem where some parameters are unknown can be transformed into an optimization problem using the estimates of the unknown parameters and the price and state variable dynamics obtained by the inference problem. In continuous time, portfolio choice under incomplete information can then be solved in two steps. First, unobservable parameters are estimated by filtering signals from the observable data. Second, the investor chooses optimal consumption and portfolio policies given these estimates.

My paper adopts their setup and considers the optimal portfolio strategy when the current values of the state variables which define the investment opportunity set are not observable.¹ In related work, Barberis (2000) and Xia (2001), consider uncertainty regarding the relation between stock returns and the state variables. Unlike Barberis and Xia, I do not focus on the possibility that assets might not be predictable, instead, I focus on how uncertainty regarding the current value predictive variable changes the composition of the investor's consumption and portfolio choice. One interpretation of the model is that business cycles, as seen by time-varying expected returns, do occur in the economy, but we are unable to pinpoint where the business cycle currently stands. The assumption of unobservable state variables also proxies for the inability of investor to accurately measure the effect of macroeconomics changes in the level of stock returns.

Section 2 discusses the structure of the economy and solves the optimization problem of the agent in a partially observable economy in a general setting. I provide a simple application of the separation theorem, the filtering theory of Lipster and Shirayev (2001), and the complete markets portfolio choice methods of Cox and Huang (1989) as it applies to my model. In Section 3, I study stock price predictability under the assumption that the instantaneous Sharpe ratio is not observable and solve for the optimal consumption and portfolio policies. In Section 4, I calibrate the model to the VAR(1) specification of Barberis (2000).² I compare my results to Wachter (2002) where the investor assumes complete markets and show incomplete information has a strong effect in the portfolio choice of the agent. In Section 5, in the context of the example considered in

¹Recent articles in operations research address some of the issues raised in this paper. Lakner (1995,1998), Karatzas and Zhao (2001), and Rishel (1999) study the asset allocation problem under incomplete information. These papers do not consider the consumption aspect of an investor's strategic asset allocation problem.

²Campbell, Chacko, Rodriguez, and Viceira (2002) show, in the context of a consumption and portfolio choice model, how to correctly relate the discrete-time model of time-varying expected returns by Campbell and Viceira (2002) to the continuous-time models in order to obtain the correct parameter values for the continuous-time model.

Section 3, I simulate how an investor, with a given prior variance for the estimation error of the unobserved variable, learns about the variable and how the variance of the estimation error changes with each new observation. I show, given the amount of data available to the investor, changes in the variance of the estimation error are negligible, such that assuming steady state in the inference process is not as strong an assumption as might be initially expected.³ Section 6 derives the model implied R^2 and their link to the observed reduction in hedging demand. Section 7 concludes and offers a variety of extensions for the methodology presented in this paper including extensions for other asset allocation models and derivative replication strategies under imperfect information.

2 The Model

I develop a model of consumption and portfolio choice when markets are incomplete and there is uncertainty regarding the current value of the state variables. As shown by Merton (1971), state variables determine the investment opportunity set faced by the investor and the optimal portfolio policy contains a component to hedge the risks associated with those changes. I assume the investor cannot accurately forecast the current value of those variables, but has information to form an estimate of the value. Once the investor determines the forecast of the state variables and the estimation error, the market is complete under the information set of the investor. Market completeness under the subjective measure of the investor allows us to apply martingale methods and obtain analytical, exact solutions to the consumption and portfolio choice problem.

Consider a finite horizon investor with horizon T . Assume the existence of a single consumption good and assume the consumption good is the numeraire. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we define a d_Z -dimensional orthogonal Brownian Motion Z and a d_W -dimensional orthogonal Brownian Motion W . Let \mathbb{F} denote the filtration generated by the Brownian Motions (Z, W) . Assume the filtration is right-continuous and the probability space is complete. Assume the existence of a d_Z -dimensional orthogonal Brownian Motion Z and a d_W -dimensional orthogonal Brownian Motion W on the probability space such that \mathbb{F} is the standard filtration generated by Z and W . The Brownian Motions Z and W are assumed to be orthogonal

³Even under steady state inference, the investor does not observe the unobservable variable because under steady state inference the variance of the estimation error is positive. Even in the steady state the investor is not able to precisely estimate the unobserved variable.

to each other. For all Ito processes in this paper assume all drift coefficients are defined in \mathcal{L}^1 and all diffusion term coefficients are defined in \mathcal{L}^2 .⁴

2.0.1 Securities Market and State Variables

The securities market consists of a riskless asset, the money market account, which pays the locally riskless rates at all times, and N risky securities which span Z , the Brownian motion related to shocks in asset prices. The money market account grows at the riskless rate of return. The price of money market account satisfies

$$dB_t = r_t B_t dt, \quad (1)$$

where r_t is the locally riskless rate of return.

The prices for the risky securities follow the multidimensional Ito process

$$dS_t = \underbrace{\text{diag}(S_t)}_{N \times N} \left[\underbrace{\mu_{S_t}}_{N \times 1} dt + \underbrace{\sigma_{S_t}}_{N \times d_Z} dZ_t \right], \quad (2)$$

where $\mu_{S_t} \in (\mathcal{L}^1)^N$ and $\sigma_{S_t} \in (\mathcal{L}^2)^{N \times d_Z}$. Assume the dimension of the Z is equal to the rank of σ_{S_t} almost surely. The drift component represents the instantaneous expected return for the asset, while the diffusion is defined as the volatility of the asset.

Changes in the investment opportunity set of the agent are represented by a vector X_t of state variables. The state variables satisfy the following multidimensional Ito process:

$$dX_t = \underbrace{\mu_{X_t}}_{S \times 1} dt + \underbrace{\sigma_{X_t}}_{S \times d_Z} dZ_t + \underbrace{\sigma_{W_t}}_{S \times d_W} dW_t, \quad (3)$$

where $\mu_{X_t} \in (\mathcal{L}^1)^S$, $\sigma_{X_t} \in (\mathcal{L}^2)^{S \times d_Z}$, and $\sigma_{W_t} \in (\mathcal{L}^2)^{S \times d_W}$ and the Brownian motion vectors Z and W are orthogonal. The market is incomplete as long as the dimension of W is greater than zero. Some of the state variables might not be observable. I will assume that the number of unobservable parameters is equal to the difference between the total number of shocks and the number of shocks spanned by market securities. In other words, the rank of σ_{W_t} is equal to d_W .

⁴ Assume the following definition for the sets described in the paper hold:

$$\mathcal{L}^1 = \left\{ X \in \mathcal{L} : \int_0^T |X_t| dt < \infty \text{ a.s.} \right\},$$

$$\mathcal{L}^2 = \left\{ X \in \mathcal{L} : \int_0^T X_t^2 dt < \infty \text{ a.s.} \right\}.$$

2.0.2 Investors Preferences and Budget Constraint

The investor's preferences are assumed to satisfy the standard constant relative risk aversion, power utility function:

$$u(C_t) = e^{-\phi t} \frac{C_t^{1-\gamma}}{1-\gamma} dt, \quad (4)$$

where γ is the coefficient of relative risk aversion and ϕ is the agent's discount rate. Denote α_t as the vector of portfolio weight for the investor's optimal investment strategy in the risky assets. The investors' budget satisfies:

$$dW_t = W_t \{ [r_t + \alpha_t' (\mu_{St} - r_t \iota)] dt + \alpha_t' \sigma_{St} dZ_t \} - C_t dt \quad (5)$$

and the investor is subject to a non-negative wealth constraint.

2.1 Solution for the Model

This section presents the solution for the investor's optimization problem. The agent optimization problem is to maximize (4) subject to (5) and the non-negative wealth constraint under the filtered processes. Similar to Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986), the investor's consumption and portfolio choice problem follows two steps: (1) an inference problem in which the investor updates his or her estimate of the unobservable state variables, (2) an optimization problem in which the investor chooses her optimal consumption and portfolio policies under the new estimate for the unobservable state variables. In this section I solve the investor's inference problem and optimization problem. A second solution method is provided in the Appendix.

2.1.1 Inference Problem

Assume the drifts of the stock price processes in (2) is given by

$$\mu_{St} = \underbrace{\beta_{0t}}_{N \times 1} + \underbrace{\beta_{Xt}}_{N \times S} X_t, \quad (6)$$

and the drifts of the state variables processes in (3) satisfy

$$\mu_{Xt} = \underbrace{a_{0t}}_{S \times 1} + \underbrace{a_{Xt}}_{S \times S} X_t. \quad (7)$$

Equations (6) and (7) represent an economy where returns are time-varying. Equation (6) assume a linear relation modeled between expected returns and the predictive variable. Since some of the state variables are not observable, the instantaneous expected return is not directly observable. One interpretation of the model is to think of the assets in this economy as being either “good” or “bad” investments, depending on whether their current expected return is above or below their long run expected return, but the investor cannot determine exactly the current expected return of the assets.

The inference problem is solved with filtering methods covered in Lipster & Shirayev (2001). I follow their treatment as it applies to our model. Assume the investor observes instantaneous returns to the money market account (1) and the equity (2). Assume the investor also knows $\sigma_{St}, \sigma_{Xt}, \sigma_{Yt}, \beta_{0t}, \beta_{Xt}, a_{0t}, a_{1t}$. However the investor does not observe the current state of X_t . In other words, prices are the only signals investors have regarding the investment opportunity set. If the investor commits to high-frequency trading, prices serve as the natural choice for information regarding the investment opportunity set.⁵

Let X_0 be the investor’s prior, such that $X_0 \sim N(\widehat{X}_0, v_0)$, where v_0 represents the investors’ prior variance-covariance matrix for the state variables. In terms of the filtering literature, equations (1) and (2) are the observation equations and (3) are the system equations. The filtering theory for continuous time developed by Lipster and Shirayev, allows us to describe the dynamics of the mean and the variance of the distribution of the unobservable stochastic process X_t . The instantaneous changes in the drift and the variance-covariance matrix of X_t are given by:

$$d\widehat{X}_t = \left[a_{0t} + a_{Xt}\widehat{X}_t \right] dt + \left[\sigma_{Xt}\sigma'_{St} + v_t\beta'_{Xt} \right] \left[\sigma_{St}\sigma'_{St} \right]^{-1} \left[\text{diag}(S_t^{-1}) dS_t - \left(\beta_{0t} + \beta_{Xt}\widehat{X}_t \right) dt \right] \quad (8)$$

$$\frac{dv_t}{dt} = a_{Xt}v_t + v_t a'_{Xt} + \sigma_{Xt}\sigma'_{Xt} + \sigma_{Wt}\sigma'_{Wt} - \left[\sigma_{Xt}\sigma'_{St} + v_t\beta'_{Xt} \right] \left[\sigma_{St}\sigma'_{St} \right]^{-1} \left[\sigma_{Xt}\sigma'_{St} + v_t\beta_{Xt} \right]' \quad (9)$$

where \widehat{X}_t is the investor’s estimate of the unobservable state variable and v_t represents the variance of the estimation error for the unobservable state variable at time t .

When the agent has incomplete information, the agent’s portfolio hedging demand needs to account for the unobserved state variables, but also for the reduction in variance the estimation error as new observations come about. I assume inference has reached a steady state. In other

⁵An exception to the low frequency issues with predictive variable is trading volume. Recently Cremers (2002), considers the role of trading volume as a predictive variable.

words, the variance of the distribution for the estimated parameter does not change with each new observation. Thus $dv_t = 0$, and v_t does not need to be considered a state variable in the consumption and portfolio choice problem. Denote the steady state variance matrix as v_{ss} . From the definition of steady state variance and equation (9), v_{ss} is a positive definite matrix such that

$$0 = a_{Xt}v_{ss} + v_{ss}a'_{Xt} + \sigma_{Xt}\sigma'_{Xt} + \sigma_{Wt}\sigma'_{Wt} - [\sigma_{Xt}\sigma'_{St} + v_{ss}\beta'_{Xt}] [\sigma_{St}\sigma'_{St}]^{-1} [\sigma_{Xt}\sigma'_{St} + v_{ss}\beta_{Xt}]'.$$

In Section 5, I discuss the merits of the steady state learning assumption and show that with a reasonable amount of data, the variance of the estimation error is very close to the variance implied by the steady state results.

The new innovation process, defined as the normalized deviation of the return from its conditional estimated mean is given by

$$\sigma_{St}d\widehat{Z}_t = \left[\mu_{St} - \left(\beta_{0t} + \beta_{Xt}\widehat{X}_t \right) \right] dt + \sigma_{St}dZ_t \quad (10)$$

Although Z_t is not observable, the innovation process \widehat{Z}_t is derived from observable processes and is thus observable. The process (10) implies that the risky securities returns (2) are observable under the form

$$dS_t = S_t \left[\left(\beta_{0t} + \beta_{Xt}\widehat{X}_t \right) dt + \sigma_{St}d\widehat{Z}_t \right] \quad (11)$$

The dynamics for the state variables also become observable under the new innovation process. The state variables dynamics are given by the equation

$$d\widehat{X}_t = \left[a_{0t} + a_{Xt}\widehat{X}_t \right] dt + \left[\sigma_{Xt}\sigma'_{St} + v_{ss}\beta'_{Xt} \right] \left(\sigma_{St}\sigma'_{St} \right)^{-1} \sigma_{St}d\widehat{Z}_t \quad (12)$$

As long as the securities span the rank of \widehat{Z} , the investor's own state price density is uniquely defined. It is this result which will allow us to tackle the optimization problem with martingale methods. The assumption of steady state variance allows us to reduce the state variable space considerably and in some cases solve the optimization problem in closed form. The assumption of steady state variance formalizes the decision not to have the variance of the estimation error as a state variable.⁶

⁶Barberis (2000) also reduces the state space by assuming the investor's learning does not reduce the variance of the estimation error once the investor starts investing.

2.1.2 Optimization Problem

As far as the investor is concerned, the stochastic changes to the price and the state variable are perfectly correlated because the price serves as the signal of the state variable. After filtering the unobservable processes and assuming the variance of the estimates of the state variables reaches its steady state, the securities span the number of observable Brownian Motions. The investor has a uniquely determined stochastic discount factor, therefore I can apply martingale methods developed in Cox-Huang (1989) to solve for the agent's optimal consumption and portfolio choice.

The agent assumes the prices of the money market account and the risky securities are given by the equations

$$dB_t = B_t [r_t dt], \quad (13)$$

$$dS_t = S_t \left[\hat{\mu}_{S_t} dt + \sigma_{S_t} d\hat{Z}_t \right], \quad (14)$$

where $\hat{\mu}_{S_t}$ is chosen to match equation (11). Also, the investor assumes the state variables satisfy the following equation

$$d\hat{X}_t = \hat{\mu}_{X_t} dt + \hat{\sigma}_{X_t} d\hat{Z}_t, \quad (15)$$

where $\hat{\mu}_{X_t}$ and $\hat{\sigma}_{X_t}$ are chosen to match equation (12).

Although the market is incomplete, a unique stochastic discount factor can be defined for the investor. Under the information set of the investor, the state variables are spanned by the securities, therefore the investor can define a stochastic discount factor. Let M_t be the stochastic discount factor, the process for M_t must satisfy the following condition: $M_t B_t$ and $M_t S_t$ are martingales. I define a stochastic discount factor that satisfies the martingale properties under the information set of the investor. Denote by \widehat{M}_t the stochastic discount factor under the investor's information set such that $\widehat{M}_t B_t$ and $\widehat{M}_t S_t$ are martingales. Assuming that \widehat{M}_t follows an Ito process, an application of Ito's Lemma given (13) and (14) yields the following process for the stochastic discount factor:

$$\frac{d\widehat{M}_t}{\widehat{M}_t} = -r_t dt - \hat{\eta}_t d\hat{Z}_t, \quad \widehat{M}_0 = 1 \quad (16)$$

where

$$\hat{\eta}_t = \left(\sigma'_{S_t} \sigma_{S_t} \right)^{-1} \sigma'_{S_t} (\hat{\mu}_{S_t} - r_t \iota)$$

and $\hat{\eta}_t$ satisfies Novikov's Condition and ι represents a vector of ones. Note $\hat{\eta}_t$ is the investor's estimate of the Sharpe ratio and a affine function of \hat{X}_t . Equation (16) can be solved to obtain \widehat{M}_t

in its exponential form:

$$\widehat{M}_t = \exp \left\{ - \int_0^t r_s ds - \int_0^t \widehat{\eta}_s d\widehat{Z}_s - \frac{1}{2} \int_0^t \|\widehat{\eta}_s\|^2 ds \right\}.$$

Similar to the incomplete markets consumption and portfolio choice model of He and Pearson (1991), the investor in my model is able to determine a unique stochastic discount factor, but unlike He and Pearson, the stochastic discount factor for the investor is straightforward to obtain and does not require the use of a dual problem. Although the investor has a unique stochastic discount factor, this does not imply it is the unique discount factor for the economy. Basak (2000) studies a dynamic equilibrium model of heterogeneous beliefs and finds individual-specific Arrow-Debreu prices can differ. Similar to Basak, the investor has a uniquely define stochastic discount factor based on their beliefs on certain parameters in the economy. Therefore even when markets are incomplete, the consumption and portfolio choice problem of the investor can be solved with martingale methods.

Let the superscript I denote operations taken under the information set of the investor. Given the process governing the dynamics of the stochastic discount factor, the agent's optimization problem can be solved with martingale methods. As stated previously in this section, the agent's optimization problem is to maximize the expected lifetime utility of consumption J_t where

$$J_t = \sup_{\{\alpha_s, C_s\}} E_t^I \left[\int_t^T e^{-\phi(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \right] \quad (17)$$

subject to the dynamic budget constraint under the estimated processes for the securities and a non-negative wealth constraint. The existence of the stochastic discount factor allows us to write the agent's dynamic budget constraint as a static budget constraint given by

$$E_s^I \left[\int_t^T \widehat{M}_s C_s ds \right] \leq W_0. \quad (18)$$

where the expectation is defined under the investor's information set as represented by the results of the inference process described previously. Equation (18) states the agent's expected consumption stream in the future appropriately discounted will be less than or equal to his current wealth.

The investor's problem can now be solved as a static optimization problem as described in Cox and Huang (1989) and Karatzas and Shreve (1998, Chapter 3). Intuitively, since the stochastic discount factor is well defined for the investor, the investor can dynamically trade the long-lived

securities to obtain the optimal consumption profile in a manner similar to that of an investor with access to complete markets. Thus, there is no uncertainty regarding the consumption and portfolio choice of the agent conditional of knowing the state, the only uncertainty that remains is the realization of a given state. The investor's portfolio allocation changes accordingly with the optimal consumption choice.

The first order condition for utility maximization under the budget constraint is given by

$$C_s = \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1}{\gamma}}, \quad (19)$$

where λ_t is the Lagrangian multiplier. Notice also $\lambda_t^{-\frac{1}{\gamma}}$ represents the choice of consumption at time t given the information of the agent at time t . Substituting the first order condition for consumption (19) into the static budget constraint (18) yields the following expression for the static budget constraint:

$$W_t = E_t^I \left[\int_t^T \frac{\widehat{M}_s}{M_t} \left(\lambda_t \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}(s-t)} ds \right]. \quad (20)$$

Equation (20) states that wealth is a function of the stochastic discount factor and the processes that drive the distribution of the stochastic discount factor. As shown in (16), the only processes that matter for the distribution of the stochastic discount factor are the interest rate and the Sharpe ratio. I assume the sharpe ratio is a function of the state variable vector \widehat{X}_t , therefore the current values for both the stochastic discount factor and the estimated state variable determine the information set the agent uses in forming conditional expectations.⁷ The wealth at time $s > t$ following the optimal policy is given by

$$\begin{aligned} W_s &= E_s^I \left[\int_s^T \frac{\widehat{M}_u}{M_s} C_u du \right] \\ &= \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1}{\gamma}} E_s^I \left[\int_s^T e^{-\frac{\phi}{\gamma}(u-s)} \left(\frac{\widehat{M}_u}{M_s} \right)^{1-\frac{1}{\gamma}} du \right] \end{aligned} \quad (21)$$

Define the function $F_{s,u}$ as follows

$$F_{s,u} = E_s^I \left[\left(\frac{\widehat{M}_u}{M_s} \right)^{1-\frac{1}{\gamma}} \right] \quad (22)$$

$F_{s,u}$ is the agent's expectation of how the investment opportunity set is going to look like at time u , with a weighing function related to the risk aversion of the investor. Notice that for $\gamma = 1$,

⁷This is due to the fact that both M_t and \widehat{X}_t are Markov processes.

the expectation yields a constant, regardless of the value of the fraction.. As will be shown in the portfolio choice of the investor, $F_{s,u}$ is directly linked to the hedging demand component of the investor's portfolio.

Define $G_{s,T}$ as

$$G_{s,T} = \int_s^T e^{-\frac{\phi}{\gamma}(u-s)} F_{s,u} du \quad (23)$$

such that $G_{s,t}$ is also function of \widehat{X}_s . $G_{s,t}$ weighs the investor reaction to changes in the future investment opportunity set by the investor's impatience and risk aversion. If the investor discounts future utility heavily, then the investor assigns greater weigh to the short-term future investment opportunity set. Highly risk averse investors will penalize longer horizon opportunity set changes less since they care about maintaining a low variance in their consumption.

Let Q denote the investor specific risk-neutral measure. The wealth of the investor under the risk-neutral measure is given by

$$W_s = \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{\widehat{M}_t} \right)^{-\frac{1}{\gamma}} \int_s^T e^{-\frac{\phi}{\gamma}(u-s) - (1-\frac{1}{\gamma}) \int_s^u r_v dv} E_s^{I,Q} \left[\left(\frac{\widehat{M}_u}{\widehat{M}_s} \right)^{-\frac{1}{\gamma}} \right] du$$

Under the investor specific risk-neutral measure, the rate of return to wealth is equal to the instantaneous rate of return for the money market account, thus the drift of the wealth process, as obtained by the applying Ito's Lemma, must equal the locally riskless rate times the agent's current wealth. The equation above implies the following partial differential equation is solved by the agent's wealth function:

$$\begin{aligned} r_s W_s &= \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{\widehat{M}_t} \right)^{-\frac{1}{\gamma}} + \frac{\partial W_s}{\partial s} - \frac{\partial W_s}{\partial \widehat{M}_s} \widehat{M}_s \left(r_s + \frac{\widehat{\eta}'_s \widehat{\eta}_s}{2} \right) + \frac{\partial W_s}{\partial \widehat{X}_s} \left(a_{0s}^Q + a_{X_s}^Q \widehat{X}_s \right) \\ &+ \frac{1}{2} \frac{\partial^2 W_s}{\partial \widehat{M}_s^2} \widehat{M}_s^2 \widehat{\eta}'_s \widehat{\eta}_s + \frac{1}{2} tr \left[\widehat{\sigma}_{X_s} \widehat{\sigma}'_{X_s} \frac{\partial^2 W_s}{\partial \widehat{X}_s^2} \right] - \frac{\partial^2 W_s}{\partial \widehat{M}_s \partial \widehat{X}_s} \widehat{M}_s \widehat{\sigma}_{X_s} \widehat{\eta}_s \end{aligned} \quad (24)$$

where $a_{\cdot s}^Q$ are the coefficients of the drift for the state variables under the investor specific risk neutral measure and $tr(\cdot)$ is the trace function. Equation (24) is solved by

$$W_s = \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{\widehat{M}_t} \right)^{-\frac{1}{\gamma}} G_{s,T}. \quad (25)$$

where the boundary condition for (24) is given by

$$G_{s,s} = 1 \quad (26)$$

A functional form to the investor's wealth can be obtained by attempting to solve (24) or alternatively, equation (25) can be simplified furthermore as shown in the Appendix. One can also use the intuition in Wachter (2002) to solve the consumption and portfolio problem by considering the portfolio problem for each period separately and scaling the solution with the Lagrange multiplier obtained from the first order condition for consumption.

2.2 Portfolio Choice

Since the investor believes the markets are complete, I follow Cox-Huang (1989) to find the optimal asset allocation strategy for the investor. In complete markets, the portfolio allocation has to be such that the magnitude and direction stochastic changes in the wealth process are hedged by the portfolio allocation. The investor's percentage allocation of wealth to the risky assets is given by

$$\alpha'_t = \underbrace{\frac{\widehat{M}_t}{W_t} \frac{\partial W_t}{\partial \widehat{M}_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\eta}_t}_{\text{myopic demand}} + \underbrace{\frac{1}{W_t} \frac{\partial W_t}{\partial \widehat{X}_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\sigma}'_{Xt}}_{\text{hedging demand}}. \quad (27)$$

The portfolio choice of the agent can be decomposed into its myopic demand, the demand due to the current state of the economy, and the hedging demand, the demand due to expected changes in the investment opportunity set. In the model, the hedging demand is due to the stochastic nature of the estimated state variables. Both the myopic and hedging components are subject to the estimation risk due to the unobserved state variables. The myopic demand of the agent is affected by the estimated state variables by how those estimates change the investors perception of the current investment opportunity set as proxied by the Sharpe ratio. The hedging demand of the agent is affected by the investor's perception of the diffusion for the estimated state variables (how the investment opportunity set changes with time) as well as by the current estimate of the unobserved state variables. Given (27) and the second boundary condition in (25), write the investors' portfolio as

$$\alpha'_t = \frac{1}{\gamma} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\eta}_t + \frac{1}{G_t} \frac{\partial G_t}{\partial \widehat{X}_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\sigma}'_{Xt} \quad (28)$$

The function G which determines the magnitude of the hedging demand is the agent's current wealth to consumption ratio. As in the complete markets framework, the relation between the agent's current consumption relative to expected future consumption is related to how the agent

determines to hedge changes in the investment opportunity set. Although our model is one of incomplete markets, investors, via the inference problem, are able to obtain individual-specific Arrow-Debreu prices, therefore their behavior maps to that of a complete markets investor.

Write the hedging demand component of the agent's portfolio choice as

$$\alpha_t^{hedging} = \frac{\frac{\partial G_t}{\partial X_t}}{G_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\sigma}'_{Xt} \quad (29)$$

The magnitude of the hedging demand is given by the sensitivity of the wealth to consumption ratio to the state variables. The duration of the hedging component will change relative to the perfect information case due to the difference between the estimate and the true value of the variable and because of the estimation error. The hedging demand also depends on the perceived covariance between the state variables and the stock prices since

$$\sigma_{St} \widehat{\sigma}'_{Xt} = [\sigma_{Xt} \sigma'_{St} + v_{ss} \beta'_{Xt}]'$$

the covariance component of the hedging demand will also change due to the variance of the estimation error.

The value function can be used to obtain the optimal consumption policy to obtain the optimal portfolio policy of the agent. Define J_t as the indirect utility function, the indirect utility function obtained via the optimal consumption and portfolio policy solves

$$J_t = \sup_{\{C_t, \alpha_t\}} E_t^I \left[\int_t^T e^{-\phi(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \right] \quad (30)$$

I substitute consumption in (30) by the first order condition (19) to obtain

$$\begin{aligned} J_t &= \frac{1}{1-\gamma} E_t^I \left[\int_t^T e^{-\phi(s-t)} \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1-\gamma}{\gamma}} ds \right] \\ &= \frac{\lambda_t^{1-\frac{1}{\gamma}}}{1-\gamma} \int_t^T e^{-\frac{\phi}{\gamma}(s-t)} E_t^I \left[\left(\frac{\widehat{M}_s}{M_t} \right)^{1-\frac{1}{\gamma}} \right] ds \\ &= \frac{\lambda_t^{1-\frac{1}{\gamma}}}{1-\gamma} G_{t,T} \end{aligned}$$

From (20),

$$\lambda_t^{-\frac{1}{\gamma}} = \frac{W_t}{G_{t,T}}$$

then the value function can be stated as

$$J_t = G_{t,T}^\gamma \frac{W_t^{1-\gamma}}{1-\gamma}$$

where $G_{t,T}$ is defined as in the previous section. The optimal portfolio allocation to risky assets is given by

$$\begin{aligned}\alpha_t &= \frac{-J_W}{W_t J_{WW}} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \hat{\eta}_t - \frac{J_{WX}}{W_t J_{WW}} (\sigma_{St} \sigma'_{St})^{-1} (\sigma_{St} \hat{\sigma}'_{Xt}) \\ &= \frac{1}{\gamma} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \hat{\eta}_t + \frac{1}{G_{t,T}} \frac{\partial G_{t,T}}{\partial \hat{X}_t} (\sigma_{St} \sigma'_{St})^{-1} (\sigma_{St} \hat{\sigma}'_{Xt})\end{aligned}$$

where, by Leibniz's Rule,

$$\frac{\partial G_{t,T}}{\partial \hat{X}_t} = \int_t^T e^{-\frac{\phi}{\gamma}(s-t)} \frac{\partial F_{t,s}}{\partial \hat{X}_t} ds. \quad (31)$$

Since $F_{t,s}$ is a function of the ratio of the state price density at time s relative to the state price density at time t , equation (31) formalizes the relation between the hedging demand and changes in the investment opportunity set. Also, notice the hedging demand is a weighted function of the expected changes in the investment opportunity set for all horizons up to retirement. The weighting function is related negatively to the investor's impatience and positively to its relative risk aversion. Therefore, the more impatient investors care more about hedging demand in a shorter horizon, while more risk averse investor care about longer horizon consumption needs.

The assumption steady state variance is not necessary to obtain an expression for the optimal portfolio policy since the Cox-Huang methodology would still apply even if the diffusion component of the state variable decays deterministically. In those cases where an analytical solution does not obtain, the investor could use the Monte Carlo methods of Detemple, Garcia, and Rindisbacher (2003) or Cvitanic, Goukasian, and Zapatero (2002) to obtain a numerical solution. Both methods require market completeness which is satisfied under the information set of the investor.

2.3 Consumption to Wealth Ratio

The consumption to wealth ratio is easily obtain by applying some algebra to equations (19) and (25)

$$\frac{C_t}{W_t} = \frac{\lambda_t^{-\frac{1}{\gamma}}}{\lambda_t^{-\frac{1}{\gamma}} G_{t,T}}, \quad (32)$$

or

$$\frac{C_t}{W_t} = G_{t,T}^{-1}. \quad (33)$$

An application of equation (33) to the portfolio hedging demand (29) yields the following expression for the investor’s hedging demand:

$$\alpha_t^{hedging} = \frac{C_t}{W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \widehat{X}_t} (\sigma_{St} \sigma'_{St})^{-1} (\sigma_{St} \widehat{\sigma}'_{Xt}). \quad (34)$$

Equation (34) shows the link between future expected consumption and the hedging strategy of the investor. When markets are complete, the investor essentially can plan the consumption strategy for each possible outcome at each possible horizon, equation (34) shows how the investor changes the portfolio strategy to maintain the desired consumption plan.

3 Portfolio Choice with Unobservable Time-Varying Expected Returns

A useful example of the strength of our technique is to analyze the consumption and portfolio choice problem when the Sharpe ratio is mean reverting. Liu (2001) and Wachter (2002) finds a closed-form solutions to the consumption and portfolio choice where the predictive variable is fully observable and markets are complete. In order to solve the model, Wachter assumes the market is complete and the shocks to the proxy for the predictive variable and the stock price are perfectly negatively correlated. The assumption of perfect negative correlation does not seem controversial given that the empirically estimated correlation for the shocks to the dividend price ratio and the stock price is -0.93. Accounting for parameter uncertainty greatly decreases the demand of the risky asset due to hedging for changes in the investment opportunity set.

In this section of the paper, I extend Wachter to account for incomplete information in the agent’s optimization problem. Unlike Wachter, I will not assume market completeness. Instead, I assume uncertainty regarding the current value of the predictive process. Note the assumption regarding steady-state estimation does not allow us to study the role of the variance of the estimation error for the unobservable parameters as a state variable in the agent’s policies.⁸ In this model the predictive relation is known, since the predictive relation in our model is given by the standard deviation of the risky asset.

⁸In a related paper, Lewellen and Shanken (2002) study the equilibrium effects of learning on asset prices. They find mean reversion in asset prices can be explained by the learning of the agents regarding the dividend process. Xia (2001) solves a similar model where learning plays a role in the hedging demand of the investor.

Assume the existence of a money market account where the risk free rate is constant and the existence of one risky securities whose price process satisfies

$$\frac{dS_t}{S_t} = (r + \sigma_s \eta_t) dt + \sigma_s dZ_S, \quad (35)$$

such that the Sharpe ratio, η_t , is mean reverting, and satisfies

$$d\eta_t = \kappa(\theta - \eta_t) dt + \sigma_\eta dZ_x. \quad (36)$$

Assume the correlation between shocks to the stock price and shocks to the Sharpe ratio are imperfectly correlated. The correlation coefficient is denoted by ρ . The imperfect correlation between (35) and (36) implies the market is incomplete. Yet, when the Sharpe ratio is not observable and under assumptions explained in Section 2.1.2, the optimization problem can be restated in a complete markets framework.

3.1 Inference Problem

I apply the filtering methods of Lipster and Shiryaev (2001) to find a observationally equivalent economy under the subjective measure of the investor. Applying the results of section 2.1.1 to the current problem yields the following processes for the stock price and the state variable dynamics respectively:

$$\frac{dS_t}{S_t} = (r + \sigma_S \hat{\eta}_t) dt + \sigma_S d\hat{Z}_S, \quad (37)$$

$$d\hat{\eta}_t = \kappa(\theta - \hat{\eta}_t) dt + \varepsilon_\eta d\hat{Z}_S, \quad (38)$$

where

$$\varepsilon_\eta = v_{ss} + \rho\sigma_\eta$$

and

$$d\hat{Z}_S = [(\eta_t - \hat{\eta}_t) dt + dZ_S]. \quad (39)$$

The measurement error (variance) of the Sharpe ratio solves the following Riccati Equation

$$\frac{dv_t}{dt} = -2\kappa v_t + \sigma_\eta^2 - [v_t + \rho\sigma_\eta]^2. \quad (40)$$

Equation (40) can be solved following the appendix of Detemple (1986).

Following the methodology presented in Section 2, when computing for the optimal consumption and portfolio policies, assume learning has reached a steady state in which new data and estimation does not reduce the measurement error of the Sharpe ratio.⁹ Let v_{ss} denote the variance of the estimation error under the steady state.¹⁰ By applying the definition of steady state filtering to (40), v_{ss} is determined by the quadratic equation

$$0 = -2\kappa v_{ss} + \sigma_\eta^2 - [v_{ss} + \rho\sigma_\eta]^2. \quad (41)$$

The resulting variance will be the positive root of the quadratic equation obtained from our assumption in (40). If two positive roots are obtained, I study both cases: the high-prior equilibrium and the low-prior equilibrium.

3.2 Consumption and Portfolio Choice

After the investor solves the inference problem and estimates the Sharpe ratio, the estimated processes for the stock price and the Sharpe ratio are perfectly correlated. The investor sees this processes as perfectly correlated because the inference problem essentially projects the unobservable variable (the Sharpe ratio) into the space of the signal (the stock price), thus the source of uncertainty for both processes after the inference is the same. As seen in (38) the true correlation is accounted for in the diffusion coefficient for the estimated Sharpe ratio. In this section I show the main steps and results of the consumption and portfolio problem. The details of the derivation are provided in the Appendix.

I derive the agent's portfolio choice by applying (28) to the model. Let α_t be the proportion of wealth allocated to the risky asset. The portfolio choice of the agent can be decomposed into its myopic and hedging component.

$$\alpha_t = \alpha_t^{myopic} + \alpha_t^{hedging} \quad (42)$$

where

$$\alpha_t^{myopic} = \frac{1}{\gamma} \frac{\hat{\eta}_t}{\sigma_S}, \quad (43)$$

⁹A closed form solution is obtainable for equation (40). Please refer to Detemple (1986, Appendix) for details.

¹⁰Barberis (2000) also reduces the state space by not considering variance of estimation error, but he does not assume this occurs due to a steady-state in the equation determining the variance of the estimates. In the Barberis model, steady state learning occurs when parameter uncertainty disappears. My setup allows for the separation of parameter uncertainty and learning about the variance of the estimation error.

and

$$\alpha_t^{hedging} = \frac{\varepsilon_\eta \int_t^T (B(s-t) + C(s-t) \hat{\eta}_t) H_{t,s} ds}{\gamma \sigma_S \int_t^T H_{t,s} ds} \quad (44)$$

The hedging demand of the investor has the usual properties found for hedging demand in the presence of excess return predictability. The Sharpe ratio does not only come into play in the assigning of relative weight for the hedging demand via the function H , the wealth to consumption ratio, it also comes into play linearly as a measure of market timing. As was shown in (29), when the solution to H is of the exponential form, the sensitivity of the log wealth to consumption ratio to the state variable determines the relative weight each period in the agent's horizon has on the hedging strategy.

The consumption to wealth ratio for the agent is given by

$$\frac{C_t}{W_t} = \left(\int_t^T H_{t,s} ds \right)^{-1}.$$

The duration or sensitivity of the wealth to consumption ratio relative to changes in the investment opportunity set is given by

$$\frac{C_t}{W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \hat{\eta}_t} = \frac{\int_t^T (B(s-t) + C(s-t) \hat{\eta}_t) H_{t,s} ds}{\gamma \int_t^T H_{t,s} ds}, \quad (45)$$

as shown generally in Section 2.3, (45) establishes the relationship between the agent's hedging demand and the sensitivity of the agent's consumption and savings decision to changes in the investment opportunity set. This relation is straightforward due to market completeness under the filtered processes and the inextricable link between the agent's hedging demands and the expected consumption in the future.

4 Calibration and Results

Campbell and Viceira (1999) study optimal consumption and portfolio choice when expected returns are mean reverting. They assume the riskless rate of return is constant and the log excess return for stocks is given by the following VAR(1) specification:

$$\begin{aligned} \log S_{t_n+\Delta t} &= r_f + x_{t_n} + \varepsilon_{t_n+\Delta t}, \\ x_{t_n+\Delta t} &= (1 - \phi) \mu + \phi x_{t_n} + \eta_{t_n+\Delta t}. \end{aligned} \quad (46)$$

Campbell and Viceira use the dividend to price ratio as a proxy for changes in the investment opportunity set. They derive parameters for (46) from quarterly data. I adopt the results from Wachter (2002), which give the monthly parameters for the models by Campbell and Viceira (1999) and Barberis (2000). As explained previously, I maintain imperfect correlation between the state variable and stock returns and set the correlation to -0.93.

Table II consider the portfolio choice of the investor with incomplete information under various assumptions for the current estimate of the Sharpe ratio. The myopic and the hedging demand of the investor seems to increase monotonically with increases in the Sharpe ratio. Yet, the percentage of the portfolio dedicated to hedging changes in the investment opportunity set decreases monotonically with increases in the Sharpe ratio. The result is highly intuitive: When the investor estimates a low value for the Sharpe ratio, the investor is more willing to time the market because he expects the returns to be higher in the future due to the mean reversion in the parameter. This effect is also stronger when the investment horizon is longer. As η_t increases so does the myopic and hedging demand of the agent, but there is a reduction in the amount of the portfolio allocated to hedging changes in the investment opportunity set. For each panel, the hedging demand of the investor increases with the time horizon and decreases with respect to relative risk aversion. Yet, the percentage of wealth held in the risky asset due to hedging demand increases with both the time horizon and risk aversion. This result implies more risk averse investor reduce their exposure to risky assets, but increase the amount of the exposure that is due to changes in the investment opportunity set. Compared to Campbell and Viceira (1999) and Wachter (2002), hedging demands for investors considering the role of parameter uncertainty are lower.

Table III compares the consumption and portfolio strategy of an investor which estimates the current Sharpe ratio and computes his or her strategy according to the methods in this paper against an investor with perfect information about the economy under the assumption of perfect negative correlation between stock returns and the predictive process, the mean-reverting Sharpe ratio. The second investor type corresponds to the model presented in Wachter (2002). In Table III I assume the Sharpe ratio, as estimated by the first investor and observed by the second investor, is the long run value. The comparison in Table III allows us to concentrate on the role of parameter uncertainty in the hedging demand of the investor and does not account for the possibility of further differences in the consumption and the portfolio strategies of both investors due to the

incomplete information structure. In other words, I do not account for further differences due to differences in each agent's belief of the current value of the Sharpe ratio. As expected, the differences in the proportion of wealth allocated to stock in both examples is due to differences in the hedging demand. The hedging demand for the investor with incomplete information is much lower. In the case where the investor has a coefficient of relative risk aversion of 5 and a 30-year investment horizon, the difference in the percentage of wealth allocated to the risky asset is 24%. As expected, the percentage of the portfolio dedicated to hedging demands is lower when accounting for parameter uncertainty. A major driver in the difference in this results is the perceived volatility of the state variable. For the calibration the standard deviation of the predictive variable is

$$\begin{aligned}
\varepsilon_\eta &= v_{ss} + \rho\sigma_\eta & (47) \\
&= 0.0035 + (-0.93) 0.0189 \\
&= -0.0141
\end{aligned}$$

As expected, this is smaller than the standard deviation that would be obtained under the assumption of complete markets and full information. The reduction of the volatility of the predictive variable is the main reason why hedging demands under parameter uncertainty are considerably smaller than those obtained under the complete markets assumption.

Although in our calibration parameter uncertainty reduces the magnitude of the hedging demand, it is possible that under a different calibration the hedging demand of the investor would be high in magnitude. The investor takes a negative position in the risky asset due to the measurement error. This can be clearly seen by substituting (47) in the hedging demand (44):

$$\begin{aligned}
\alpha_t^{hedging} &= \frac{\varepsilon_\eta C_t}{\sigma_S W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \hat{\eta}_t} \\
&= \underbrace{\frac{v_{ss} C_t}{\sigma_S W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \hat{\eta}_t}}_{<0} + \underbrace{\frac{\rho\sigma_\eta C_t}{\sigma_S W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \hat{\eta}_t}}_{>0}
\end{aligned}$$

Had the negative hedging component due to estimation error dominated the positive hedging component due to changes in the investment opportunity set, the results could have been different. A second lever in which the perceived volatility determines the hedging demand is in the value of the functions $B(\cdot)$ and $C(\cdot)$. Under the complete markets assumption, Wachter (2002) shows $B(\cdot) < 0$

and $C(\cdot) < 0$. Under incomplete information the perceived covariance is smaller, but the functions $B(\cdot)$ and $C(\cdot)$ are still negative. It follows that another possible reason for the reduction in the hedging demand of the investor is the reduction in the duration of the hedging demand as measure by the sensitivity of the consumption to wealth ratio to changes in the state variable. In Table IV, I compare changes in hedging demand due to changes in the duration measure and changes in the perceived volatility of the state variable. I do the comparison via the following decomposition

$$\alpha_t^{hedging,CM} - \alpha_t^{hedging} = \underbrace{\frac{-\sigma_\eta}{\sigma_S} (Dur^{CM} - Dur)}_{Duration\ Effect} + \underbrace{\left(\frac{-\sigma_\eta}{\sigma_S} - \frac{\varepsilon_\eta}{\sigma_S} \right) Dur}_{Covariance\ Effect} \quad (48)$$

where the superscript CM represents the complete markets investor assuming perfect negative correlation between the state variable and stock returns and Dur^{CM} and Dur is given by

$$Dur^{CM} = \frac{\int_t^T (B^{CM}(s-t) + C^{CM}(s-t)\eta_t) H_{t,s}^{CM} ds}{\gamma \int_t^T H_{t,s}^{CM} ds},$$

$$Dur = \frac{\int_t^T (B(s-t) + C(s-t)\hat{\eta}_t) H_{t,s} ds}{\gamma \int_t^T H_{t,s} ds}.$$

The solutions to the functions $A^{CM}(\cdot)$, $B^{CM}(\cdot)$, and $C^{CM}(\cdot)$ are provided in Wachter (2002). To concentrate on the role of the perceived volatility in the hedging demand, I assume the current Sharpe ratio and the estimate of the second investor for the Sharpe ratio are equal. I analyze equation (48) in Table IV for various assumptions of the current Sharpe ratio, relative risk aversion, and investment horizon. The covariance effect seems to dominate the change in the hedging demand for short-horizon investors.¹¹ In general, for the longer-horizon investors, it is the duration effect that carries the most weight in the change in hedging demands. For the low risk aversion investor, the covariance effect dominates. As risk aversion increases, the covariance effect is reduced and the duration effect increases.

Using the decomposition of the portfolio demand into its market timing and non-market timing components allows the comparison of the market timing aggressiveness implied by the Wachter

¹¹Barberis (2000) also mentions the reduction in sensitivity to the state variable in the portfolio demand when parameter uncertainty is taken into account. Yet, his model does not allow for the separate analysis of the reduction in the hedging demand as it pertains to changes in the perceived covariance and changes in the duration, the sensitivity of hedging demands to changes in the state variable.

model relative to my model. The agent's portfolio in terms of its timing component and its non-timing component are

$$\alpha_t^{NMT} = \frac{\theta}{\gamma\sigma_S} + \frac{\varepsilon_\eta}{\sigma_S} \frac{\int_t^T (B(s-t) + C(s-t)\theta) H_{t,s} ds}{\gamma \int_t^T H_{t,s} ds}, \quad (49)$$

$$\alpha_t^{MT} = \left[\frac{1}{\gamma\sigma_S} + \frac{\varepsilon_\eta}{\sigma_S} \frac{\int_t^T C(s-t) H_{t,s} ds}{\gamma \int_t^T H_{t,s} ds} \right] (\hat{\eta}_t - \theta). \quad (50)$$

Similar to Campbell and Viceira (1999) equations (49) is the intercept of the portfolio demand and the coefficient in (50) is the slope of the portfolio. Let the coefficient of the market timing component divided by the non-market timing component be the measure of market timing aggressiveness for the investor. For various assumptions of the degree of relative risk aversion and retirement horizon, Table V shows the market timing and non-market timing component of the two models when the expected return is equal to the long-run return. The table also shows the measure of market timing aggressiveness. As expected from the previous discussion about the hedging demand differences, Table V shows a reduction in the slope and intercept of the portfolio allocation when parameter uncertainty is considered. Our measure of market timing aggressiveness shows the parameter uncertainty investor to have a higher slope (when normalized to its non-market timing demand) than the complete markets investors. The results follows from the subjective distribution attributed to the predictive variable by the investor accounting for parameter uncertainty. The investor has a more precise signal than the complete markets investor due to how the variance of the estimation error reduces the implied variance of the state variable. Therefore, the investor is willing to market time more aggressively than an investor who does not consider the role of parameter uncertainty.

5 The Longevity of Learning

A crucial assumption made to obtain closed-form solution to the investors consumption and investment problem is that learning has reached a steady-state process. In other words, any new observation of the securities will be accounted for by the agent is his new estimates of the unobservable parameters, but the new observation will not contribute in reducing the estimation risk, the variance of the estimates. This assumption begs two questions: (1) How quickly would an agent on average, regardless of prior, reach the steady state in the learning process? (2) Can the estimation risk in the steady state significantly change the investment strategy of the agent? This

section provides answers to both questions in the context of the model presented in this section.

To answer the first question, I construct a simulation of how the estimation risk of the agent changes after each observation through time. For the case of stock price predictability, I first obtain the steady state variance of the measurement error and then simulate the learning of the agent under the assumption of priors that are multiples of the steady-state estimation risk. I assume changes in the variance of the estimation error follow (40). I assume that new observations are made every 1/10th of a quarter. As expected, from Merton (1980), our results are not sensitive to the assumption of the sampling frequency. Figure 1 shows how the agent's estimation risk under the assumption that the prior is two-times, five-times, ten-times, and twenty-times the steady state value. Notice that by the time 15 years (180 months) pass by, all variance estimates, regardless of prior, are lower than even two-times the steady state variance. By the 30 year mark (360 months), the agent is not significantly far away from the steady state regardless of the assumption of the prior. The figure provides strong evidence that our assumption of steady-state learning is not out of line and makes sense given the amount of data the agent has available to estimate these parameters. Assume the agent has access to the CRSP database, then it is fair to state agents have about 40 years (480 months) of daily data and about 75 years (900 months) of monthly data to earn from before deciding on their consumption and portfolio strategies, thus it is quite believable that a rational agent would achieve a level of learning such that the steady state assumption is innocuous.

In order to understand how the estimation error is reduced with each new observation, I check the magnitude in which the estimation error variance is reduced with each new observation. Figure 2 plots the instantaneous reduction in variance for a given point in time. By the time the agent has observed 15 years (180 months) of data, the reduction in the variance of the estimates of the unobservable variables are negligible. This implies, the learning effect should be negligible in the hedging component of the agent's portfolio for our model. Our results imply parameter uncertainty, not learning, drives the changes in the portfolio composition in comparison to the portfolio model under perfect observability of all processes.

6 Return Predictability and the model implied R^2

In this section I discuss the implication of parameter uncertainty for the forecasting of prices. Campbell (1991) offers a description of how predictive variables seems to increase the forecasting power of returns.¹² For the parameters in Table I, I solve analytically the model implied R^2 under three different specifications: (1) the parameters of the model, (2) the model under the assumption of perfect correlation, (3) the model when parameter uncertainty is considered. Let $R(t, \tau)$ be the return on the stock from time t to time $t + \tau$. Let $E_t^i [R(t, \tau)]$ be the expected return conditional on the information set implied by model i . Define the model implied R^2 as the fraction of the variance of returns explained by the model. The model implied R^2 at time t for forecast horizon τ is given by

$$R^2(t, \tau) = \frac{\text{Var}(E_t^i [R(t, \tau)])}{\text{Var}(R(t, \tau))} = \frac{\text{Var}(E_t^i [R(t, \tau)])}{\text{Var}(E_t^i [R(t, \tau)]) + \text{Var}(\varepsilon^i(t, \tau))}$$

where $\varepsilon^i(t, \tau)$ is the variance not explained under model i . For the model presented in section 3, the model implied R^2 can be obtained analytically:

$$R^2(t, \tau) = \frac{\frac{\sigma_\eta^2}{2\kappa^3} (1 - e^{-\kappa\tau})^2}{\frac{\sigma_\eta^2}{2\kappa^3} (1 - e^{-\kappa\tau})^2 + \frac{\sigma_\eta^2}{2\kappa^3} [\kappa\tau - 2(1 - e^{-\kappa\tau}) + \frac{1}{2}(1 - e^{-2\kappa\tau})] + 2\frac{\rho\sigma_\eta}{\kappa^2} [\kappa\tau + e^{-\kappa\tau} - 1] + \tau}$$

Details of the derivation are provided in the appendix.

Figure 3 presents the model implied R^2 under three different specifications for various horizons. **table1** is the model implied R^2 under the parameters of Table I. **cm** is the model implied R^2 under the assumption of perfect negative correlation between the predictive variable and stock prices. **pu** is the model implied R^2 under the estimation of the model provided by the filtering process. The model implied R^2 peak at around a 10-15 year horizon. Notice that for both the complete markets assumption and the parameter uncertainty model, the implied R^2 decay slower than under the values obtained from the VAR. The model implied R^2 under parameter uncertainty peak at around 0.15, and at 0.32 under the complete markets assumption.

The model implied R^2 allows us to gain a sense of the difference in hedging for both models. The reduction in hedging for investors accounting for parameter uncertainty is linked to their model implied R^2 . The lower R^2 in the parameter uncertainty case should come as no surprise since

¹²Mamaysky (2002) also uses model implied R^2 to study the forecasting power of return predictability.

parameter uncertainty allows us to quantify the reduction in forecasting power due to the noise of the current estimate for the predictive variable.

7 Conclusion

I study the incomplete markets consumption and portfolio choice optimization problem under partially observable parameters. Under suitable assumptions regarding the number of securities in the market as well as which parameters are unobservable, I can transform the problem into one where the market is observationally complete after estimating unobservable parameters and accounting for parameter uncertainty. Obtaining an observationally complete market, allows us to solve exactly the investor's optimization problem and obtain exact consumption and portfolio rules. I consider an example for which the assumption of parameter uncertainty is sensible: the mean reverting Sharpe ratio model. For both examples, I show how the separation theorem of Simon (1956) as extended to continuous time by Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) and the Cox-Huang (1989) method allow us to solve the model and find analytical solutions for both the consumption and portfolio strategy.

I apply the methodology of the paper to the consumption and portfolio choice problem under mean reverting returns, when the current value for the Sharpe ratio, our proxy for the investment opportunity set, is not observable. I calibrate the results to the parameters in Wachter (2002) and compare the investor's policies under parameter uncertainty to those of an investor with complete information as modeled by Wachter (2002). I find significant quantitative changes in the demand for the risky asset when parameter uncertainty is considered. Yet, the qualitative portfolio choice implications of the model are not different from those of Barberis (2000) and Wachter. I complete the analysis with a study of the longevity of learning to validate our assumption regarding steady-state in the learning process and a discussion of the link between hedging demands and the investor's model forecasting ability.

The methodology of this paper could be extended to consider stochastic volatility and the role of derivatives in strategic asset allocation. The model would allow us to find exact consumption and portfolio policies when volatility follows a central tendency model similar to the one I have used in this paper to describe interest rate dynamics in Balduzzi, Das, and Foresi (1998). The results

would complement those of Liu and Pan (2002). The focus of their papers is the disentanglement of volatility and jump risk in an investor's portfolio through the use of derivative securities in the dynamic asset allocation strategy. The proposed model would allow for a similar risk disentanglement (between market and volatility risk) and also provide us with normative exact consumption rules under the assumption of stochastic volatility. I can also consider the implications of incomplete information when interest rates are time-varying and the market includes a long-term bonds. The results of this model would complement those of Campbell and Viceira (2001) for the optimal allocation of long-term bonds for long-lived investors.

An exciting research question would be to study the combined effect of imperfect information and uncertainty aversion in the hedging demand of investors. Recent work by Knox (2002), considers the consumption and investment problem with both learning and uncertainty aversion. He obtains an analytical solution to the consumption and portfolio choice of an investor learning about the expected return of a risky asset. His work can be extended to find analytical solutions for the consumption and portfolio problem when markets are incomplete by utilizing the framework presented in this paper.

Currently, Rodriguez (2002b) applies the model presented to this paper to a consumption and portfolio choice variable with multiple predictive variables and considers the role for hedging demands for various linear combinations of such variables. In this companion paper, I apply the techniques presented in Campbell, Chacko, Rodriguez, and Viceira (2002) to obtain accurate continuous time parameters values for the parameters obtained from the discrete time VAR. The versatility of the model is enhanced in this setting where the author can use predictive variables with low correlations with the stock returns and still obtain closed-form solutions for the optimal consumption and portfolio demand.

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A Alternative Solution for Section 2

A solution for the wealth process presented in Section 2 is obtained by an application of the Feynman-Kac formula. First, notice that under the investor specific “risk-neutral” measure, the wealth of the agent at time s is

$$\begin{aligned} W_s &= E_s^{I,Q} \left[\int_s^T e^{-r(u-s)} \left(\lambda_t e^{\phi(u-t)} \frac{\widehat{M}_u}{M_t} \right)^{-\frac{1}{\gamma}} du \right] \\ &= \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1}{\gamma}} \int_s^T \exp \left(- \int_s^u r_v dv - \frac{\phi}{\gamma} (u-s) \right) E_s^{I,Q} \left[\left(\frac{\widehat{M}_u}{M_s} \right)^{-\frac{1}{\gamma}} \right] du \end{aligned}$$

Define $\widehat{M}_u = \exp \left(- \int_0^u r_v dv \right) \widehat{\xi}_u$. Under the risk-neutral measure of the investor,

$$\begin{aligned} \frac{\widehat{\xi}_u}{\widehat{\xi}_s} &= \exp \left(\frac{1}{2} \int_s^u \|\widehat{\eta}_v\|^2 dv - \int_s^u \widehat{\eta}'_v d\widehat{Z}_v^Q \right) \\ d\widehat{Z}_v^Q &= d\widehat{Z}_v - \widehat{\eta}_t dt \end{aligned}$$

where \widehat{Z}_s^Q is the Brownian motion under this measure as defined by Girsanov’s theorem.

Write the wealth process as follows:

$$W_s = \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1}{\gamma}} \left\{ \int_s^T \exp \left(- \left(1 - \frac{1}{\gamma} \right) \left(\int_s^u r_v dv \right) - \frac{\phi}{\gamma} (u-s) \right) E_s^{I,Q} \left[\left(\frac{\widehat{\xi}_u}{\widehat{\xi}_s} \right)^{-\frac{1}{\gamma}} \right] du \right\}. \quad (51)$$

Solving (51) requires finding a solution for the conditional expectation. The solution can be found by applying Girsanov’s theorem to find an alternate equivalent martingale measure:

$$\begin{aligned} E_s^{I,Q} \left[\left(\frac{\widehat{\xi}_u}{\widehat{\xi}_s} \right)^{-\frac{1}{\gamma}} \right] &= E_s^{I,Q} \left[\exp \left(- \frac{1}{2\gamma} \int_s^u \|\widehat{\eta}_v\|^2 dv + \frac{1}{\gamma} \int_s^u \widehat{\eta}'_v d\widehat{Z}_v^Q \right) \right] \\ &= E_s^{I,Q} \left[\exp \left(\frac{1-\gamma}{2\gamma^2} \int_s^u \|\widehat{\eta}_v\|^2 dv - \frac{1}{2\gamma^2} \int_s^u \|\widehat{\eta}_v\|^2 dv + \frac{1}{\gamma} \int_s^u \widehat{\eta}'_v d\widehat{Z}_v^Q \right) \right] \\ &= E_s^{I,Q^\gamma} \left[\exp \left(\frac{1-\gamma}{2\gamma^2} \int_s^u \|\widehat{\eta}_v\|^2 dv \right) \right] \end{aligned} \quad (52)$$

Define the equivalent measure Q^γ such that $\widehat{Z}_S^{Q^\gamma}$, the brownian motion defined under the new measure is given by

$$d\widehat{Z}_v^{Q^\gamma} = d\widehat{Z}_v^Q - \frac{1}{\gamma} \widehat{\eta}'_v.$$

The equality between the second and third equation follow from Bayes’ rule.

Let $D(\widehat{X}_s, s) = E_s^{I, Q^\gamma} \left[\exp \left(\frac{1-\gamma}{2\gamma^2} \int_s^u \|\widehat{\eta}_v\|^2 dv \right) \right]$. The Feynman-Kac formula states equation () is solved by the partial differential equation

$$-\frac{1-\gamma}{2\gamma^2} \eta'_s \eta_s = \frac{\partial D}{\partial s} + \frac{\partial D}{\partial \widehat{X}_s} \left(a_0^{Q^\gamma} + a_X^{Q^\gamma} \widehat{X}_s \right) + \frac{1}{2} tr \left[\widehat{\sigma}_{X_s} \widehat{\sigma}'_{X_s} \frac{\partial^2 D}{\partial \widehat{X}_s^2} \right]$$

where $a_{\cdot s}^{Q^\gamma}$ are the coefficients of the drift for the state variables under the measure Q^γ and the terminal condition is given by

$$D(\widehat{X}_u, u) = 0$$

The derivation above shown the expectation is indeed a function of the state variables and time. The wealth process can be stated now as

$$W_s = \left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{M_t} \right)^{-\frac{1}{\gamma}} \int_s^T \left[\exp \left(- \int_s^u r_v dv - \frac{\phi}{\gamma} (u-s) \right) D(\widehat{X}_s, s) \right] du \quad (53)$$

From (27), the demand for the risky asset is

$$\alpha'_t = \frac{\widehat{M}_t}{W_t} \frac{\partial W_t}{\partial \widehat{M}_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\eta}_t + \frac{1}{W_t} \frac{\partial W_t}{\partial \widehat{X}_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\sigma}'_{Xt}$$

Substituting (53) in (27) gives

$$\begin{aligned} \alpha'_t &= \frac{1}{\gamma} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\eta}_t + \frac{\int_s^T \left[\exp \left(- \int_s^u r_v dv - \frac{\phi}{\gamma} (u-s) \right) \frac{\partial D(\widehat{X}_s, s)}{\partial \widehat{X}_t} \right] du}{\int_s^T \left[\exp \left(- \int_s^u r_v dv - \frac{\phi}{\gamma} (u-s) \right) D(\widehat{X}_s, s) \right] du} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\sigma}'_{Xt}, \\ &= \frac{1}{\gamma} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\eta}_t + \frac{1}{G_t} \frac{\partial G_t}{\partial \widehat{X}_t} (\sigma_{St} \sigma'_{St})^{-1} \sigma_{St} \widehat{\sigma}'_{Xt}. \end{aligned}$$

The second equality follows from the definition of G_t .

B Derivation of Optimal Consumption and Portfolio Policy for Unobservable Time-Varying Expected Returns

The solution to the optimal consumption and portfolio problem under time-varying returns follows closely the derivations by Munk (2002) and Wachter (2002). Following Lipster and Shirvayev (2001), the optimal filter results in the following dynamics for the stock price and the state variable:

$$\begin{aligned}\frac{dS_t}{S_t} &= (r + \sigma_S \hat{\eta}_t) dt + \sigma_S d\hat{Z}_S, \\ d\hat{\eta}_t &= \kappa(\theta - \hat{\eta}_t) dt + (\rho\sigma_\eta + v_t) d\hat{Z}_S,\end{aligned}$$

where v_t , the variance of the estimation error, satisfies the Ricatti Equation

$$dv_t = \left[-2\kappa v_t + \sigma_\eta^2 - (\rho\sigma_\eta + v_t)^2 \right] dt$$

and

$$d\hat{Z}_S = (\eta_t - \hat{\eta}_t) dt + dZ_S$$

Under the filtered dynamics, the market is complete. I can apply martingale methods to solve for the optimal consumption and portfolio choice.

The investors' value function at time t is defined as

$$J(W_t, \hat{\eta}_t, T - t) = \sup_{\{\alpha_S, C_S\}} E_t^I \left[\int_t^T e^{-\phi(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \right],$$

subject to the dynamic budget constraint

$$dW_t = (rW_t - C_t) dt + \alpha_t W_t \left[\sigma_S \hat{\eta}_t dt + \sigma_S d\hat{Z}_S \right].$$

Define \hat{M}_t as the state price density at time t , such that

$$\frac{d\hat{M}_t}{\hat{M}_t} = -r dt - \hat{\eta}_t d\hat{Z}_S,$$

The consumption strategy is financially if

$$E_t^I \left[\int_t^T \frac{\hat{M}_s}{\hat{M}_t} C_s ds \right] \leq W_t.$$

The first order condition for consumption is given by

$$\begin{aligned}e^{-\phi(s-t)} C_s^{-\gamma} &= \lambda_t \frac{\hat{M}_s}{\hat{M}_t}, \\ C_s &= \left(\lambda_t e^{\phi(s-t)} \frac{\hat{M}_s}{\hat{M}_t} \right)^{-\frac{1}{\gamma}}\end{aligned}$$

The wealth at time t under the optimal consumption policy can be expressed as

$$\begin{aligned} W_t &= E_t^I \left[\int_t^T \lambda_t^{-\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}(s-t)} \left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{1-\frac{1}{\gamma}} ds \right] \\ &= \lambda_t^{-\frac{1}{\gamma}} \int_t^T e^{-\frac{\phi}{\gamma}(s-t)} E_t^I \left[\left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{1-\frac{1}{\gamma}} \right] ds \end{aligned}$$

Let $\widehat{M}_t = e^{-rt} \widehat{\xi}_t$,

$$W_t = \lambda_t^{-\frac{1}{\gamma}} \int_t^T e^{-\frac{\phi}{\gamma}(s-t) - r(1-\frac{1}{\gamma})(s-t)} E_t^I \left[\left(\frac{\widehat{\xi}_s}{\widehat{\xi}_t} \right)^{1-\frac{1}{\gamma}} \right] ds$$

where

$$\frac{\widehat{\xi}_s}{\widehat{\xi}_t} = \exp \left(-\frac{1}{2} \int_t^s \widehat{\eta}_u^2 du - \int_t^s \widehat{\eta}_u d\widehat{Z}_S \right).$$

Since

$$E_t^I \left[\frac{\widehat{\xi}_s}{\widehat{\xi}_t} \right] = 1$$

I can apply Bayes' rule to obtain

$$W_t = \lambda_t^{-\frac{1}{\gamma}} \int_t^T e^{-\frac{\phi}{\gamma}(s-t) - r(1-\frac{1}{\gamma})(s-t)} E_t^{I,Q} \left[\left(\frac{\widehat{\xi}_s}{\widehat{\xi}_t} \right)^{-\frac{1}{\gamma}} \right] ds$$

where I, Q represents an equivalent martingale measure to the investor's original information set.

For I, Q , the dynamics of the stock price and the state variable are given by

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma_S d\widehat{Z}_S^Q, \\ d\widehat{\eta}_t &= (\kappa(\theta - \widehat{\eta}_t) - (\rho\sigma_\eta + v_t)\widehat{\eta}_t) dt + (\rho\sigma_\eta + v_t) d\widehat{Z}_S^Q, \end{aligned}$$

where by Girsanov's Theorem,

$$d\widehat{Z}_S^Q = \widehat{\eta}_t dt + d\widehat{Z}_S.$$

is a Brownian motion under the measure I, Q .

The ratio $\frac{\widehat{\xi}_s}{\widehat{\xi}_t}$ is given by.

$$\begin{aligned} \frac{\widehat{\xi}_s}{\widehat{\xi}_t} &= \exp \left(-\frac{1}{2} \int_t^s \widehat{\eta}_u^2 du - \int_t^s \widehat{\eta}_u d\widehat{Z}_S \right), \\ &= \exp \left(-\frac{1}{2} \int_t^s \widehat{\eta}_u^2 du - \int_t^s \widehat{\eta}_u (d\widehat{Z}_S^Q - \widehat{\eta}_u du) \right), \\ &= \exp \left(\frac{1}{2} \int_t^s \widehat{\eta}_u^2 du - \int_t^s \widehat{\eta}_u d\widehat{Z}_S^Q \right). \end{aligned}$$

Thus

$$\begin{aligned}
\left(\frac{\widehat{\xi}_s}{\widehat{\xi}_t}\right)^{-\frac{1}{\gamma}} &= \exp\left(-\frac{1}{2\gamma}\int_t^s \widehat{\eta}_u^2 du + \frac{1}{\gamma}\int_t^s \widehat{\eta}_u d\widehat{Z}_S^Q\right) \\
E_t^{I^*}\left[\left(\frac{\widehat{\xi}_s}{\widehat{\xi}_t}\right)^{-\frac{1}{\gamma}}\right] &= E_t^{I,Q}\left[\exp\left(-\frac{1}{2\gamma}\int_t^s \widehat{\eta}_u^2 du + \frac{1}{\gamma}\int_t^s \widehat{\eta}_u d\widehat{Z}_S^Q\right)\right] \\
&= E_t^{I,Q}\left[\exp\left(\frac{1-\gamma}{2\gamma^2}\int_t^s \widehat{\eta}_u^2 du - \frac{1}{2\gamma^2}\int_t^s \widehat{\eta}_u^2 du + \frac{1}{\gamma}\int_t^s \widehat{\eta}_u d\widehat{Z}_S^Q\right)\right] \\
&= E_t^{I,Q^\gamma}\left[\exp\left(\frac{1-\gamma}{2\gamma^2}\int_t^s \widehat{\eta}_u^2 du\right)\right]
\end{aligned}$$

where

$$d\widehat{Z}_S^{Q^\gamma} = d\widehat{Z}_S^Q - \frac{1}{\gamma}\widehat{\eta}_t dt$$

is a Brownian Motion under the measure I, Q^γ . The equality between the second and third equation follow from Bayes' rule.

The state variable process is now given by

$$\begin{aligned}
d\widehat{\eta}_t &= (\kappa(\theta - \widehat{\eta}_t) - (\rho\sigma_\eta + v_t)\widehat{\eta}_t) dt + (\rho\sigma_\eta + v_t)\left(d\widehat{Z}_S^{Q^\gamma} + \frac{1}{\gamma}\widehat{\eta}_t dt\right), \\
&= (a - b\widehat{\eta}_t) dt + (\rho\sigma_\eta + v_t) d\widehat{Z}_S^{Q^\gamma},
\end{aligned}$$

where

$$\begin{aligned}
a &= \kappa\theta, \\
b &= \kappa + \left(1 - \frac{1}{\gamma}\right)(\rho\sigma_\eta + v_t).
\end{aligned}$$

Assume $v_t = v_{ss}$ such that $dv_{ss} = 0$ and $b = \kappa + \left(1 - \frac{1}{\gamma}\right)(\rho\sigma_\eta + v_{ss})$. Following Munk (2002) we apply Feynman-Kac formula and define the function $D(\widehat{\eta}_t, s-t)$ as

$$D(\widehat{\eta}_t, s-t) = E_t^{I,Q^\gamma}\left[\exp\left(\frac{1-\gamma}{2\gamma^2}\int_t^s \widehat{\eta}_u^2 du\right)\right]. \quad (54)$$

Equation (54) solves the differential equation

$$-\frac{1-\gamma}{2\gamma^2}\widehat{\eta}_t^2 D = \frac{\partial D}{\partial t} + (a - b\widehat{\eta}_t)\frac{\partial D}{\partial \widehat{\eta}_t} + \frac{1}{2}(\rho\sigma_\eta + v_{ss})^2 \frac{\partial^2 D}{\partial \widehat{\eta}_t^2}. \quad (55)$$

under the terminal condition $D(\widehat{\eta}_t, 0) = 1$. The differential equation (55) admits a exponential quadratic solution

$$D(\widehat{\eta}_t, s-t) = \exp\left[\frac{1}{\gamma}\left\{A_D(s-t) + B(s-t)\widehat{\eta}_t + \frac{1}{2}C(s-t)\widehat{\eta}_t^2\right\}\right]$$

where

$$A'_D(\tau) = B(\tau)\kappa\theta + \frac{1}{2}C(\tau)(\rho\sigma_\eta + v_{ss})^2 + \frac{1}{2\gamma}B^2(\tau)(\rho\sigma_\eta + v_{ss})^2, \quad (56)$$

$$B'(\tau) = B(\tau)\left[\frac{1-\gamma}{\gamma}(\rho\sigma_\eta + v_{ss}) - \kappa\right] + C(\tau)\kappa\theta + \frac{1}{\gamma}B(\tau)C(\tau)(\rho\sigma_\eta + v_{ss})^2, \quad (57)$$

$$C'(\tau) = 2C(\tau)\left[\frac{1-\gamma}{\gamma}(\rho\sigma_\eta + v_{ss}) - \kappa\right] + \frac{1}{\gamma}C^2(\tau)(\rho\sigma_\eta + v_{ss})^2 + \frac{1-\gamma}{\gamma}. \quad (58)$$

and the terminal condition for D can be restated in terms of the terminal conditions for A_D, B , and C :

$$A_D(0) = B(0) = C(0) = 0.$$

Equations (56), (57), and (58) are solved in similar fashion to Kim and Omberg (1996), Wachter (2002), and Chacko and Viceira (2001). The solutions to (56), (57), and (58) are

$$\begin{aligned} A_D(\tau) &= \left[\frac{1-\gamma}{\gamma} \left(\frac{2\kappa^2\theta^2}{\delta^2} + \frac{\varepsilon_\eta^2}{\delta - 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)} \right) \right] \tau \\ &+ \frac{4\frac{1-\gamma}{\gamma}\kappa^2\theta^2 \left[\left(4\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right) + \delta\right) e^{-\delta\tau} - 8\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right) e^{-\delta\tau/2} + 4\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right) - \delta \right]}{\delta^3 \left[\left(\delta - 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) + \left(\delta + 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) e^{-\delta\tau} \right]} \\ &+ \frac{2\left(\frac{1-\gamma}{\gamma}\right)\varepsilon_\eta^2}{\delta^2 - 4\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)^2} \ln \left| \frac{\left(\delta - 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) + \left(\delta + 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) e^{-\delta\tau}}{2\delta} \right|, \\ B(\tau) &= \frac{4\frac{1-\gamma}{\gamma}\kappa\theta(1 - e^{-\delta\tau/2})^2}{\delta \left[\left(\delta - 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) + \left(\delta + 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) e^{-\delta\tau} \right]}, \\ C(\tau) &= \frac{2\frac{1-\gamma}{\gamma}(1 - e^{-\delta\tau})}{\left(\delta - 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) + \left(\delta + 2\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)\right) e^{-\delta\tau}}. \end{aligned}$$

where

$$\delta^2 = 4\left(\frac{1-\gamma}{\gamma}\varepsilon_\eta - \kappa\right)^2 - 4\frac{1-\gamma}{\gamma^2}\varepsilon_\eta^2.$$

The wealth process is now

$$W_t = \lambda_t^{-\frac{1}{\gamma}} \int_t^T e^{-\frac{\phi}{\gamma}(s-t) - r\left(1-\frac{1}{\gamma}\right)(s-t)} G(\hat{\eta}_t, s-t) ds$$

Define $H(\hat{\eta}_t, s-t)$ as

$$H(\hat{\eta}_t, s-t) = e^{-\frac{\phi}{\gamma}(s-t) - r\left(1-\frac{1}{\gamma}\right)(s-t)} G(\hat{\eta}_t, s-t).$$

Then $H(\hat{\eta}_t, s-t)$ also has an exponential quadratic solution of the form

$$H(\hat{\eta}_t, s-t) = \exp\left[\frac{1}{\gamma}\left\{A(s-t) + B(s-t)\hat{\eta}_t + \frac{1}{2}C(s-t)\hat{\eta}_t^2\right\}\right],$$

where

$$A(\tau) = -\phi\tau + r(1-\gamma)\tau + A_D(\tau).$$

I can now apply use the methods in Cox-Huang (1989) to obtain the optimal portfolio policy.

Using the results above, we can write the value function as:

$$\begin{aligned} J(W_t, \hat{\eta}_t, T-t) &= E_t^I \left[\int_t^T e^{-\phi(s-t)} \frac{\left(\lambda_t e^{\phi(s-t)} \frac{\widehat{M}_s}{\widehat{M}_t}\right)^{-\frac{1-\gamma}{\gamma}}}{1-\gamma} ds \right] \\ &= \frac{\lambda_t^{-\frac{1-\gamma}{\gamma}}}{1-\gamma} \int_t^T e^{-\frac{\phi}{\gamma}(s-t)} E_t^I \left[\left(\frac{\widehat{M}_s}{\widehat{M}_t}\right)^{1-\frac{1}{\gamma}} \right] ds \\ &= \frac{W_t^{1-\gamma}}{1-\gamma} \left[\int_t^T e^{-\frac{\phi}{\gamma}(s-t)} E_t^I \left[\left(\frac{\widehat{M}_s}{\widehat{M}_t}\right)^{1-\frac{1}{\gamma}} \right] ds \right]^\gamma \\ &= \left[\int_t^T H(\hat{\eta}_t, s-t) ds \right]^\gamma \frac{W_t^{1-\gamma}}{1-\gamma} \\ &= G_{t,T}^\gamma \frac{W_t^{1-\gamma}}{1-\gamma} \end{aligned}$$

The optimal portfolio policy for the investor as a function of derivatives of the value function is

$$\begin{aligned} \alpha_t &= -\frac{J_W}{W J_{WW}} \frac{\hat{\eta}_t}{\sigma_S} - \frac{J_{W\eta}}{W J_{WW}} \frac{\varepsilon_\eta}{\sigma_S}, \\ &= \frac{\hat{\eta}_t}{\gamma \sigma_S} + \frac{\varepsilon_\eta}{\gamma \sigma_S} \frac{1}{G_{t,T}} \frac{\partial G_{t,T}}{\partial \hat{\eta}_t}, \end{aligned} \tag{59}$$

where the derivative of $G_{t,T}$ with respect to the estimate of the Sharpe ratio can be expressed as

$$\begin{aligned} \frac{\partial G_{t,T}}{\partial \eta} &= \frac{\partial \left[\int_t^T H(\hat{\eta}_t, s-t) ds \right]}{\partial \eta}, \\ &= \int_t^T \frac{\partial H(\hat{\eta}_t, s-t)}{\partial \hat{\eta}_t} ds, \\ &= \frac{1}{\gamma} \int_t^T (B(s-t) + C(s-t)\hat{\eta}_t) H(\hat{\eta}_t, s-t) ds. \end{aligned} \tag{60}$$

Applying (60) to (59) results in the following expression for the allocation to the risky asset

$$\alpha_t = \frac{\hat{\eta}_t}{\gamma \sigma_S} + \frac{\varepsilon_\eta}{\gamma \sigma_S} \frac{\int_t^T (B(s-t) + C(s-t)\hat{\eta}_t) H(\hat{\eta}_t, s-t) ds}{\int_t^T H(\hat{\eta}_t, s-t) ds}.$$

Which is indeed the result for (42) when the myopic demand is given by (43) and the hedging demand by (44).

C Derivation of Model Implied R^2

Assume the model presented in Section 3. Define $R(t, \tau)$ as follows

$$\begin{aligned} R(t, \tau) &= \log \frac{S_{t+\tau}}{S_t} \\ &= \int_0^\tau \left(r - \frac{\sigma_S^2}{2} + \sigma_S \eta_{t+v} \right) dv + \int_0^\tau \sigma_S dZ_{S,t+v} \end{aligned}$$

Then

$$\begin{aligned} E_t [R(t, \tau)] &= (r - \sigma_S^2/2) \tau + \sigma_S \left(\theta \tau + \frac{\eta_t - \theta}{\kappa} (1 - e^{-\kappa \tau}) \right), \\ Var (E_t [R(t, \tau)]) &= \frac{\sigma_S^2 \sigma_\eta^2}{2\kappa^3} (1 - e^{-\kappa \tau})^2. \end{aligned}$$

Let $\varepsilon(t, \tau)$ be the unexplained component of returns such that $\varepsilon(t, \tau) = R(t, \tau) - E_t [R(t, \tau)]$.

Then,

$$\begin{aligned} \varepsilon(t, \tau) &= \sigma_S \left(\int_0^\tau \eta_{t+v} dv - \theta \tau - \frac{\eta_t - \theta}{\kappa} (1 - e^{-\kappa \tau}) \right) + \int_0^\tau \sigma_S dZ_{S,t+v} \\ &= \sigma_S \left(\int_0^\tau \left[\rho \int_0^v \sigma_\eta e^{-\kappa(v-s)} dZ_{S,t+s} + (1 - \rho^2)^{\frac{1}{2}} \int_0^v \sigma_\eta e^{-\kappa(v-s)} dZ_{\eta,t+s} \right] dv \right) \\ &\quad + \int_0^\tau \sigma_S dZ_{S,t+v} \end{aligned}$$

The variance of the unexplained component is given by

$$Var (\varepsilon(t, \tau)) = \sigma_S^2 \left[\frac{\sigma_\eta^2}{\kappa^3} \left[\kappa \tau - 2(1 - e^{-\kappa \tau}) + \frac{1}{2}(1 - e^{-2\kappa \tau}) \right] + 2 \frac{\rho \sigma_\eta}{\kappa^2} [\kappa \tau + e^{-\kappa \tau} - 1] + \tau \right]$$

Define $R^2(t, \tau)$ as the ratio of variance of returns explained by the model relative to the total variance of returns, then

$$\begin{aligned} R^2(\tau) &= \frac{Var (E_t [R(t, \tau)])}{Var (R(t, \tau))} = \frac{Var (E_t [R(t, \tau)])}{Var (E_t [R(t, \tau)]) + Var (\varepsilon(t, \tau))} \\ &= \frac{\frac{\sigma_\eta^2}{2\kappa^3} (1 - e^{-\kappa \tau})^2}{\frac{\sigma_\eta^2}{2\kappa^3} (1 - e^{-\kappa \tau})^2 + \frac{\sigma_\eta^2}{2\kappa^3} \left[\kappa \tau - 2(1 - e^{-\kappa \tau}) + \frac{1}{2}(1 - e^{-2\kappa \tau}) \right] + 2 \frac{\rho \sigma_\eta}{\kappa^2} [\kappa \tau + e^{-\kappa \tau} - 1] + \tau} \end{aligned}$$

TABLE I

Continuous Time Parameter Values

Taken from Barberis (2000), Campbell and Viceira (1999), and Wachter (2002)

Model:

$$\begin{aligned}\frac{dB_t}{B_t} &= r dt \\ \frac{dS_t}{S_t} &= (r + \sigma_s \hat{\eta}_t) dt + \sigma_s d\hat{Z}_S, \\ d\hat{\eta}_t &= \kappa(\theta - \hat{\eta}_t) dt + \varepsilon_\eta d\hat{Z}_S, \\ \varepsilon_\eta &= v_{ss} + \rho\sigma_\eta, \\ v_{ss} : 0 &= -2\kappa v_{ss} + \sigma_\eta^2 - [v_{ss} + \rho\sigma_\eta]^2.\end{aligned}$$

Parameter values at monthly frequency:

$$\begin{aligned}r &= 0.0014, \\ \sigma_S &= 0.0436, \\ \kappa &= 0.0226, \\ \theta &= 0.0788, \\ \sigma_\eta &= 0.0189, \\ \rho &= -0.93, \\ v_{ss} &= 0.0035,\end{aligned}$$

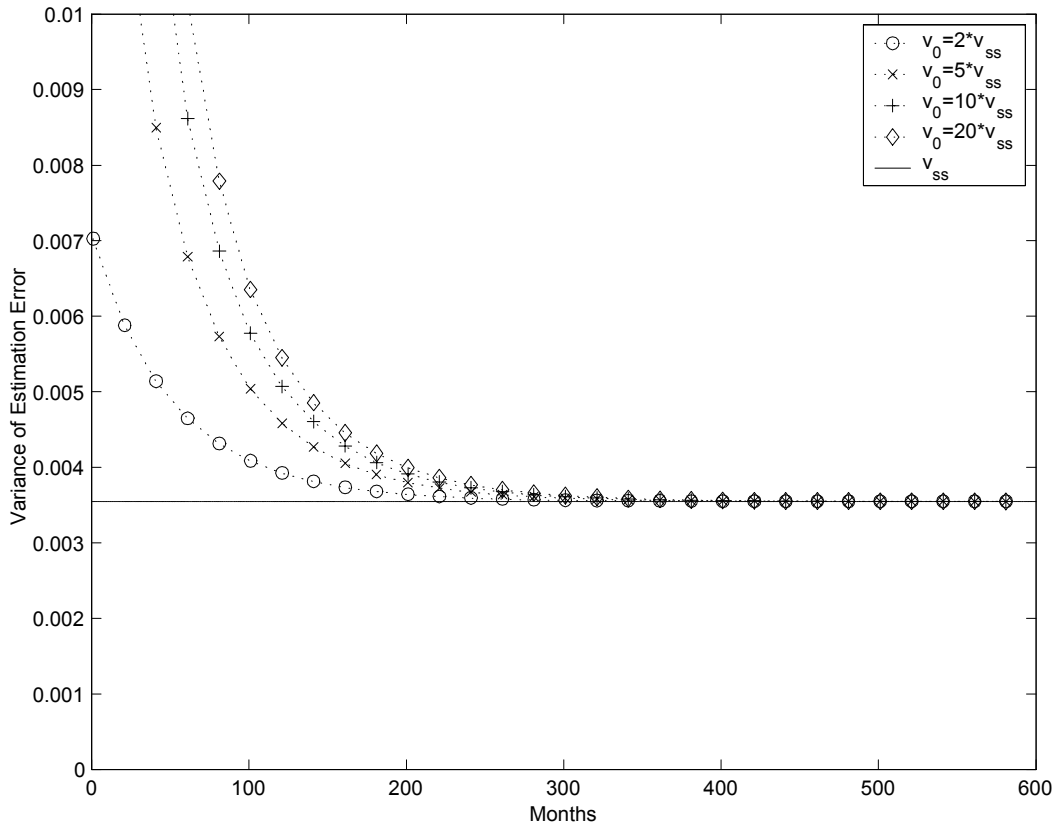


Figure 1: Path of the variance v_t of the estimation error of the unobservable Sharpe ratio for various assumptions on the prior v_0 .

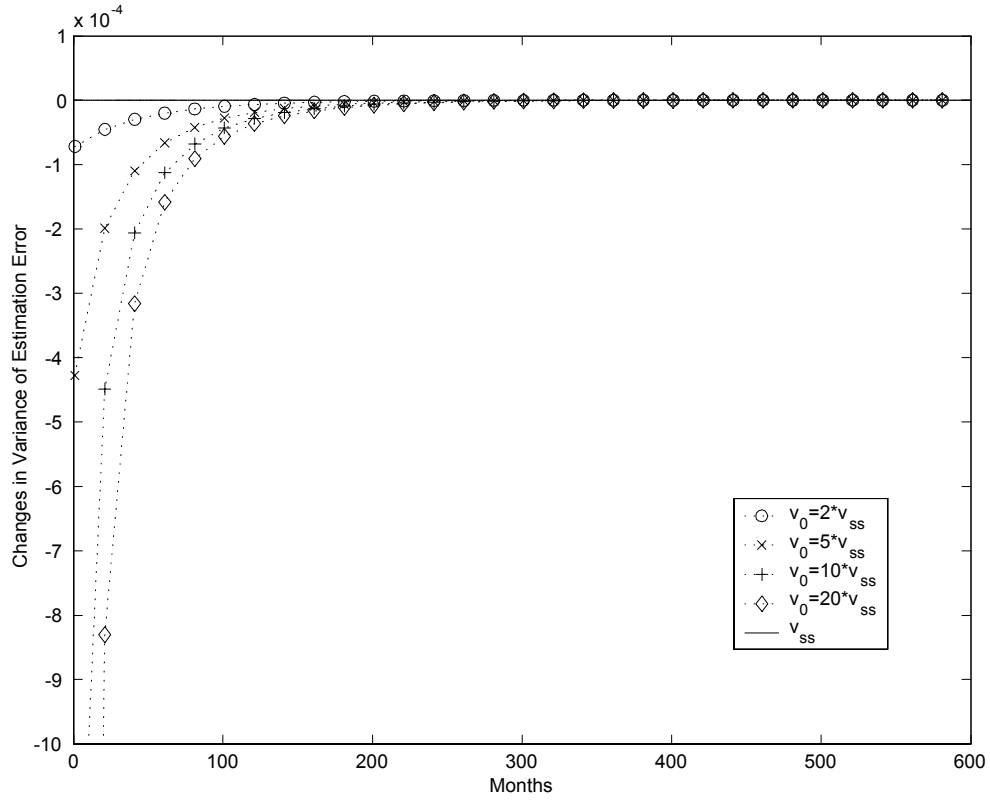


Figure 2: Changes in the variance v_t of the estimation error for the unobservable Sharpe ratio under various assumption for the prior v_0 .

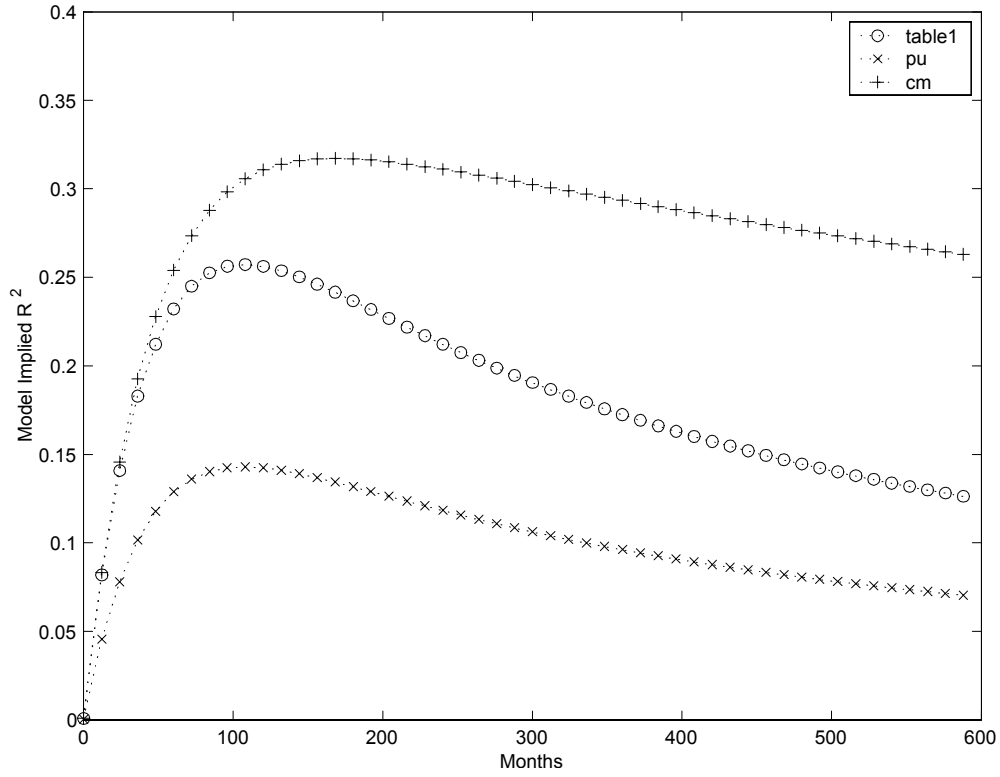


Figure 3: Comparison of Model Implied R^2 for various horizons. **table1** is the model implied by the parameters in Table I without parameter uncertainty. **pu** is the model implied by Table I accounting for parameter uncertainty. **cm** is the model implied by Table I assuming $\rho = -1$.

Table II
Optimal Portfolio Policies under Parameter Uncertainty ^a

γ	Myopic Demand				Hedging Demand				Hedging as % of Total Demand			
	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
Panel A: $\eta = \theta_\eta - \sigma_\eta$												
T=	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
2	68.69	68.69	68.69	68.69	10.37	16.43	24.94	26.99	13.12	19.30	26.64	28.20
5	27.48	27.48	27.48	27.48	8.40	14.21	24.88	28.96	23.40	34.08	47.52	51.31
10	13.74	13.74	13.74	13.74	5.32	9.24	17.40	21.28	27.90	40.22	55.88	60.76
25	5.50	5.50	5.50	5.50	2.49	4.41	8.76	11.19	31.19	44.50	61.45	67.06
Panel B: $\eta = \theta_\eta$												
T=	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
2	90.37	90.37	90.37	90.37	11.98	18.83	28.21	30.43	11.71	17.24	23.80	25.19
5	36.15	36.15	36.15	36.15	9.53	16.01	27.65	32.00	20.86	30.69	43.34	46.96
10	18.07	18.07	18.07	18.07	5.98	10.33	19.19	23.30	24.86	36.37	51.49	56.32
25	7.23	7.23	7.23	7.23	2.78	4.89	9.60	12.18	27.78	40.37	57.05	62.75
Panel C: $\eta = \theta_\eta + \sigma_\eta$												
T=	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
2	112.04	112.04	112.04	112.04	13.58	21.21	31.46	33.85	10.81	15.91	21.93	23.20
5	44.82	44.82	44.82	44.82	10.65	17.80	30.39	35.02	19.20	28.42	40.41	43.87
10	22.41	22.41	22.41	22.41	6.64	11.41	20.96	25.32	22.85	33.74	48.33	53.05
25	8.96	8.96	8.96	8.96	3.07	5.38	10.44	13.16	25.51	37.51	53.82	59.49

^aThis table describes portfolio policies of an investor with parameter uncertainty regarding stock price predictability under assumption about the current value of η . We assume the parameters presented in Table I hold. The portfolio choice of the investor is given by (43) where the myopic component satisfies (44) and the hedging component is given by (45). The table presents the results of the calibration of the model to the parameters in Table I for $\eta = \theta_\eta - \sigma_\eta, \theta_\eta, \theta_\eta + \sigma_\eta$. We consider the asset allocation of the investor for various parameter values for the coefficient of relative risk aversion and the investment horizon (in years). The first column presents the myopic demand under the various assumption for η . The second column presents the hedging demand. The third column shows the how much of the total demand for the stock is due to hedging for changes in the investment opportunity set. Both myopic and hedging demand increase monotonically with increases in the sharpe ratio, but the percentage of the portfolio dedicated to hedging decreases monotonically.

Table III
Comparison of Optimal Portfolio Policies ^a

γ	Parameter Uncertainty				Complete Markets				Difference	
	2.5	5	15	30	2.5	5	15	30		
Panel A: Myopic Demand as % of Wealth										
T=	2.5	5	15	30	2.5	5	15	30		
2	90.37	90.37	90.37	90.37	90.37	90.37	90.37	90.37		
5	36.15	36.15	36.15	36.15	36.15	36.15	36.15	36.15		
10	18.07	18.07	18.07	18.07	18.07	18.07	18.07	18.07		
25	7.23	7.23	7.23	7.23	7.23	7.23	7.23	7.23		
Panel B: Hedging Demand as % of Wealth										
T=	2.5	5	15	30	2.5	5	15	30	2.5	5
2	11.98	18.83	28.22	30.43	17.07	27.29	41.56	44.74	-5.09	-8.46
5	9.53	16.01	27.65	32.00	14.87	26.02	47.89	55.98	-5.34	-10.01
10	5.98	10.32	19.19	23.30	10.21	18.59	38.42	48.41	-4.23	-8.26
25	2.78	4.89	9.60	12.18	5.44	10.16	23.18	32.06	-2.66	-5.27
Panel C: Hedging Demand as % of Total Demand										
T=	2.5	5	15	30	2.5	5	15	30	2.5	5
2	11.71	17.24	23.80	25.19	15.89	23.20	31.50	33.12	-4.18	-5.95
5	20.86	30.69	43.34	46.96	29.14	41.86	56.99	60.77	-8.29	-11.16
10	24.86	36.37	51.49	56.32	36.11	50.71	68.01	72.81	-11.25	-14.34
25	27.78	40.37	57.05	62.75	42.93	58.44	76.23	81.60	-15.16	-18.07

^aThis table compares the consumption and portfolio policies of an investor with parameter uncertainty regarding stock price predictability to an investor which has complete information about the economy and assumes perfect negative correlation between stock returns and the predictive variable, i.e. the investor modeled in Wachter (2002). Both investor assume the parameters presented in Table I hold. The portfolio choice of the investor is given by (42) where the myopic component satisfies (43) and the hedging component is given by (44). The table presents the results of the calibration of the model to the parameters in Table I under the assumption the current value for η_t is the long-run value and the first investor has reached the steady state in the learning process. We consider the asset allocation of the investor for various parameter values for the coefficient of relative risk aversion and the investment horizon (in years). The first column presents the optimal consumption and portfolio policies for the investor with incomplete information. The second column presents the calibration of Wachter (2002) for the parameters in Table I. The third column highlights the differences between each strategy.

Table IV
Changes in the Hedging Demand due to Parameter Uncertainty ^a

γ	Difference in Hedging Demand				% due to Duration Effect				% due to Covariance Effect			
	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
Panel A: $\eta = \theta_\eta - \sigma_\eta$												
2	4.46	7.47	11.92	12.82	19.17	23.56	27.30	26.91	80.83	76.44	72.70	73.09
5	4.86	9.14	18.68	22.24	39.94	45.98	53.74	54.77	60.06	54.02	46.26	45.22
10	3.94	7.70	18.04	23.67	53.12	58.28	66.48	68.77	46.88	41.71	33.52	31.22
25	2.53	5.00	12.90	18.97	65.74	69.38	76.42	79.51	34.26	30.62	23.58	20.49
Panel B: $\eta = \theta_\eta$												
2	5.09	8.46	13.34	14.32	18.23	22.72	26.53	26.16	81.76	77.28	73.47	73.84
5	5.34	10.01	20.25	23.98	38.07	44.45	52.56	53.64	61.93	55.55	47.44	46.36
10	4.23	8.26	19.23	25.10	50.97	56.59	65.34	67.75	49.03	43.41	34.66	32.25
25	2.66	5.27	13.58	19.88	63.67	67.75	75.43	78.72	36.33	32.25	24.57	21.28
Panel C: $\eta = \theta_\eta + \sigma_\eta$												
2	5.72	9.45	14.74	15.78	17.50	22.03	25.86	25.51	82.50	77.97	74.13	74.49
5	5.82	10.87	21.77	25.67	36.48	43.13	51.50	52.60	63.52	56.86	48.50	47.40
10	4.53	8.83	20.40	26.50	49.08	55.09	64.31	66.81	50.92	44.91	35.69	33.19
25	2.79	5.54	14.24	20.78	61.79	66.26	74.53	78.00	38.21	33.74	25.47	22.00

^aThis table describes the differences in the hedging demand of an investor who assumes the market is complete as in Wachter (2002) to an investor who considers parameter uncertainty an follows the optimal portfolio policy in (43). We consider three possible current values for the Sharpe ratio. The first column shows the total difference in hedging demand for different values of relative risk aversion and different investment horizons. The second column shows the percentage of the difference due to the “duration” effect and the third column shows the percentage of the difference due to the “covariance” effect as defined in equation (48).

Table V
Market Timing for Optimal Portfolio Policies ^a

γ	Parameter Uncertainty				Complete Markets				Difference			
	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
Panel A: Intercept (*100)												
T=	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
2	102.63	110.01	121.35	125.14	107.84	118.87	136.26	141.91	-5.22	-8.87	-14.91	-16.77
5	45.76	52.44	64.98	70.27	51.16	62.66	86.39	96.47	-5.40	-10.22	-21.41	-26.23
10	24.08	28.50	37.68	42.19	28.34	36.85	57.50	68.62	-4.26	-8.35	-19.82	-26.43
25	10.02	12.14	16.92	19.58	12.68	17.43	30.66	39.91	-2.66	-5.29	-13.74	-20.33
Panel B: Slope												
T=	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
2	12.35	12.81	13.38	13.54	12.70	13.38	14.25	14.48	-0.35	-0.57	-0.86	-0.94
5	5.20	5.57	6.15	6.33	5.46	6.07	7.10	7.43	-0.27	-0.50	-0.95	-1.10
10	2.65	2.88	3.28	3.42	2.81	3.20	3.99	4.32	-0.16	-0.32	-0.72	-0.90
25	1.07	1.18	1.37	1.45	1.14	1.32	1.76	2.00	-0.07	-0.15	-0.39	-0.54
Panel C: Market-Timing Aggressiveness												
T=	2.5	5	15	30	2.5	5	15	30	2.5	5	15	30
2	12.03	11.65	11.03	10.82	11.77	11.25	10.46	10.20	0.26	0.39	0.57	0.62
5	11.36	10.63	9.46	9.01	10.68	9.69	8.22	7.70	0.68	0.94	1.24	1.30
10	11.00	10.11	8.69	8.10	9.91	8.68	6.94	6.29	1.08	1.43	1.75	1.81
25	10.70	9.70	8.11	7.42	9.02	7.61	5.75	5.00	1.68	2.09	2.37	2.41

^aThis table shows the slope and intercept of the portfolio policies as represented in equations (43) and (44). We assume the parameters presented in Table I hold. The portfolio choice of the investor is given by (43) where the myopic component satisfies (44) and the hedging component is given by (45). The table presents the results of the calibration of the model to the parameters in Table I. We consider the asset allocation of the investor for various parameter values for the coefficient of relative risk aversion and the investment horizon (in years). The first panel presents intercept of the portfolio strategy. The second panel presents the slope of the portfolio strategy. The third panel shows the measure of market timing aggressiveness which is the slope of the portfolio policy divided by the intercept.