

Asset Allocation under Imperfect Information:

Closed-Form Consumption and Portfolio Choice under Incomplete Markets

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Abstract

This paper studies consumption and portfolio choice under incomplete markets and parameter uncertainty. I establish the necessary conditions under which the investor's optimization problem under incomplete markets can be transformed into a complete markets problem. This paper extends the literature on asset allocation by obtaining closed form solutions to the consumption and portfolio problem of an investor with incomplete information about variables which determine the changes in the investment opportunity set and by stating a set of conditions which allow the treatment of the incomplete markets consumption and portfolio choice problem with martingale methods developed in the complete markets framework. The methodology presented in this paper bridges the gap between the numerical and approximate solutions found in incomplete markets consumption and portfolio choice problems and the closed-form solutions found in complete markets consumption and portfolio choice problems.

1 Introduction

This paper studies consumption and portfolio choice under incomplete markets and parameter uncertainty. I establish the necessary conditions under which the investor's optimization problem under incomplete markets can be transformed into a complete markets problem. This paper extends the literature on asset allocation by obtaining closed form solutions to the consumption and portfolio problem of an investor with incomplete information about variables which determine the changes in the investment opportunity set and by stating a set of conditions which allow the treatment of the incomplete markets consumption and portfolio choice problem with martingale methods developed in the complete markets framework.

Under assumptions on the dimension of the security space and the unobserved state variable process, together with a restricted form of parameter uncertainty, we can apply the certainty equivalence of Simon (1956) and solve the consumption and portfolio choice problem substituting the unknown parameters by their estimates.¹ Under certainty equivalence, the separation of the inference problem and the optimization problem applied initially to continuous time asset allocation models by Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) holds. These authors study the consumption and investment problem under incomplete information arising from an unobserved state variable. They show that the optimization problem where some parameters are unknown can be transformed into an optimization problem using the estimates of the unknown parameters and the price and state variable processes obtained by the inference problem. In continuous time, portfolio choice under parameter uncertainty can then be solved in two steps. First, unobservable parameters are estimated by filtering signals for unobservable parameters from the observable data. Second, the investor chooses optimal consumption and portfolio policies given these estimates. I follow their algorithm under the plausible assumption that the rank of price processes is equal to the dimension of observable shocks in the economy, ensuring market completeness after the investor solves the inference problem. This implies that prices are the only signals investor used to estimate unobservable state variables as would be expected if the semi-strong form of market efficiency holds. I apply the Cox-Huang (1989) technique to the consumption and portfolio choice problem. I assume steady-state dynamics in the inference problem to obtain closed-form solutions to the consumption and portfolio policies and find substantial reductions in the hedging demand of an investor relative to models where the role of parameter uncertainty is not considered.² We can also determine, under preferences over terminal wealth, a certainty equivalent wealth

¹Gennotte (1986) provides the following definition for certainty equivalence: The certainty equivalence principle states that if the objective function is quadratic and if the controlled process is a linear function of the unobservable state variables, then the optimization problem can be solved assuming the unobserved variable are known and equal to their conditional expectation.

²In the inference problem, steady state dynamics are satisfied when the variance of the estimation error for

based measure of the benefits of full information. Furthermore, I show steady-state learning is not a strong assumption given the amount of price data currently available. In our calibration, we show an agent with 10 to 15 years of data essentially displays the same inference problem as the steady-state inference problem.

Evidence of predictability in asset markets has revived the consumption and portfolio choice literature. Recently, economists have focussed on hedging demand due to changes in the investment opportunity set. Merton (1971) derives the existence of a hedging portfolio that accounts for changes in variables determining the attractiveness of future investment opportunities. At the time, the empirical evidence was unable to reject the hypothesis that asset prices followed a random walk. Without time varying returns, it followed naturally that portfolio choice should be entirely myopic. More recently, Poterba and Summers (1988), Campbell and Shiller (1988) and Fama and French (1989) find evidence of predictability in the time series of asset prices. Lewellen (2001a) show that mean reversion in stock return may be even stronger than previously perceived. He shows that mean reverting component comprises more than 25% of stock returns. With abundant evidence that expected returns are time varying, Kim and Omberg (1996) study the role return predictability on optimal asset allocation problem, finding closed form solutions for the hedging demands. More recently, Brennan (1998), Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2002), Chacko and Viceira (2001), and Liu (2001), extend this work in a variety of directions. For example, Wachter (2002) shows that in a complete markets model, hedging demand due to mean reversion in expected excess returns can explain why investment advisors suggest younger investors should have more of their wealth allocated to the risky asset than older investors. This result has been dubbed the investment horizon effect.

While there is abundant evidence of predictability, there is good reason to question the stability of the time series relationships seen in historical data. Any reasonable normative model of portfolio choice must hence acknowledge a role for parameter uncertainty. Bawa and Klein (1976) and Bawa, Brown, and Klein (1979) study the role of uncertainty in asset allocation. Kandel and Stambaugh (1996) extend the theory to consider uncertainty about the predictability in asset prices. They find that the predictive relation between returns and the dividend to price ratio, although statistically weak, is economically significant even in the presence of estimation risk. In other words, investors should account for predictability in the portfolio decision, hence it would be suboptimal for the investor to invest under the assumption of a random walk process for asset prices and ignore the role predictability should play in asset allocation even when the evidence of predictability is statistically weak. Balduzzi and Lynch (2000) find that even in the presence of transaction costs,

the unobserved variables does not change with additional observations. The assumption of steady-state dynamics essentially reduces the state space because, under steady-state inference, the variance of the estimation error is not a state variable.

ignoring predictability in asset prices comes at a substantial cost, in terms of utils, to the investor. Barberis (2000) extends Kandel and Stambaugh to a dynamic asset allocation problem. In his model, the hedging demand created by the interaction between learning and the current value of the state variable is ignored. Xia (2001) extends Barberis (2000) to consider such interaction and finds that this component of the hedging demand is non-monotonic and can change signs depending on whether the dividend-price ratio is above or below the assumed long run value for the ratio. Her paper is the first one to question the investment horizon effect.

In this paper, I develop a normative model of portfolio choice under the assumption that the agent understands the degree of predictability, but not the value of the predictive parameter. This setup is motivated by the standard assumption that excess returns to risky assets are a function of the volatility and the market price of risk. Several empirical studies have shown volatility is easily estimated, hence, the source of parameter uncertainty we should care about lies in the parametrization of the market price of risk. In this case, under the assumption of steady-state inference, closed-form solutions are readily obtainable. My results are novel in two dimensions. First, I show how parameter uncertainty, a reasonable assumption to make given the empirical evidence presented above, can help us simplify the consumption and portfolio choice problem. Second, I find closed form solutions to both the consumption and the portfolio policies when markets are incomplete. Recent articles in operations research address some of the issues raised in this paper. Lakner (1995,1998), Karatzas and Zhao (2001), and Rishel (1999) study the asset allocation problem under incomplete information. The focus of these papers is methodology. My approach focuses on how partial information can substantially change the economic intuition regarding consumption and portfolio choice. My paper is an extension of Liu (2001) and Wachter (2002) to incomplete markets in a partially observable economy. Similarly, it can be framed as an extension to Barberis (2000) by allowing for exact solutions to the consumption and portfolio choice of the investor under parameter uncertainty. Our paper also offers an alternative to the approximate solution methods for consumption and portfolio choice in incomplete markets presented by Campbell and Viceira (1999,2002), and Chacko and Viceira (2001).

Section 2 discusses the structure of the economy and solves the optimization problem of the agent in a partially observable economy in a general setting. I provide a simple application of the separation theorem as it applies to our model, the filtering theory of Lipster and Shiriyayev (2000), and the complete markets portfolio choice methods of Cox and Huang (1989). In Section 3, I study stock price predictability under the assumption that the instantaneous Sharpe ratio is not observable and solve for the optimal consumption and portfolio policies. This section can be considered an extension of Liu (2001) and Wachter (2002) to incomplete markets in a partially observable economy. We calibrate the model to the VAR of Campbell and Viceira (2000), extended to continuous time in Campbell, Rodriguez, and Viceira (2002). I compare our results to Wachter

(2002) where the investor assumes complete markets and show parameter uncertainty has a strong effect in the portfolio choice of the agent. We also compare our investor to an investor with full information about the process driving the Sharpe ratio and obtain a measure of the cost of incomplete information. In Section 4, in the context of the example considered in Section 3, I simulate how an investor, with a given prior variance for the estimation error of the unobserved variable, learns about the variable and how the variance of the estimation error changes with each new observation. I show the variance of the estimation error should not be a major consideration in these problems such that parameter uncertainty, the estimate of the unobserved variable, drives differences between the policies obtained under incomplete information and those obtained under complete information. I show, given the amount of data available to the investor, changes in the variance of the estimation error are negligible, such that assuming steady state in the inference process is not as strong an assumption as might be initially expected.³ Section 5 concludes and offers a variety of extensions for the methodology presented in this paper including extensions for other asset allocation models and option replication portfolios.

2 The Model

The economy consists of a set of securities with price processes defined in continuous time and a representative agent with utility over lifetime consumption. Assume time t takes values in the set $\mathcal{T} = [0, T]$. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration \mathbb{F} . Assume the filtration is right-continuous and the probability space is complete. Assume the existence of a d_Z -dimensional orthogonal Brownian Motion Z and a d_W -dimensional orthogonal Brownian Motion W on the probability space such that \mathbb{F} is the standard filtration generated by Z and W . The Brownian Motions Z and W are assumed to be orthogonal to each other. For all Ito processes in this paper assume all drift coefficients are defined in \mathcal{L}^1 and all diffusion term coefficients are defined in \mathcal{L}^2 .⁴

2.0.1 Securities Market and State Variables

The securities market consists of a riskless asset, the money market account, which pays the locally riskless rates at all times, and N risky securities which span Z , the Brownian motion related to shocks in asset prices. The money market account grows at the riskless rate of return. The price

³Even under steady state inference, the investor does not observe the unobservable variable because under steady state inference the variance of the estimation error is positive such that the estimate might not equal the true value for the variable.

⁴Assume the following definition for the sets described in the paper hold:

$$\begin{aligned}\mathcal{L}^1 &= \left\{ X \in \mathcal{L} : \int_0^T |X_t| dt < \infty \text{ a.s.} \right\}, \\ \mathcal{L}^2 &= \left\{ X \in \mathcal{L} : \int_0^T X_t^2 dt < \infty \text{ a.s.} \right\}.\end{aligned}$$

of money market account satisfies

$$dB_t = B_t [r_t dt], \quad (1)$$

where r_t is the locally riskless rate of return.

The prices for the risky securities are given by the following multidimensional Ito process

$$dS_t = S_t [\mu_{S_t} dt + \sigma_{S_t} dZ_t], \quad (2)$$

where $\mu_{S_t} \in (\mathcal{L}^1)^N$ and $\sigma_{S_t} \in (\mathcal{L}^2)^{N \times d_Z}$. Assume the dimension of the Z is equal to the rank of σ_{S_t} almost surely. The drift component represents the instantaneous expected return for the asset, while the diffusion is defined as the volatility of the asset.

Changes in the investment opportunity set of the agent are represented by a vector X_t of state variables. The state variables satisfy the following multidimensional Ito process:

$$dX_t = \mu_{X_t} dt + \sigma_{X_t} dZ_t + \sigma_{W_t} dW_t, \quad (3)$$

where $\mu_{X_t} \in (\mathcal{L}^1)^N$, $\sigma_{X_t} \in (\mathcal{L}^2)^{N \times d_Z}$, and $\sigma_{W_t} \in (\mathcal{L}^2)^{N \times d_W}$. The market is incomplete since the dimension of Z plus the dimension of W is greater than the diffusion coefficient of the security prices. Some of the state variables might not be observable. We will assume that the number of unobservable parameters is equal to the difference between the total number of shocks and the number of shocks spanned by market securities. In other words, the rank of σ_{W_t} is equal to d_W .

2.0.2 Investors Preferences and Budget Constraint

The economy consists of a representative investor with utility over intertemporal consumption. The investor's preferences are assumed to satisfy the standard constant relative risk aversion, power utility function:

$$E_0 \left[\int_0^T e^{-\phi t} \frac{C_t^{1-\gamma} - 1}{1-\gamma} dt \right], \quad (4)$$

where γ is the coefficient of relative risk aversion and ϕ is the agent's discount rate. We assume the agent budget satisfies:

$$dW_t = W_t \{ [r_t + \alpha_t (\mu_{S_t} - r_t \iota)] dt + \alpha_t \sigma_{S_t} dZ_t \} - C_t dt \quad (5)$$

and the agent is subject to a non-negative wealth constraint.

2.1 Solution for the Model

This section solves the investor's optimization problem. The agent optimization problem is to maximize (4) subject to (5) and the non-negative wealth constraint under the filtered processes. Similar to Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986), the investor's consumption and portfolio choice problem follows two major steps: (1) an inference problem in which the investor updates his or her estimate of the unobservable state variables, (2) an optimization problem in which the agent chooses her optimal consumption and portfolio policies under the new estimate for the unobservable state variables. We can use the separation theorem stated above as long as certainty equivalence holds.⁵

The rest of this section I solve the investor's inference problem and optimization problem. I provide conditions which satisfy the continuous-time equivalent of the certainty equivalence principle to hold and apply the separation theorem to our problem.

2.1.1 Inference Problem

Certainty equivalence is satisfied when the power of any variable in the value function is at most two. In order to satisfy the conditions for certainty equivalence, we assume the drifts of the stock price processes in (2) is given by

$$\mu_{St} = \beta_{0t} + \beta_{Xt}X_t, \quad (6)$$

and the drifts of the state variables processes in (3) satisfy

$$\mu_{Xt} = a_{0t} + a_{1t}X_t. \quad (7)$$

Since the state variables are linearly related to the stock processes, it follows the linear-quadratic structure required for certainty equivalence holds for continuous time consumption and portfolio optimization problem.

An example of a model that would satisfy (6) and (7) is to assume the dividend to price is incorrectly measured since dividends are obtained at most at quarterly frequency and prices are updated continuously and expected returns on stock prices are predictable by the dividend to price ratio. For the example, X_t , would be the dividend to price ratio. We can also assume X_t is the earnings to price ratio and do the same analysis.

⁵Certainty equivalence (Simon (1956)) allows us to replace the unobservable variables with their least squares estimates in the optimization problem. Gennotte explains how the certainty equivalence argument clearly holds in continuous time because instantaneous changes in the optimization problem depend solely on the first and second moments of the processes. When certainty equivalence holds we can separate the investor's problem into an inference problem and an optimization problem subject to the estimated values for the unobservable parameters obtained in the inference stage.

The inference problem is solved with filtering methods covered in Lipster & Shiriyayev (2000). We follow their treatment as it applies to our model. Assume the investor observes instantaneous returns to the money market account (1) and the equity (2). Assume the investor also knows $\sigma_{St}, \sigma_{Xt}, \sigma_{Yt}, \beta_{0t}, \beta_{Xt}, a_{0t}, a_{1t}$. However the investor does not observe the current state of X_t . Consequently, none of the Brownian Motions in the economy are observable.

Agents begin with the same prior $X_0 \sim N(\hat{X}_0, v_0)$. In terms of the filtering literature, we can think of (1) and (2) as the observation equations and (3) as the system equations. The filtering theory for continuous time developed by Lipster and Shiriyayev, allows us to describe the dynamics of the mean and the variance of the distribution of the unobservable stochastic process X_t . The instantaneous changes in the drift and the variance-covariance matrix of X_t are given by:

$$d\hat{X}_t = \left[a_{0t} + a_{1t}\hat{X}_t \right] dt + \left[\sigma_{Xt}\sigma'_{St} + \beta_{Xt}v_t \right] \left[\sigma_{St}\sigma'_{St} \right]^{-1} \left[S_t^{-1}dS_t - \left(\beta_{0t} + \beta'_{Xt}\hat{X}_t \right) dt \right], \quad (8)$$

$$\frac{dv_t}{dt} = a_{1t}v_t + v_t a'_{1t} + \sigma_{Xt}\sigma'_{Xt} + \sigma_{Yt}\sigma'_{Yt} - \left[\sigma_{Xt}\sigma'_{St} + \beta_{Xt}v_t \right] \left[\sigma_{St}\sigma'_{St} \right]^{-1} \left[\sigma_{Xt}\sigma'_{St} + \beta_{Xt}v_t \right]' \quad (9)$$

Note that v_t is a multi-dimensional Riccati Equation and can be solved following Reid (1972).

The new innovation process, defined as the normalized deviation of the return from its conditional mean is given by

$$d\hat{Z}_t = \sigma_{St}^{-1} \left[\mu_{St} - \left(\beta_{0t} + \beta_{Xt}\hat{X}_t \right) \right] dt + dZ_t \quad (10)$$

Although Z_t is not observable, the innovation process \hat{Z}_t is derived from observable processes and is thus observable. The process (10) implies that the risky securities returns (2) are observable under the form

$$dS_t = S_t \left[\left(\beta_{0t} + \beta_{Xt}\hat{X}_t \right) dt + \sigma_{St}d\hat{Z}_t \right] \quad (11)$$

The dynamics for the state variables also become observable under the new innovation process. The state variables dynamics are given by the equation

$$d\hat{X}_t = \left[a_{0t} + a_{1t}\hat{X}_t \right] dt + \left[\sigma_{Xt}\sigma'_{St} + \beta_{Xt}v_t \right] \left(\sigma'_{St} \right)^{-1} d\hat{Z}_t \quad (12)$$

As long as the securities span the rank of \hat{Z} , we achieve market completeness. It is this result which will allow us to tackle the optimization problem by means of the Cox-Huang (1989) technique.

2.1.2 Optimization Problem

As far as the investor is concerned, the stochastic changes to the price and the state variable are perfectly correlated because the price serves as the signal of the state variable, hence the market is complete relative to her information set. After filtering the unobservable processes, the securities

span the number of observable Brownian Motions. The markets are observationally complete, therefore we can apply the Cox-Huang technique to solve for the agent's optimal consumption and portfolio choice.

The agent assumes the prices of the money market account and the risky securities are given by the equations

$$dB_t = B_t [r_t dt], \quad (13)$$

$$dS_t = S_t [\hat{\mu}_{S_t} dt + \hat{\sigma}_{S_t} d\hat{Z}_t], \quad (14)$$

where $\hat{\mu}_{S_t}$ and $\hat{\sigma}_{S_t}$ are chosen to match equation (11). Also, the investor assumes the state variables satisfy the following equation

$$d\hat{X}_t = \hat{\mu}_{X_t} dt + \hat{\sigma}_{X_t} d\hat{Z}_t, \quad (15)$$

where $\hat{\mu}_{X_t}$ and $\hat{\sigma}_{X_t}$ are chosen to match equation (12).

When the agent has incomplete information, the agent's portfolio hedging demand needs to account for the unobserved state variables, but also for the reduction in variance the estimation error as new observations come about. In our model, we assume inference has reached a steady state. In other words, the variance of the distribution for the estimated parameter does not change with each new observation. Thus $dv_t = 0$, and v_t does not need to be considered a state variable in the consumption and portfolio choice problem. In Section 4, we discuss the merits of the steady state learning assumption and show that with a reasonable amount of data, about 10 to 12 years of observations, the variance of the estimation error is very close to the variance implied by the steady state results.

Under complete markets and the assumption of no arbitrage, we can define a unique stochastic discount factor for the economy. Let M_t be the stochastic discount factor, the process for M_t must satisfy the following condition: $M_t B_t$ and $M_t S_t$ are martingales under the risk-neutral probability measure \mathbb{Q} , also known as the equivalent martingale measure. Assuming that $\log M_t$ follows an Ito process, an application of Ito's Lemma given (13) and (14) yields the following process for the stochastic discount factor:

$$\frac{dM_t}{M_t} = -r_t dt - \hat{\eta}_t d\hat{Z}_t, \quad (16)$$

where

$$\hat{\eta}_t = \left(\sigma_{S_t} \sigma'_{S_t} \right)^{-1} \sigma_{S_t} (\hat{\mu}_{S_t} - r_t)$$

and $\hat{\eta}_t$ satisfies Novikov's Condition. Note $\hat{\eta}_t$ is the estimated Sharpe ratio.

Now that we have found the process governing the dynamics of the stochastic discount factor, we can solve the agent's optimization problem via martingale methods. As stated previously in this

section, the agent's optimization problem is to maximize the expected lifetime utility of consumption (4) subject to the dynamic budget constraint under the estimated processes for the securities and a non-negative wealth constraint. The dynamic budget constraint under the estimated processes is given by

$$dW_t = W_t \left\{ [r_t + \alpha_t (\hat{\mu}_{St} - r_t)] dt + \alpha_t \hat{\sigma}_{St} d\hat{Z}_t \right\} - C_t dt$$

The existence of the stochastic discount factor allows us to write the agent's dynamic budget constraint as a static budget constraint given by

$$E_0 \left[\int_0^T M_t C_t \right] \leq W_0. \quad (17)$$

Equation (17) states the agent's expected consumption stream in the future appropriately discounted will be less than or equal to his current wealth.

The investor's problem can now be solved as a static optimization problem as described in Cox and Huang (1989) and Karatzas and Shreve (1998, Chapter 3). Intuitively, since the market is complete, we can construct Arrow-Debreu securities such that we accomplish the amount of consumption desired in each possible state. Thus, there is no uncertainty regarding the consumption and portfolio choice of the agent conditional of knowing the state, the only uncertainty that remains is the realization of a given state. The investor shifts the allocation to the risky asset according to future consumption expectations.

The first order condition for utility maximization under the budget constraint is given by

$$C_t = \left(\lambda e^{\phi t} M_t \right)^{-\frac{1}{\gamma}}, \quad (18)$$

where λ is the Lagrangian multiplier. Substituting the first order condition for consumption (18) into the static budget constraint (17) yields the following expression for the static budget constraint:

$$W_t = \frac{1}{M_t} E_t \left[\int_t^T M_s (\lambda M_s)^{-\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma} s} ds \right]. \quad (19)$$

Equation (19) states that wealth is a function of the stochastic discount factor and the processes that drive the distribution of the stochastic discount factor. As shown in (16), the only processes that matter for the distribution of the stochastic discount factor are the interest rate and the Sharpe ratio. I assume both process are driven by the estimated state variable vector \hat{X}_t , therefore the current values for both the stochastic discount factor and the estimated state variable determine the information set the agent uses in forming conditional expectations.⁶

⁶This is due to the fact that both M_t and \hat{X}_t are Markov processes.

Let $Q_t = (\lambda M_t)^{-1}$ such that

$$W_t = Q_t E_t \left[\int_t^T Q_s^{\frac{1}{\gamma}-1} e^{-\frac{\phi}{\gamma}s} ds \right] \quad (20)$$

Notice for an asset allocation problem under terminal wealth at time s with the same Lagrangian multiplier λ and discount factor we would obtain

$$W_t = Q_t E_t \left[Q_s^{\frac{1}{\gamma}-1} e^{-\frac{\phi}{\gamma}s} \right] = F \left(Q_t, \hat{X}_t, s - t \right) \quad (21)$$

such that for the consumption problem we can write current wealth as

$$W_t = \int_t^T F \left(Q_t, \hat{X}_t, s - t \right) ds = G \left(Q_t, \hat{X}_t, T - t \right) \quad (22)$$

Under the risk-neutral measure, the rate of return to wealth is equal to the instantaneous rate of return for the money market account, thus the drift of the wealth process, as obtained by the applying Ito's Lemma, must equal the locally riskless rate times the agent's current wealth. The condition above implies the following partial differential equation is solved by the agent's wealth function:

$$\begin{aligned} r_t G &= Q_t^{\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}t} + \frac{\partial G}{\partial t} + \frac{\partial G}{\partial Q_t} \mathcal{D}Q_t + \frac{\partial G}{\partial \hat{X}_t} \mathcal{D}\hat{X}_t \\ &+ \frac{1}{2} \frac{\partial^2 G}{\partial Q_t^2} \mathcal{D} \langle Q_t, Q_t \rangle + \frac{1}{2} \frac{\partial^2 G}{\partial \hat{X}_t^2} \mathcal{D} \langle \hat{X}_t, \hat{X}_t \rangle + \frac{\partial^2 G}{\partial Q_t \partial \hat{X}_t} \mathcal{D} \langle Q_t, \hat{X}_t \rangle \end{aligned} \quad (23)$$

where $\mathcal{D}f$ is defined as the drift under the risk-neutral measure of the process f , and $\langle f, g \rangle$ denotes the quadratic variation of the processes f and g . The boundary condition for (23) is given by

$$G \left(Q_T, \hat{X}_T, 0 \right) = 0.$$

From Wachter (2002), we know that F can be written in the following form:

$$F \left(Q_t, \hat{X}_t, T - t \right) = Q_t^{\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}t} H \left(\hat{X}_t, T - t \right) \quad (24)$$

such that the total wealth of the agent can be written as an integral of the F pertaining to the remaining life of the agent:

$$G \left(Q_t, \hat{X}_t, T - t \right) = Q_t^{\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}t} \int_t^T H \left(\hat{X}_t, s - t \right) ds \quad (25)$$

Solutions for the problem are obtained by guessing an exponential affine or exponential quadratic function for $H \left(\hat{X}_t, T - t \right)$, substituting the guessed form into the PDE (23) and matching terms to obtain a system of ODEs from which we obtain closed-form solutions.

2.2 Portfolio Choice

The portfolio choice of the agent is obtain using the Cox-Huang (1989) methodology. Similar to equation (23) in order to find the portfolio strategy of the agent we must have that the portfolio allocation of the agent is such that the magnitude and direction stochastic changes due to consumption needs are matched by shocks to the assets in the agent's portfolio. In simpler terms, the agent's consumption risk is fully hedged by the portfolio strategy. It follows

$$\alpha_t G \sigma_{St} = \underbrace{\frac{\partial G}{\partial Q_t} \mathcal{V} Q_t}_{\text{myopic demand}} + \underbrace{\frac{\partial G}{\partial \hat{X}_t} \mathcal{V} \hat{X}_t}_{\text{hedging demand}}, \quad (26)$$

where $\mathcal{V}f$ denotes the coefficient vector to the diffusion process for function f . We substitute (25) into (26) to obtain:

$$\begin{aligned} \alpha_t &= \frac{1}{G} \frac{\partial G}{\partial Q_t} \Sigma^{-1} \sigma_Q + \frac{1}{G} \frac{\partial G}{\partial \hat{X}_t} \Sigma^{-1} \sigma_X \\ &= \frac{1}{\gamma} (\sigma_{St} \sigma'_{St})^{-1} (\mu_{St} - r_t) + \frac{\int_t^T \frac{\partial H(\hat{X}_t, s-t)}{\partial \hat{X}_t} ds}{\int_t^T H(\hat{X}_t, s-t) ds} \Sigma^{-1} \sigma_X \end{aligned} \quad (27)$$

where $\Sigma = \sigma_{St} \sigma'_{St}$ and $\sigma_f = (\mathcal{V}f) \sigma'_{St}$.

The portfolio choice of the agent can be decomposed into its myopic demand, the demand due to the current state of the economy, and the hedging demand, the demand due to expected changes in the investment opportunity set. In the model, the hedging demand is due to the stochastic nature of the estimated state variables. As we will see in the next section when we discuss the consumption to wealth ratio, the function H which determines the magnitude of the hedging demand is the agent's current wealth to consumption ratio. Not surprisingly, the relation between the agent's current consumption relative to expected future consumption is inextricably related to how the agent determines to hedge changes in the investment opportunity set. This relation is made clearer in the complete markets setting where we know the agent can trade in the securities available in order to achieve an optimal Arrow-Debreu allocation of resources.

When $H(\hat{X}_t, s-t)$ is assumed to be exponential ($H(X_t, s-t) = \exp[h(X_t, s-t)]$), we can write the hedging demand component of the agent's portfolio choice as

$$\alpha_t^{\text{hedging}} = \frac{\int_t^T \frac{\partial h(\hat{X}_t, s-t)}{\partial \hat{X}_t} H(\hat{X}_t, s-t) ds}{\int_t^T H(\hat{X}_t, s-t) ds} \Sigma^{-1} \sigma_X \quad (28)$$

The magnitude of the hedging demand is the weighted average of the sensitivity of the log wealth to consumption ratio to the variables defining the changes in the investment opportunity set where the weights are given by the amount of expected consumption in a given period relative to total expected consumption in the remaining periods.

2.3 Consumption to Wealth Ratio

The consumption to wealth ratio is easily obtain by applying some algebra to equations (24) and (25)

$$\frac{C_t}{W_t} = \frac{F(Q_t, \hat{X}_t, 0)}{G(Q_t, \hat{X}_t, T-t)}, \quad (29)$$

it can easily be shown that $F(Q_t, \hat{X}_t, 0) = Q_t^{\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}t}$ such that

$$\frac{C_t}{W_t} = \left[\int_t^T H(\hat{X}_t, s-t) ds \right]^{-1}. \quad (30)$$

An application of equation (30) to the portfolio hedging demand (28) yields the following expression for the investor's hedging demand:

$$\alpha_t^{hedging} = \frac{C_t}{W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \hat{X}_t} \Sigma^{-1} \sigma_X. \quad (31)$$

Equation (31) shows the link between future expected consumption and the hedging strategy of the investor. When markets are complete, the investor essentially can plan the consumption strategy for each possible outcome at each possible horizon, equation (31) shows how the investor changes the portfolio strategy to maintain the desired consumption plan.

3 Predictability in Stock Prices

Another useful example of the strength of our technique is to analyze the consumption and portfolio choice problem when the Sharpe ratio is mean reverting. The academic literature refers to this property as the predictability of asset prices. The evidence of predictability in stock returns is extensive. Campbell and Shiller (1988) and Fama and French (1989) find that the dividend to price ratio has predictive power over the excess returns of equity over the risk free rate. Recent papers by Lewellen (2001a, 2001b) confirm the predictive power of dividend-yield, book-to-market, and the earnings-price ratio and show economically significant temporary reversals in stock prices. His full sample evidence suggests that 25% to 40% of annual returns are temporary, reversing within 18 months and that mean reversion is stronger in larger stocks.

Barberis (2000) uses the dividend to price ratio to solve numerically for the asset allocation strategy of long-term investors. He finds asset price predictability can explain the common advice that young investors should allocate a greater proportion of their wealth to stocks relative to older investors. Furthermore, Barberis shows that estimation risk due to parameter uncertainty does have

a tempering effect in the investor's market timing. He concludes that ignoring either estimation risk or predictability has a sizable negative effect in the agent's utility. Xia (2001) extends Barberis to the case where the agent has utility over intertemporal consumption and the agent does is uncertain about the degree of predictability in stock returns.

Liu (2001) and Wachter (2002) finds a closed-form solutions to the consumption and portfolio choice where the process that "predicts" excess returns is fully observable. In order to solve the model, Wachter assumes the market is complete and the shocks to the proxy for the predictive variable and the stock price are perfectly negatively correlated. The assumption of perfect negative correlation does not seem controversial given that the empirically estimated correlation for the shocks to the dividend price ratio and the stock price is -0.93. Campbell, Rodriguez, and Viceira (2002) show that in a correctly specified continuous-time model given the discrete time parameters of Campbell and Viceira (2000), the correlation between the dividend price ratio and the stock returns is -0.96. Still, accounting for parameter uncertainty greatly decreases the demand of the risky asset due to hedging for changes in the investment opportunity set.

In this section of the paper, we will essentially extend Wachter to account for parameter uncertainty in the agent's optimization problem. Unlike Wachter, I will not assume market completeness. Instead, I assume parameter uncertainty regarding the current value of the predictive process and obtain an observationally complete market and find exact consumption and portfolio rules. Note that our assumption regarding steady-state estimation does not allow us to study the role of the variance of the estimation error for the unobservable parameters as a state variable in the agent's policies.⁷ Our model assumes the predictive relation is known, since the predictive relation in our model is given by the standard deviation of the risky asset.⁸

We assume the existence of a money market account where the risk free rate is constant and the existence of one risky securities whose price process satisfies

$$\frac{dS_t}{S_t} = (r + \sigma_s \eta_t) dt + \sigma_s dZ_S, \quad (32)$$

such that the Sharpe ratio, η_t , is mean reverting, and whose dynamics are given by

$$d\eta_t = \kappa_\eta (\theta_\eta - \eta_t) dt + \sigma_\eta dZ_\eta. \quad (33)$$

We assume the correlation between shocks to the stock price and shocks to the Sharpe ratio are imperfectly correlated. The correlation coefficient is denoted by $\rho_{S\eta}$.

⁷In a related paper, Lewellen and Shanken (2002) study the equilibrium effects of learning on asset prices. They find mean reversion in asset prices can be explained by the learning of the agents regarding the dividend process.

⁸It is well known from the econometrics literature that variances can be estimated very precisely with a fixed horizon of data and high sampling frequency. Thus, this assumption regarding full information about the predictive relation is consistent with our model.

The imperfect correlation between (32) and (33) implies the market is incomplete. Yet, when the Sharpe ratio is not observable and under assumptions explained in Section 2.1.2, we show the optimization problem can be restated in a complete markets framework.

3.1 Inference Problem

The assumption regarding the processes governing the stock price and the Sharpe ratio as well as the assumption that the instantaneous Sharpe ratio is unobservable allow us to apply the filtering methods of Lipster and Shirayev to find a observationally equivalent economy. Applying the results of section 2.1.1 to the current problem yields the following processes for the stock price and the state variable dynamics respectively:

$$\frac{dS_t}{S_t} = (r + \sigma_s \hat{\eta}_t) dt + \sigma_s d\hat{Z}_S, \quad (34)$$

$$d\hat{\eta}_t = \kappa_\eta (\theta_\eta - \hat{\eta}_t) dt + \varepsilon_{S\eta} d\hat{Z}_S, \quad (35)$$

where

$$\varepsilon_{S\eta} = v_t + \rho_{S\eta} \sigma_\eta$$

and

$$d\hat{Z}_S = [(\eta_t - \hat{\eta}_t) dt + dZ_S].$$

The measurement error (variance) of the Sharpe ratio solves the following Riccati Equation

$$\frac{dv_t}{dt} = -2\kappa_\eta v_t + \sigma_\eta^2 - [v_t + \rho_{S\eta} \sigma_\eta]^2. \quad (36)$$

Following our methodology, when computing for the optimal consumption and portfolio policies, we assume learning has reached a steady state in which new data and estimation does not reduce the measurement error of the Sharpe ratio. Let v_{ss} denote the variance of the estimation error under the steady state. By applying the definition of steady state filtering to (36), v_{ss} is determined by the quadratic equation

$$0 = -2\kappa_\eta v_{ss} + \sigma_\eta^2 - [v_{ss} + \rho_{S\eta} \sigma_\eta]^2.$$

The resulting variance will be positive root of the quadratic equation obtain from our assumption in (36).

3.2 Consumption and Portfolio Choice

After the agent solves the inference problem and estimates the Sharpe ratio, we have a complete market under the information set of the investor. In other words, the estimated processes for the stock price and the Sharpe ratio are seen to be perfectly correlated. The investor sees this processes as perfectly correlated because the inference problem essentially projects the unobservable variable (the Sharpe ratio) into the space of the signal (the stock price), thus the source to both processes after the inference is the same, markets are complete. As seen in (35) the true correlation is accounted for in the diffusion coefficient for the estimated Sharpe ratio.

Under the procedure developed in Section 2.2, the solution to the partial differential equation representing the wealth of the investor (23) is solved by (25) as it applies to the variable in this economy. The solution is given by

$$G(Q_t, \hat{\eta}_t, T-t) = Q_t^{\frac{1}{\gamma}} e^{-\frac{\phi}{\gamma}t} \int_t^T H(\hat{\eta}_t, s-t) ds, \quad (37)$$

where

$$H(\hat{\eta}_t, \tau) = \exp \left[A(\tau) + B(\tau) \hat{\eta}_t + \frac{1}{2} C(\tau) \hat{\eta}_t^2 \right], \quad (38)$$

with boundary conditions

$$A(0) = B(0) = C(0) = 0.$$

Applying (37) to the partial differential equation in (23) and separating according to coefficients for each variable yields the following system of ordinary differential equations (38) must satisfy:

$$\begin{aligned} A'(\tau) &= -\frac{\phi}{\gamma} + B(\tau) \kappa_\eta \theta_\eta + \frac{1-\gamma}{\gamma} r + \frac{1}{2} (C(\tau) + B^2(\tau)) \varepsilon_{S\eta}^2, \\ B'(\tau) &= -B(\tau) [\kappa_\eta + \varepsilon_{S\eta}] + C(\tau) \kappa_\eta \theta_\eta + B(\tau) C(\tau) \varepsilon_{S\eta}^2 + \frac{1}{\gamma} B(\tau) \varepsilon_{S\eta}, \\ C'(\tau) &= -2C(\tau) [\kappa_\eta + \varepsilon_{S\eta}] + C^2(\tau) \varepsilon_{S\eta}^2 + \frac{2}{\gamma} C(\tau) \varepsilon_{S\eta} + \frac{1-\gamma}{\gamma^2}. \end{aligned} \quad (39)$$

We can solve (39) in similar fashion to Kim and Omberg (1996), Wachter (2002), and Chacko and Viceira (2001). The solution to (39) is given by

$$\begin{aligned} A(\tau) &= \int_0^\tau \left(-\frac{\phi}{\gamma} + B(s) \kappa_\eta \theta_\eta + \frac{1-\gamma}{\gamma} r + \frac{1}{2} (C(s) + B^2(s)) \varepsilon_{S\eta}^2 \right) ds, \\ B(\tau) &= \frac{4 \frac{1-\gamma}{\gamma^2} \kappa_\eta \theta_\eta (1 - e^{-\delta\tau/2})^2}{\delta \left[\left(\delta - 2 \left(\frac{1-\gamma}{\gamma} \varepsilon_{S\eta} - \kappa_\eta \right) \right) + \left(\delta + 2 \left(\frac{1-\gamma}{\gamma} \varepsilon_{S\eta} - \kappa_\eta \right) e^{-\delta\tau} \right) \right]}, \\ C(\tau) &= \frac{2 \frac{1-\gamma}{\gamma^2} (1 - e^{-\delta\tau})}{\left(\delta - 2 \left(\frac{1-\gamma}{\gamma} \varepsilon_{S\eta} - \kappa_\eta \right) \right) + \left(\delta + 2 \left(\frac{1-\gamma}{\gamma} \varepsilon_{S\eta} - \kappa_\eta \right) e^{-\delta\tau} \right)}. \end{aligned}$$

where

$$\delta^2 = 4 \left(\frac{1-\gamma}{\gamma} \varepsilon_{S\eta} - \kappa_\eta \right)^2 - 4 \frac{1-\gamma}{\gamma^2} \varepsilon_{S\eta}^2 > 0$$

We derive the agent's portfolio choice by applying (26) and (28) to this example. Let α_t be the proportion of wealth allocated to the risky asset. The portfolio choice of the agent can be decomposed into its myopic and hedging component.

$$\alpha_t = \alpha_t^{myopic} + \alpha_t^{hedging} \quad (40)$$

where

$$\alpha_t^{myopic} = \frac{1}{\gamma} \frac{\hat{\eta}_t}{\sigma_S}, \quad (41)$$

and

$$\alpha_t^{hedging} = \frac{\varepsilon_{S\eta} \int_t^T (B(s-t) + C(s-t) \hat{\eta}_t) H(\hat{\eta}_t, s-t) ds}{\sigma_S \int_t^T H(\hat{\eta}_t, s-t) ds} \quad (42)$$

The hedging demand of the investor has the usual properties found for hedging demand in the presence of excess return predictability. The Sharpe ratio does not only come into play in the assigning of relative weight for the hedging demand via the function H , the wealth to consumption ratio, it also comes into play linearly as a measure of market timing. As was shown in (28), when the solution to H is of the exponential form, the sensitivity of the log wealth to consumption ratio to the state variable determines the relative weight each period in the agent's horizon has on the hedging strategy.

The consumption to wealth ratio for the agent is given by

$$\frac{C_t}{W_t} = \left(\int_t^T H(\hat{\eta}_t, s-t) ds \right)^{-1}.$$

The duration or sensitivity of the wealth to consumption ratio relative to changes in the investment opportunity set is given by

$$\frac{C_t}{W_t} \frac{\partial \left(\frac{W_t}{C_t} \right)}{\partial \hat{\eta}_t} = \frac{\int_t^T (B(s-t) + C(s-t) \hat{\eta}_t) H(\hat{\eta}_t, s-t) ds}{\int_t^T H(\hat{\eta}_t, s-t) ds},$$

as shown generally in Section 2.3, we notice the relationship between the agent's hedging demand and the sensitivity of the agent's consumption and savings decision to changes in the investment opportunity set. This relation is straightforward due to market completeness under the filtered processes and the inextricable link between the agent's hedging demands and the expected consumption in the future.

3.3 Calibration and Results

We calibrate our model to the parameters calculated in Campbell and Viceira (1999). The calibration requires us to convert the discrete time parameters of the Campbell and Viceira (1999) model into continuous time parameters. Campbell, Rodriguez, and Viceira (2002) show how to correctly parametrize the continuous time model given the discrete time parameters. The methods used to recalibrate the model for continuous-time parameters follows the treatment in Bergstrom (1984). We present a synopsis of the main results. Interested readers should refer to Campbell, Rodriguez, and Viceira for completeness.

Campbell and Viceira (1999) study optimal consumption and portfolio choice when expected returns are mean reverting. They assume the riskless rate of return is constant and the log excess return for stocks is given by the following VAR(1) specification:

$$\begin{aligned}\log S_{t_n+\Delta t} &= r_f + x_{t_n} + \varepsilon_{t_n+\Delta t}, \\ x_{t_n+\Delta t} &= (1 - \phi)\mu + \phi x_{t_n} + \eta_{t_n+\Delta t}.\end{aligned}\tag{43}$$

Campbell and Viceira use the dividend to price ratio as a proxy for changes in the investment opportunity set. They derive parameters for (43) for quarterly data. The results of the discrete-time VAR can be obtained in Campbell and Viceira (2000). Campbell, Rodriguez, and Viceira (2002), applies the time-aggregation methods of Bergstrom (1984) to obtain an equivalent continuous-time specification for (43). They show that the continuous time VAR given by

$$\begin{aligned}d\log S_t &= \left(\mu_t - \frac{\sigma_S^2}{2}\right)dt + \sigma_S dZ_S, \\ d\mu_t &= \kappa(\theta - \mu_t)dt + \sigma_\mu dZ_\mu, \\ dZ_S dZ_\mu &= \rho dt,\end{aligned}$$

is equivalent to (43) when

$$\begin{aligned}\phi &= e^{-\kappa\Delta t}, \\ r_f &= r\Delta t, \\ \mu &= \left(\theta - \frac{\sigma_S^2}{2} - r\right)\Delta t,\end{aligned}$$

and the variance-covariance matrix for $(\varepsilon_{t_n+\Delta t}, \eta_{t_n+\Delta t})$ satisfy equations (17), (18), and (19) in Campbell, Rodriguez, and Viceira (2002). Assume for the model presented in our paper that $\mu_t = r + \sigma_S \eta_t$, such that $\theta_\eta = \frac{\theta - r}{\sigma_S}$, and $\sigma_\eta = \frac{\sigma_\mu}{\sigma_S}$. The parameter values for the continuous time calibration of our model for predictability in stock prices are presented in Table I.

3.3.1 Comparison with Wachter (2002)

Table II compares the consumption and portfolio strategy of an investor which estimates the current Sharpe ratio and computes his or her strategy according to the methods in this paper against an investor with perfect information about the economy under the assumption of perfect negative correlation between stock returns and the predictive process, the mean-reverting Sharpe ratio. The second investor type corresponds to the model presented in Wachter (2002). In Table II we assume the Sharpe ratio, as estimated by the first investor and observed by the second investor, is the long run value. The comparison in Table II allows us to concentrate on the role of parameter uncertainty in the hedging demand of the investor and does not account for the possibility of further differences in the consumption and the portfolio strategies of both investors due to the incomplete information structure. In other words, we do not account for further differences due to differences in each agent's belief of the current value of the Sharpe ratio.

As expected, the differences in the proportion of wealth allocated to stock in both examples is due to differences in the hedging demand. The hedging demand for the investor with incomplete information is much lower. In the case where the investor has a coefficient of relative risk aversion of 5 and a 30-year investment horizon, the difference in the percentage of wealth allocated to the risky asset is 61.38%. Another striking result is that under parameter uncertainty the composition of the risky portfolio dedicated to hedging changes in the investment opportunity set fluctuates between 2% and 17% under the various assumed coefficient of relative risk aversion and investment horizon. On the other hand, the complete markets investor dedicates 13% of the portfolio to hedging concerns when her coefficient of relative risk aversion is 2 and her investment horizon is 2.5 years, and 79% of the portfolio when her coefficient of relative risk aversion is 30 and her investment horizon is 50 years. The economic intuition for this result is the following: Since the investor is uncertain about his current estimate of the variable used for market timing purposes, *ceteris paribus*, the investor chooses a portfolio strategy which times the market less aggressively. In our calibration, we still obtain the horizon effect for stock allocation such that the investor still optimally allocates more wealth to stocks when their investment horizon is longer.

Table III consider the portfolio choice of the investor with incomplete information under various assumptions for the current estimate of the Sharpe ratio. The myopic and the hedging demand of the investor seems to increase monotonically with increases in the Sharpe ratio. Yet, the percentage of the portfolio dedicated to hedging changes in the investment opportunity set decreases monotonically with increases in the Sharpe ratio. The result is highly intuitive: When the investor estimates a low value for the Sharpe ratio, the investor is more willing to time the market because he expects the returns to be higher in the future due to the mean reversion in the parameter. This effect is also stronger when the investment horizon is longer. Figure 3 shows the hedging and myopic demand for an agent with a coefficient of relative risk aversion of 5 and an investment horizon

of 30 years for various values of η_t . As η_t increases so does the myopic and hedging demand of the agent, but as shown in Table III, we see a reduction in the amount of the portfolio allocated to hedging changes in the investment opportunity set.

3.3.2 Comparison with Terminal Wealth Models

We now consider an incomplete markets model where the agent wants to maximize the utility of terminal wealth. Under mean-reverting returns, closed-form solution have been found by Kim and Omberg (1996) and Liu (1999). We consider a model where data is calibrated to the parameters in Table I, but one agent assumes he has perfect knowledge of the state variable. This agent will not account for parameter uncertainty in the portfolio choice. We will conduct welfare analysis for the case where there is asymmetric information, such that one agent is fully informed about the instantaneous Sharpe Ratio while the other one solves for the optimal policies by using the estimate provided through the filtering methods.

Let $V_i(W_t, \eta_t, T - t)$ be the value function of the perfectly informed investor and $V_u(W_t, \hat{\eta}_t, T - t)$ be the value function of the investor with imperfect information about the Sharpe ratio. The value function for the informed investor is given by

$$V_i(W_t, \eta_t, \tau) = \phi_i(\eta_t, \tau) \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (44)$$

where the function $\phi_i(\eta_t, \tau)$ is given by

$$\phi_i(\eta_t, \tau) = \exp \left[A_i(\tau) + B_i(\tau) \eta_t + \frac{1}{2} C_i(\tau) \eta_t^2 \right]. \quad (45)$$

The coefficients A , B , and C satisfy the following system of ordinary differential equations:

$$\begin{aligned} A'_i(\tau) &= -\frac{1}{2} \sigma_\eta^2 C_i(\tau) + \left[-\frac{1}{2} \sigma_\eta^2 + \frac{1-\gamma}{2\gamma} \rho_{S\eta}^2 \sigma_\eta^2 \right] B_i^2(\tau) + \kappa_\eta \theta_\eta B_i(\tau) + (1-\gamma) r, \\ B'_i(\tau) &= -\left[\kappa_\eta + \frac{1-\gamma}{\gamma} \rho_{S\eta} \sigma_\eta \right] B_i(\tau) + \kappa_\eta \theta_\eta C_i(\tau) + \left[\sigma_\eta^2 + \frac{1-\gamma}{\gamma} \rho_{S\eta}^2 \sigma_\eta^2 \right] B_i(\tau) C_i(\tau), \\ C'_i(\tau) &= -2 \left[\kappa_\eta + \frac{1-\gamma}{\gamma} \rho_{S\eta} \sigma_\eta \right] C_i(\tau) + \left[\sigma_\eta^2 + \frac{1-\gamma}{\gamma} \rho_{S\eta}^2 \sigma_\eta^2 \right] C_i^2(\tau) + \frac{1-\gamma}{\gamma}, \end{aligned}$$

with boundary conditions

$$A_i(0) = B_i(0) = C_i(0) = 0.$$

The portfolio strategy of the informed investor satisfies

$$\alpha_{S,t}^i = \frac{\eta_t}{\gamma \sigma_S} + \frac{\rho_{S\eta} \sigma_\eta}{\gamma \sigma_S} [B_i(\tau) + C_i(\tau) \eta_t]. \quad (46)$$

Similarly the value function of the uninformed agent is given by

$$V_u(W_t, \hat{\eta}_t, \tau) = \exp \left[A_u(\tau) + B_u(\tau) \hat{\eta}_t + \frac{1}{2} C_u(\tau) \hat{\eta}_t^2 \right] \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (47)$$

where the function $\phi_u(\hat{\eta}_t, \tau)$ is given by

$$\phi_u(\hat{\eta}_t, \tau) = \exp \left[A_u(\tau) + B_u(\tau) \hat{\eta}_t + \frac{1}{2} C_u(\tau) \hat{\eta}_t^2 \right]. \quad (48)$$

The coefficients A , B , and C satisfy the following system of ordinary differential equations:

$$\begin{aligned} A'_u(\tau) &= -\frac{1}{2\gamma} \varepsilon_{S\eta}^2 B_u^2(\tau) + \frac{1}{2} \varepsilon_{S\eta}^2 C_u(\tau) + \kappa_\eta \theta_\eta B_u(\tau) + (1-\gamma)r, \\ B'_u(\tau) &= -\left[\kappa_\eta + \frac{1-\gamma}{\gamma} \varepsilon_{S\eta} \right] B_u(\tau) + \kappa_\eta \theta_\eta C_u(\tau) + \frac{1}{\gamma} \varepsilon_{S\eta}^2 B_u(\tau) C_u(\tau), \\ C'_u(\tau) &= -2 \left[\kappa_\eta + \frac{1-\gamma}{\gamma} \varepsilon_{S\eta} \right] C_u(\tau) + \frac{1}{\gamma} \varepsilon_{S\eta}^2 C_u^2(\tau) + \frac{1-\gamma}{\gamma}, \end{aligned}$$

with boundary conditions

$$A_u(0) = B_u(0) = C_u(0) = 0.$$

The portfolio choice of the uninformed investor has the form

$$\alpha_{S,t}^u = \frac{\hat{\eta}_t}{\gamma \sigma_S} + \frac{\varepsilon_{S\eta}}{\gamma \sigma_S} [B_u(\tau) + C_u(\tau) \hat{\eta}_t]. \quad (49)$$

The strategies of the investors will differ in most periods, the difference between the allocations to the risky asset by both types of investors is

$$\alpha_{S,t}^i - \alpha_{S,t}^u = \frac{(\eta_t - \hat{\eta}_t)}{\gamma \sigma_S} + \left[\frac{\rho \sigma_\eta}{\gamma \sigma_S} B_i(\tau) - \frac{\varepsilon_{S\eta}}{\gamma \sigma_S} B_u(\tau) + \frac{\rho \sigma_\eta}{\gamma \sigma_S} C_i(\tau) \eta_t - \frac{\varepsilon_{S\eta}}{\gamma \sigma_S} C_u(\tau) \hat{\eta}_t \right]. \quad (50)$$

For equation (50) the difference in the myopic demands comes from the fact the informed and the uninformed investors assume different dynamics for changes in the Sharpe ratio. The difference is mainly due to the uninformed investor accounting for estimation error. The differences in the hedging demand are due to the differences in the problems both investors optimize. Once again the uninformed investor optimally accounting for estimation error is the main factor in the differences between the asset allocation strategies of each investor.

The certainty equivalent wealth of the investor is

$$CEW_a = E_0^a [U(W_T)], a = i, u. \quad (51)$$

We can calculate the certainty equivalent wealth by looking at the value function of each investor under the optimal portfolio strategy. Assume both investors have power utility over terminal

wealth, their initial wealth is the same and their coefficient of relative risk aversion as well as their investment horizon is the same. Following Liu and Pan (2002), we quantify the portfolio improvement due to better information in terms of the difference in certainty equivalent quarterly, continuously compounded returns. The portfolio improvement in terms of this measure is

$$\mathcal{R}^{CEW} = \frac{\ln CEW_i}{T} - \frac{\ln CEW_u}{T} = \frac{1}{1-\gamma} \left[\frac{\ln \phi_i(\eta_0, T)}{T} - \frac{\ln \phi_u(\hat{\eta}_0, T)}{T} \right], \quad (52)$$

where $\phi_i(\cdot)$ and $\phi_u(\cdot)$ are defined in (45) and (48) respectively. Table IV shows the improvement the investor obtains when full information about the state variable process is available. The improvement is greater when the investor is less risk averse. Also the improvement seems to be non-monotonic with the horizon. Figure IV shows the value of (52) for an agent with relative risk aversion coefficient of five for various horizons. Indeed, the figure shows the non-linear effect in the measure created by the difference in hedging demand of fully informed investors and incompletely informed investors. Note that for relative risk aversion of two, the curve does show a hump around the 7.5 year mark (30 quarters), while for relative risk aversion of five the effect is monotonic and decreasing. The figure shows the benefits of full information to be greater in the short term.

4 Discussion: The longevity of learning

A crucial assumption we have made to obtain closed-form solution to the investors consumption and investment problem is that learning has reached a steady-state process. In other words, any new observation of the securities will be accounted for by the agent is his new estimates of the unobservable parameters, but the new observation will not contribute in reducing the estimation risk, the variance of the estimates. This assumption begs two questions: (1) How quickly would an agent on average, regardless of prior, reach the steady state in the learning process? (2) Can the estimation risk in the steady state significantly change the investment strategy of the agent? This section provides answers to both questions with the evidence from the applications to the theory.

To answer the first question, we construct a simulation of how the estimation risk of the agent changes after each observation through time. For the case of stock price predictability, we first obtain the steady state variance of the measurement error and then simulate the learning of the agent under the assumption of priors that are multiples of the steady-state estimation risk. We assume changes in the variance of the estimation error follow (36). We assume that new observations are made every 1/10th of a quarter. Our results are not sensitive to this assumption.⁹ Figure 1 shows how the agent's estimation risk under the assumption that the prior is two-times, five-times, ten-times, and twenty-times the steady state value. Notice that by the time 5 years (20 quarters)

⁹ As a robustness check, the author did the simulations with observation made every 1/100th of a quarter, 1/30th of a quarter, and once a quarter without significant change to the results.

pass by, all variance estimates, regardless of prior, are lower than even two-times the steady state variance. By the 10 year mark (40 quarters), the agent is not significantly far away from the steady state regardless of the assumption of the prior. The figure provides strong evidence that our assumption of steady-state learning is not out of line and makes sense given the amount of data the agent has available to estimate these parameters. If we assume the agent has access to the CRSP database then it is fair to state agents have about 40 years (160 quarters) of daily data and about 75 years (300 quarters) of monthly data to learn from before deciding on their consumption and portfolio strategies, thus it is quite believable that a rational agent would achieve a level of learning such that the steady state assumption is innocuous.

In order to understand how the estimation error is reduced with each new observation, we would like to see the magnitude in which the estimation error variance is reduced with each new observation. Figure 2 plots the instantaneous reduction in variance for a given point in time. By the time the agent has observed 10 years (40 quarters) of data, the reduction in the variance of the estimates of the unobservable variables are negligible. This implies, the learning effect should be negligible in the hedging component of the agent's portfolio for our model. Our results imply parameter uncertainty, not learning, drives the changes in the portfolio composition in comparison to the portfolio model under perfect observability of all processes.¹⁰

5 Conclusion

We study the incomplete markets consumption and portfolio choice optimization problem under partially observable parameters. Under suitable assumption of the number of securities in the market as well as which parameters are unobservable, we can transform the problem into one where the market is observationally complete after estimating unobservable parameters and accounting for parameter uncertainty. Obtaining an observationally complete market, allows us to solve exactly the investor's optimization problem and obtain exact consumption and portfolio rules. We consider an example for which the assumption of parameter uncertainty is sensible: the mean reverting Sharpe ratio model. For both examples, we show how the separation theorem of Simon (1956) as extended to continuous time by Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) and the Cox-Huang (1989) method allow us to solve the model and find parametrization for both the consumption and portfolio strategy.

We apply the methodology of the paper to the consumption and portfolio choice problem under mean reverting returns, when the current value for the Sharpe ratio, our proxy for the investment

¹⁰We run the same diagnostic test on the longevity of learning for the Xia (2001) model where the predictive relation is unobservable but assumed to be a constant. In this setting, the reduction of the variance of the estimation error is indeed slower. Therefore, our assumption of steady-state learning might not be adequate for Xia's model. Still, with 80 years of data, the posterior variance for the estimation error is not very far from 0.

opportunity set, is not observable. We calibrate the results to a similar model by Campbell and Viceira (1999) and compare the investor's policies under parameter uncertainty to those of an investor with complete information as modeled by Wachter (2002). We find significant quantitative changes in the demand for the risky asset when parameter uncertainty is considered. Yet, the qualitative portfolio choice implications of the model are not different from those of Barberis (2000) and Wachter. We complete our analysis with a study of the longevity of learning to validate our assumption regarding steady-state in the learning process.

The methodology of this paper could be extended to consider stochastic volatility and the role of derivatives in strategic asset allocation. Our model would allow us to find exact consumption and portfolio policies when volatility follows a central tendency model similar to the one we have used in this paper to describe interest rate dynamics. Our result would complement those of Liu and Pan (2002). The focus of their papers is the disentanglement of volatility and jump risk in an investor's portfolio through the use of derivative securities in the dynamic asset allocation strategy. Our paper would allow for a similar risk disentanglement (between market and volatility risk) and also provide us with normative exact consumption rules under the assumption of stochastic volatility.

The methods of this paper can also be applied to derivative pricing in incomplete markets. For example, we can consider the unobservable mean reverting Sharpe ratio model to price equity options under the risk neutral pricing, and find the replicating strategy utilizing the estimated processes for the underlying securities. Since our model yields an observationally complete market, not only can we price the option, we can also find the replication strategy with only the stock and the bond as the underlying securities. This would allow us to extend the results of Lo and Wang (1995) and study the replication strategy in the estimated economy when prices have a mean reverting component.

References

- [1] Lynch, Anthony. and Pierluigi Balduzzi, 2000, “Predictability and Transaction Costs: The Impact on Rebalancing Rules and Behavior”, *Journal of Finance* 55, 2285-2310.
- [2] Barberis, Nicholas, 2000, “Investing for the Long Run when Returns are Predictable,” *Journal of Finance* 55, 225-264.
- [3] Bawa, Vijay S. and Roger W. Klein, 1976, “The Effect of Estimation Risk on Optimal Portfolio Choice,” *Journal of Financial Economics* 3, 215-232.
- [4] Bawa, Vijay S., Stephen J. Brown, and Roger W. Klein, 1979, *Estimation Risk and Optimal Portfolio Choice*, North-Holland, New York.
- [5] Bergstrom A.R., 1984, “Continuous Time Stochastic Methods and Issues of Aggregation over Time,” in Z. Griliches and M.D. Intriligator eds. *Handbook of Econometrics Vol. 2* North Holland, Amsterdam.
- [6] Brennan, Michael J., 1998, “The Role of Learning in Dynamic Portfolio Decisions,” *European Economic Review* 1, 295-306.
- [7] Brennan, Michael J., Eduardo Schwartz, and R. Lagnado, 1997, “Strategic Asset Allocation,” *Journal of Economic Dynamics and Control* 21, 1377-1403.
- [8] Canner, Niko, N.G. Mankiw, and David N. Weil, 1997, “An Asset Allocation Puzzle,” *American Economic Review*, 87, 181-191.
- [9] Campbell, John Y., Jorge Rodriguez, and Luis Viceira, 2002, “Strategic Asset Allocation in a Continuous-Time VAR Model,” mimeo, Harvard University.
- [10] Campbell, John Y. and Robert Shiller, 1988, “The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors,” *Review of Financial Studies* 1, 195-227.
- [11] Campbell, John Y. and Luis Viceira, 1999, “Consumption and Portfolio Decisions when Expected Returns are Time-Varying,” *Quarterly Journal of Economics* 114, 433-495.
- [12] Campbell, John Y. and Luis Viceira, 2000, “Erratum: Consumption and Portfolio Decisions when Expected Returns are Time-Varying,” mimeo, Harvard University.
- [13] Campbell, John Y. and Luis Viceira, 2002, *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*, Oxford University Press.
- [14] Chacko, George and Luis Viceira, 2002, “Perturbation Methods for Dynamic Investment-Consumption Problems,” unpublished paper, Harvard University.

- [15] Cox, John C. and Chi-Fu Huang, 1989, "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process," *Journal of Political Economy* 98, 519-543.
- [16] Cox, John C. and Chi-Fu Huang, 1991, "A Variational Problem Arising in Financial Economics," *Journal of Mathematical Economics* 20, 465-487.
- [17] Detemple, Jerome, 1986, "Asset Pricing in a Production Economy with Incomplete Information," *Journal of Finance*, 383-391.
- [18] Dothan, Michael U. and David Feldman, 1986, "Equilibrium Interest Rates and Multiperiod Bonds in a Partially Observable Economy," *Journal of Finance* 41, 369-382.
- [19] Duffie, Darrell and Rui Kan, 1996, "A Yield Factor Model of Interest Rates," *Mathematical Finance* 6, 379-406.
- [20] Gennotte, Gerard, 1986, "Optimal Portfolio Choice under Incomplete Information," *Journal of Finance* 41, 733-746.
- [21] Kandel, Shmuel and Robert F. Stambaugh, 1996, "On the Predictability of Stock Returns: An Asset Allocation Perspective," *The Journal of Finance*, 51, 385-424.
- [22] Karatzas, Ioannis and Steven Shreve, 1998, *Methods of Mathematical Finance, Application of Mathematics* 39, Springer-Verlag.
- [23] Karatzas, Ioannis, and X. Zhao, 2001, "Bayesian Adaptive Portfolio Optimization," *Advances in Mathematical Finance*, Cambridge University Press.
- [24] Kim, Tong Suk and Edward Omberg, 1996, "Dynamic Nonmyopic Portfolio Behavior," *Review of Financial Studies* 9, 141-161.
- [25] Lakner, P., 1995, "Utility Maximization with Partial Information," *Stochastic Processes and their Applications* 56, 247-273.
- [26] Lakner, P., 1998, "Optimal Trading Strategy for an Investor: The Case of Partial Information," *Stochastic Processes and their Applications* 76, 77-97.
- [27] Lewellen, Jonathan, 2001, "Predicting Returns with Financial Ratios," mimeo, MIT Sloan School of Management.
- [28] Lewellen, Jonathan, 2001, "Temporary Movements in Stock Prices," mimeo, MIT Sloan School of Management.

- [29] Lewellen, Jonathan and Jay Shanken, 2002, "Learning, Asset Pricing Tests, and Market Efficiency," *Journal of Finance*, forthcoming.
- [30] Lipster, R. and A. Shiryaev, 2000, *Statistics of Random Processes II: Applications*, Springer-Verlag, New York.
- [31] Liu, Jun, 2001, "Portfolio Selection with a Dynamic Choice Set," working paper, Anderson School of Management, UCLA.
- [32] Liu, Jun and Jun Pan, 2002, "Dynamic Derivative Strategies," mimeo, MIT Sloan School of Management.
- [33] Lo, Andrew and Jiang Wang, 1995, "Implementing Option Pricing Models When Asset Returns Are Predictable," *Journal of Finance* 50(1995), 87-129.
- [34] Markowitz, Harry, 1952, "Portfolio Selection," *Journal of Finance* 7, 77-91.
- [35] Merton, Robert C., 1969, "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics* 51, 247-257.
- [36] Merton, Robert C., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory* 3, 373-413.
- [37] Merton, Robert C., 1973, "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41, 867-87.
- [38] Merton, Robert C., 1980, "On Estimating the Expected Return on the Market: an Exploratory Investigation," *Journal of Financial Economics* 8, 323-362.
- [39] Mossin, Jan, 1968, "Optimal Multiperiod Portfolio Policies," *Journal of Business* 41, 205-25.
- [40] Poterba, James and Larry Summers, 1988, "Mean Reversion in Stock Prices: Evidence and Implications," *Journal of Financial Economics* 22, 27-59.
- [41] Reid, W. T., 1972, *Riccati Differential Equations*, Academic Press, New York.
- [42] Rishel, R., 1999, "Optimal Portfolio Management with Partial Observations and Power Utility Function," manuscript, University of Kentucky.
- [43] Sorensen, Carsten, 2000, "Dynamic Asset Allocation and Fixed Income Management," *Journal of Financial and Quantitative Analysis*, 34, 513-532.
- [44] Vasicek, Oldrich, 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177-188.

- [45] Wachter, Jessica, 2002, “An Exact Solution to the Consumption and Portfolio Choice Problem under Mean Reverting Returns,” *Journal of Financial and Quantitative Analysis*, forthcoming.
- [46] Williams, Joseph, 1977, “Capital Asset Prices in Heterogeneous Beliefs,” *Journal of Financial Economics* 5, 219-239.
- [47] Xia, Yihong, 2001, “Learning about Predictability: The Effect of Parameter Uncertainty on Dynamic Asset Allocation,” *The Journal of Finance* 56, 205-246.

TABLE I

Continuous Time Parameter Values
From Campbell, Rodriguez, Viceira (2002) VAR

Model:

$$\begin{aligned}\frac{dB_t}{B_t} &= r dt \\ \frac{dS_t}{S_t} &= (r + \sigma_s \hat{\eta}_t) dt + \sigma_s d\hat{Z}_S, \\ d\hat{\eta}_t &= \kappa_\eta (\theta_\eta - \hat{\eta}_t) dt + \varepsilon_{S\eta} d\hat{Z}_S, \\ \varepsilon_{S\eta} &= v_{ss} + \rho_{S\eta} \sigma_\eta, \\ v_{ss} : 0 &= -2\kappa_\eta v_{ss} + \sigma_\eta^2 - [v_{ss} + \rho_{S\eta} \sigma_\eta]^2.\end{aligned}$$

Parameter values at quarterly frequency:

$$\begin{aligned}r &= 0.0818 * 10^{-2}, \\ \sigma_S &= 7.8958 * 10^{-2}, \\ \kappa_\eta &= 4.3875 * 10^{-2}, \\ \theta_\eta &= 0.1667, \\ \sigma_\eta &= 0.0727, \\ \rho &= -0.9626, \\ v_{ss} &= 0.0589,\end{aligned}$$

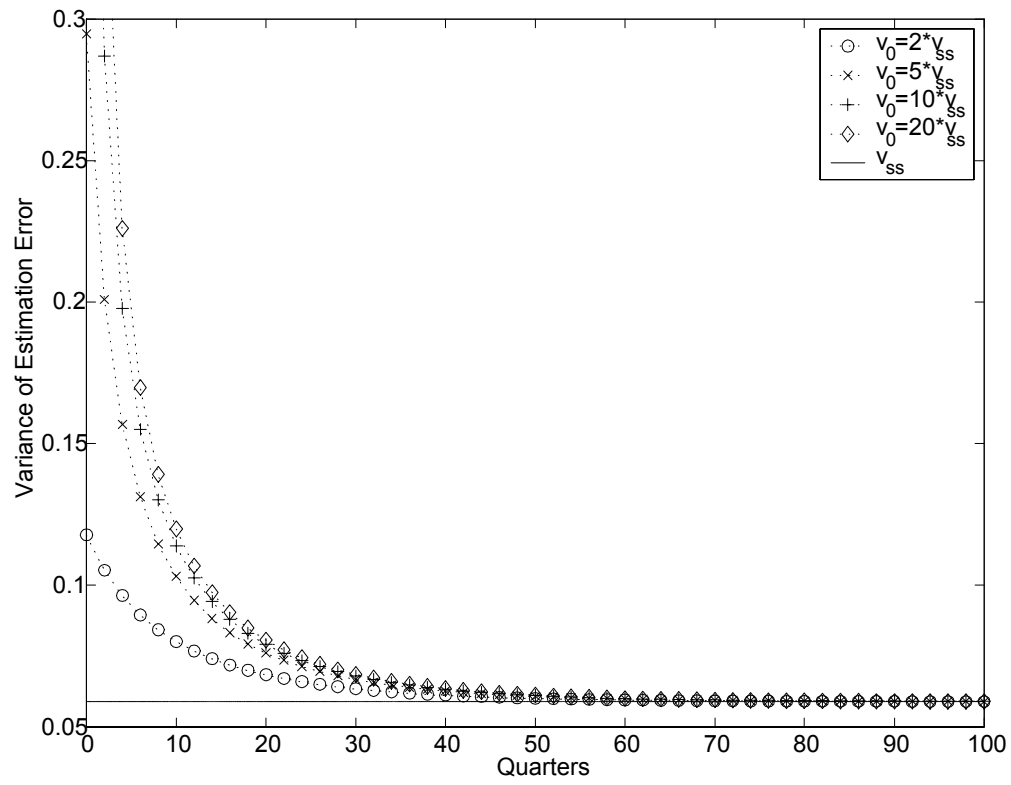


Figure 1: Path of the variance v_t of the estimation error of the unobservable Sharpe ratio for various assumptions on the prior v_0 .

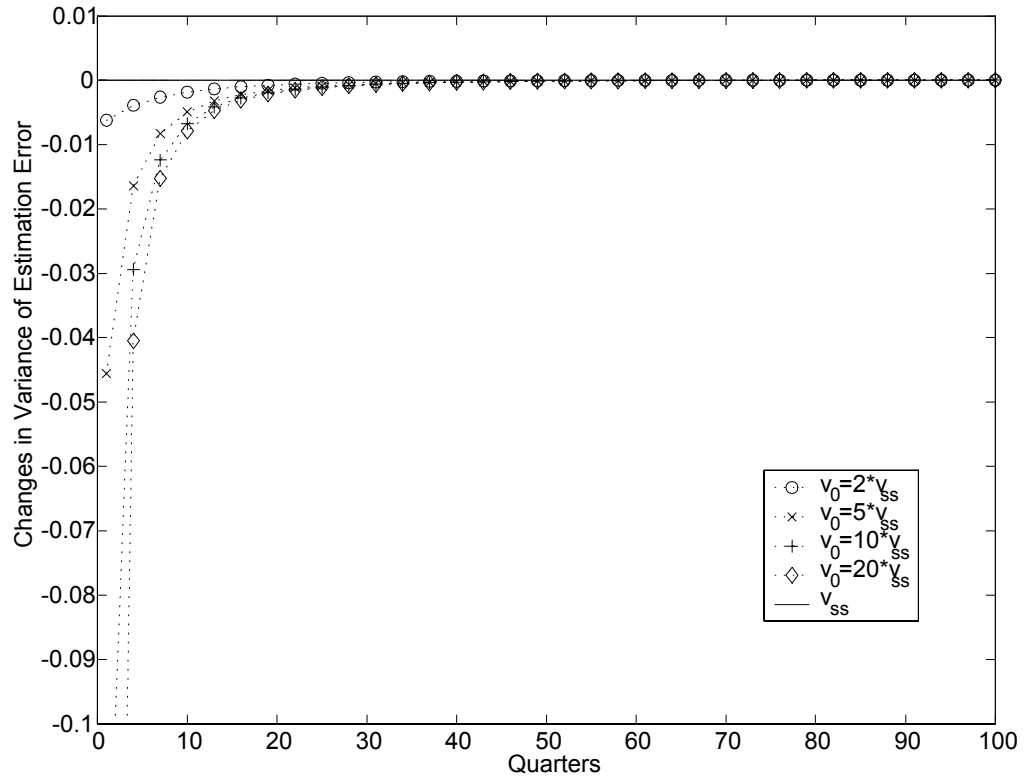


Figure 2: Changes in the variance v_t of the estimation error for the unobservable Sharpe ratio under various assumption for the prior v_0 .

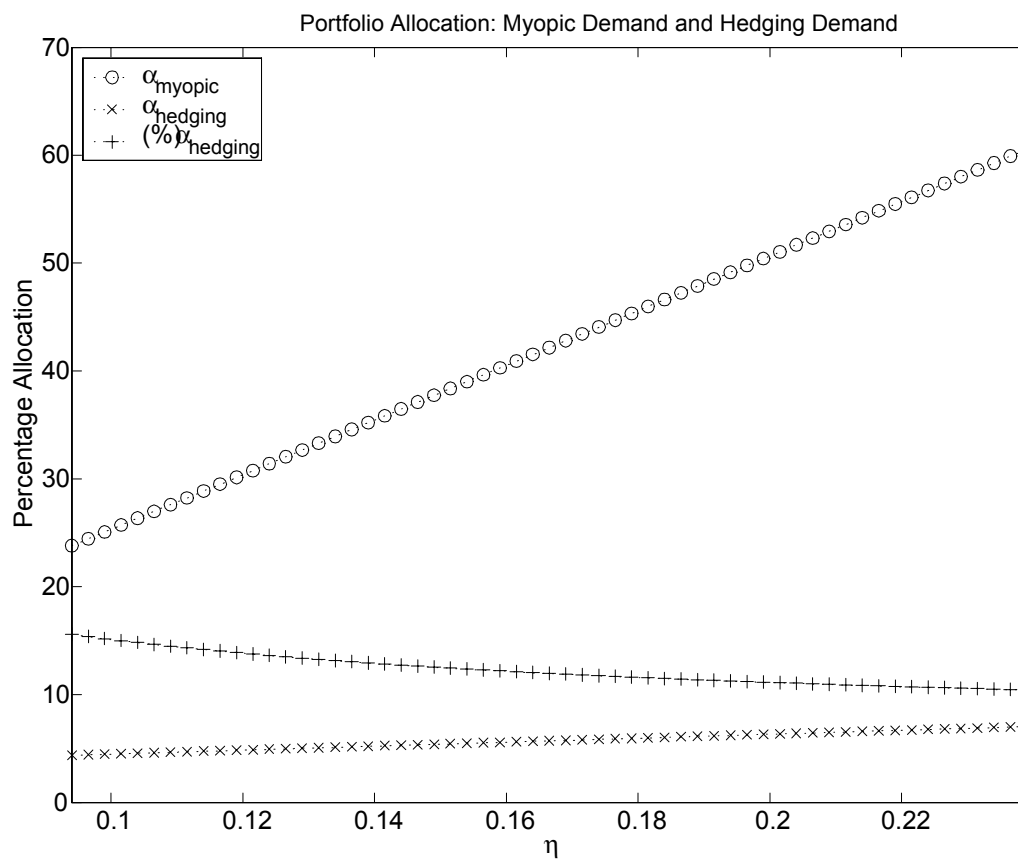


Figure 3: Portfolio Allocation for an investor with $\gamma = 5$ and $T = 30$ years.

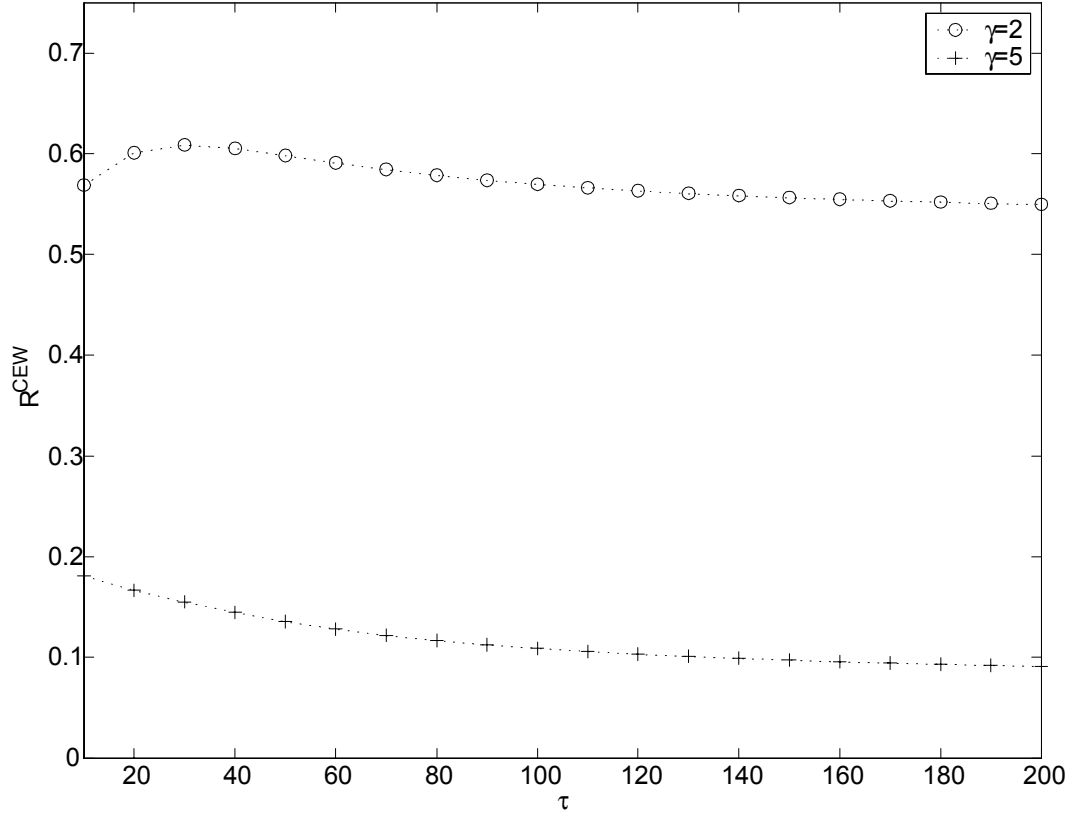


Figure 4: Comparison of asset allocation strategies for an informed and uninformed investor. \mathcal{R}^{CEW} for the parameters of Table I for various values of τ (in quarters of a year), the investment horizon of the investor.

Table II
Optimal Consumption and Portfolio Policies ^a

γ	Parameter Uncertainty					Complete Markets					Difference				
	2.5	10	30	50		2.5	10	30	50		2.5	10	30	50	
Panel A: Myopic Demand as % of Wealth															
T=															
2	105.56	105.56	105.56	105.56		105.56	105.56	105.56	105.56						
5	42.22	42.22	42.22	42.22		42.22	42.22	42.22	42.22						
10	21.11	21.11	21.11	21.11		21.11	21.11	21.11	21.11						
25	8.44	8.44	8.44	8.44		8.44	8.44	8.44	8.44						
Panel B: Hedging Demand as % of Wealth															
T=															
2	2.18	5.25	7.64	8.18		16.50	44.01	63.49	66.52		2.5	10	30	50	
5	1.45	3.69	5.74	6.31		12.31	40.78	67.31	72.35		-14.32	-38.76	-55.85	-58.34	
10	0.83	2.15	3.44	3.83		7.36	27.54	50.34	56.05		-10.86	-37.08	-61.38	-66.04	
25	0.36	0.94	1.53	1.72		3.26	13.47	27.26	31.47		-6.53	-25.39	-46.90	-52.22	
											-2.91	-12.53	-25.72	-29.75	
Panel C: Hedging Demand as % of Total Demand															
T=															
2	2.02	4.74	6.75	7.19		13.52	29.42	37.56	38.66		2.5	10	30	50	
5	3.32	8.04	11.97	13.00		22.57	49.13	61.38	63.15		-11.50	-24.69	-30.81	-31.46	
10	3.77	9.25	14.03	15.37		25.84	56.61	70.45	72.64		-19.25	-41.09	-49.42	-50.14	
25	4.04	10.02	15.36	16.92		27.88	61.47	76.35	78.84		-22.07	-47.35	-56.43	-57.27	
											-23.84	-51.45	-60.99	-61.92	
Panel D: Consumption as a % of Wealth															
T=															
2	10.56	3.08	1.49	1.23		10.68	3.38	2.00	1.84		2.5	10	30	50	
5	10.29	2.79	1.15	0.85		10.39	3.07	1.66	1.47		-0.12	-0.30	-0.51	-0.62	
10	10.17	2.67	1.02	0.70		10.23	2.86	1.36	1.13		-0.10	-0.28	-0.51	-0.62	
25	10.10	2.60	0.93	0.60		10.12	2.68	1.10	0.81		-0.06	-0.18	-0.34	-0.43	
											-0.03	-0.09	-0.17	-0.21	

^aThis table compares the consumption and portfolio policies of an investor with parameter uncertainty regarding stock price predictability to an investor which has complete information about the economy and assumes perfect negative correlation between stock returns and the predictive variable, i.e. the investor modeled in Wachter (2002). Both investor assume the parameters presented in Table I hold. The portfolio choice of the investor is given by (40) where the myopic component satisfies (41) and the hedging component is given by (42). The table presents the results of the calibration of the model to the parameters in Table I under the assumption the current value for η_t is the long-run value and the first investor has reached the steady state in the learning process. We consider the asset allocation of the investor for various parameter values for the coefficient of relative risk aversion and the investment horizon (in years). The first column presents the optimal consumption and portfolio policies for the investor with incomplete information. The second column presents the calibration of Wachter (2002) for the parameters in Table I. The third column highlights the differences between each strategy.

Table III
Optimal Portfolio Policies under Various Estimates for the Sharpe Ratio ^a

γ	Myopic Demand					Hedging Demand					Hedging as % of Total Demand				
	2.5	10	30	50		2.5	10	30	50		2.5	10	30	50	
Panel A: $\eta = \theta_\eta - \sigma_\eta$															
T=															
2	59.54	59.54	59.54	59.54		1.35	3.68	5.78	6.27		2.21	5.83	8.85	9.53	
5	23.82	23.82	23.82	23.82		0.90	2.60	4.40	4.92		3.62	9.85	15.61	17.14	
10	11.91	11.91	11.91	11.91		0.51	1.52	2.66	3.01		4.11	11.31	18.24	20.19	
25	4.76	4.76	4.76	4.76		0.22	0.66	1.19	1.36		4.41	12.22	19.93	22.18	
Panel B: $\eta = \theta_\eta$															
T=															
2	105.56	105.56	105.56	105.56		2.18	5.25	7.64	8.18		2.02	4.74	6.75	7.19	
5	42.22	42.22	42.22	42.22		1.45	3.69	5.74	6.31		3.32	8.04	11.97	13.00	
10	21.11	21.11	21.11	21.11		0.83	2.15	3.44	3.83		3.77	9.25	14.03	15.37	
25	8.44	8.44	8.44	8.44		0.36	0.94	1.53	1.72		4.04	10.02	15.36	16.92	
Panel C: $\eta = \theta_\eta + \sigma_\eta$															
T=															
2	151.58	151.58	151.58	151.58		3.00	6.79	9.48	10.08		1.94	4.29	5.89	6.23	
5	60.63	60.63	60.63	60.63		2.00	4.77	7.06	7.69		3.19	7.30	10.43	11.25	
10	30.32	30.32	30.32	30.32		1.14	2.78	4.23	4.65		3.63	8.41	12.24	13.30	
25	12.13	12.13	12.13	12.13		0.49	1.22	1.88	2.08		3.90	9.12	13.42	14.65	

^aThis table describes portfolio policies of an investor with parameter uncertainty regarding stock price predictability under assumption about the current value of η . We assume the parameters presented in Table I hold. The portfolio choice of the investor is given by (40) where the myopic component satisfies (41) and the hedging component is given by (42). The table presents the results of the calibration of the model to the parameters in Table I for $\eta = \theta_\eta - \sigma_\eta, \theta_\eta + \sigma_\eta$. We consider the asset allocation of the investor for various parameter values for the coefficient of relative risk aversion and the investment horizon (in years). The first column presents the myopic demand under the various assumption for η . The second column presents the hedging demand. The third column shows the how much of the total demand for the stock is due to hedging for changes in the investment opportunity set. Both myopic and hedging demand increase monotonically with increases in the sharpe ratio, but the percentage of the portfolio dedicated to hedging decreases monotonically.

Table IV
Portfolio Improvements due to Better Information ^a

γ	Portfolio Demand - Informed					Portfolio Demand - Uninformed					\mathcal{R}^{CEW}				
Panel A: $\eta = \theta_\eta - \sigma_\eta$															
T=	2.5	10	30	50		2.5	10	30	50		2.5	10	30	50	
2	76.79	88.18	90.04	90.05		62.04	65.88	67.98	68.08		0.552	0.663	0.599	0.571	
5	33.59	37.56	37.83	37.83		25.49	28.29	30.08	30.19		0.196	0.188	0.127	0.106	
10	17.14	18.96	19.05	19.05		12.86	14.51	15.63	15.71		0.092	0.082	0.046	0.034	
25	6.93	7.61	7.64	7.64		5.17	5.90	6.41	6.45		0.036	0.030	0.014	0.009	
Panel B: $\eta = \theta_\eta$															
T=	2.5	10	30	50		2.5	10	30	50		2.5	10	30	50	
2	132.51	145.38	147.26	147.27		109.45	114.03	116.19	116.30		0.569	0.605	0.563	0.550	
5	57.55	61.95	62.22	62.22		44.83	48.18	50.02	50.14		0.181	0.145	0.103	0.091	
10	29.33	31.33	31.42	31.42		22.60	24.58	25.73	25.81		0.081	0.057	0.033	0.025	
25	11.85	12.60	12.63	12.63		9.08	9.95	10.48	10.52		0.030	0.019	0.008	0.005	
Panel C: $\eta = \theta_\eta + \sigma_\eta$															
T=	2.5	10	30	50		2.5	10	30	50		2.5	10	30	50	
2	188.24	202.58	204.48	204.49		156.87	162.18	164.41	164.51		0.594	0.537	0.524	0.526	
5	81.51	86.33	86.61	86.61		64.17	68.06	69.98	70.10		0.160	0.092	0.076	0.074	
10	32.33	34.64	35.83	35.91		1.14	2.78	4.23	4.65		0.065	0.027	0.017	0.016	
25	16.77	17.58	17.62	17.62		13.00	14.01	14.55	14.59		0.022	0.006	0.002	0.001	

^aThis table shows the portfolio improvements due to full information under various parameter values for the investors horizon and coefficient of relative risk aversion. The measure used to calculate the improvement due to information is given in (). We calibrate the model to the parameters presented in table I. We shows the results for three different values of $\eta = \theta_\eta - \sigma_\eta$, θ_η , $\theta_\eta + \sigma_\eta$. The first column presents the allocation of wealth to risky asset by the informed investor. The second column presents the allocation of wealth to the risky asset by the uninformed investor. The third column presents the certainty equivalent wealth improvements due to better information in terms of the measure in ().