A Simple Theory of Asset Pricing
under Model Uncertainty

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Abstract

The focus of our paper is on the implications of model uncertainty for the cross-sectional properties of returns. We perform our analysis in the context of a tractable single-period mean-variance framework. We show that there is an uncertainty premium in equilibrium expected returns on financial assets and study how the premium varies across the assets. In particular, the cross-sectional distribution of expected returns can be formally described by a two-factor model, where expected returns are derived as compensation for the asset's contribution to the equilibrium risk and uncertainty of the market portfolio. In light of the large empirical literature on the cross-sectional characteristics of asset returns, understanding the implications of model uncertainty even in such a simple setting would be of significant value. By characterizing the cross-section of returns we are also able to address some of the observational equivalence issues raised in the literature. That is, whether model uncertainty in financial markets can be distinguished from risk, and whether uncertainty aversion at an individual level can be distinguished from risk aversion.

1   Introduction

The purpose of this paper is to study the implications of model uncertainty for the cross-sectional properties of asset prices in a simplest possible equilibrium setting.

The focus on model uncertainty is motivated by the difficulty of reconciling existing asset pricing theories with the empirical data. Limited success of the standard theories could be in part due to the commonly made assumption that economic agents possess perfect knowledge of the data generating process. For instance, the classical theories of Sharpe (1964), Lucas (1978), Breeden (1979) and Cox, Ingersoll and Ross (1985), assume that, while the payoffs of financial assets are random, agents know the underlying probability law exactly. In reality this is often not the case. The natural question is how the properties of financial assets are affected by the fact that investors are not certain of the correctness of the model they use to describe the probability laws in the economy.

The importance of model uncertainty has long been recognized in finance. While the literature appears under different names, such as parameter uncertainty, Knightian uncertainty, the defining characteristic of that literature is the recognition of the fact that the agents of the economy do not have a perfect knowledge of the probability law that governs the realization of the states of the world. Various issues have been studied. Dow and Werlang (1992) use the uncertainty averse preference model developed by Schmeidler (1989) to study a single period portfolio choice problem. Maenhout (1999) examines a similar problem in a continuous-time economy, but from the point of view of robust portfolio rules. Kandel and Stambaugh (1996), Brennan (1998), Barberis (2000), and Xia (2001) show that parameter uncertainty can affect significantly investors' portfolio choice. Frost and Savarino (1986), Gennotte (1986), Balduzzi and Liu (1999), Pastor (2000) and Uppal and Wang (2001) examine the implication of model uncertainty for portfolio choices when there are multiple risky assets. Detemple (1986), Epstein and Wang (1994), Chen and Epstein (2001), Epstein and Miao (2001), and Brennan and Xia (2001) study the implications for equilibrium asset prices in the representative agent and heterogenous agent economies respectively. Routledge and Zin (2002) examine the connection between model uncertainty and liquidity. There is also a significant literature, for example Lewellen and Shanken (2001), on the effect of learning about an unknown parameter on the equilibrium asset prices.

The focus of our paper is on the cross-sectional properties of returns. We perform our analysis in the context of a tractable single-period mean-variance framework. We show that there is an uncertainty premium in equilibrium expected returns on financial assets and study how the premium varies across the assets. In light of the large empirical literature on the cross-sectional characteristics of asset returns, understanding the implications of model uncertainty and uncertainty aversion even in such a simple setting would be of significant value. While prior research on model uncertainty has been concerned with its implications for the time-series of asset prices, by characterizing the cross-section of returns we are able to address some of the observational equivalence issues raised in the literature. That is, whether model uncertainty in financial markets can be distinguished from risk, and whether uncertainty aversion at an individual level can be distinguished from risk aversion (Anderson, Hansen and Sargent (1999)).

In the rest of this introduction, we will describe briefly our approach to formalizing model uncertainty and its relation to the the literature. The most common way of modelling imperfect knowledge of the model and parameters is in the Bayesian framework (Kandel and Stambaugh (1996), Lewellen and Shanken (2001), Barberis (2000) and Pástor (2000)). The key feature of this approach is that if a parameter of the model is unknown, a prior distribution of the parameter is introduced. The second approach, adopted by Dow and Werlang (1992), Epstein and Wang (1994, 1995), Chen and Epstein (2001), Epstein and Miao (2001), and the third approach, adopted by Maenhout (1999), Uppal and Wang (2001), follow the view of Knight (1921) that model uncertainty, or more precisely, the decision makers' view of model uncertainty, cannot be represented by a probability prior. Such view is supported by evidence exhibited in the Ellsberg experiment (Ellsberg (1963)). In the Ellsberg experiment, the decision maker is presented with two urns each containing 100 balls. In the first urn there are 50 red balls and 50 white balls. In the second, the proportion of red balls is not known. The decision maker is asked to rank two sets of bets. In the first set, the first bet is: win $100 if a randomly drawn ball from the first urn is red, otherwise zero. The second bet is similar, but wins if the ball drawn is white. The second set of bets is the same as the first set except that the ball is drawn from the second urn. In the experiment, it is found that the decision maker is typically indifferent between the two bets in any of the two sets, but prefers any bet from the first set to any bet from the second

set. Such behavior is inconsistent with any expect utility preference, and more generally any probabilistically sophisticated preference (Machina and Schmeidler (1992)).

The second and third approaches differ in how uncertainty and uncertainty aversion are modelled. Maenhout (1999), Uppal and Wang (2001), use the preference first introduced by Anderson, Hansen and Sargent (1999) in their study of the implications of preference for robustness for macroeconomic and general asset pricing issues.[1] This class of preferences has been extended in Uppal and Wang (2001), and axiomatized in a static setting in Wang (2001). For this class of preferences, uncertainty is described by a set of priors and the investor's aversion to it is introduced through a penalty function. Dow and Werlang (1992), Epstein and Wang (1994, 1995), Chen and Epstein (2001) and Epstein and Miao (2001) use the multi-prior expected utility developed by Gilboa and Schmeidler (1989).[2] Here both uncertainty and uncertainty aversion are introduced through a set of priors. This paper is based on the multi-prior expected utility preferences with a careful design of the set of priors to distinguish between the uncertainty and uncertainty aversion aspects of the set.

The rest of the paper is organized as follows. Section 2 describes the model. Section 4 presents the main result of this paper, the asset pricing implication of model uncertainty. Section 5 discusses several issues related to model uncertainty. Finally, Section 6 concludes.

2 The Model

2.1 The Setting

We assume a one-period representative agent economy. Consumption takes place only at the end of the period. The agent is endowed with an initial wealth W_0. Without loss of generality, we assume $W_0 = 1$.

There are N risky assets and one risk-free asset in the economy, which is available in zero net supply. As indicated in the introduction, the investors do not have perfect knowledge of

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[1] See Hansen and Sargent (2001) for more on this type of preferences.

[2] Dow and Werlang is based more directly on the Choquet expected utility developed by Schmeidler (1989). However, for the case they studied, Choquet expected utility coincides with multi-prior expected utility.

the distribution of the returns of the N risky assets. More specifically, they know that the returns $R = (R_1, \ldots, R_N)^\top$ follow a joint normal distribution with density function

$$f(R) = (2\pi)^{-n/2} |\Omega_{\mathbf{r}}|^{-1/2} \exp\left\{-\frac{1}{2}(R - \mu)^\top \Omega_{\mathbf{r}}^{-1}(R - \mu)\right\}$$

where

$$\mu = \mathrm{E}[R], \quad \Omega_{\mathbf{r}} = \mathrm{E}[(R - \mu)(R - \mu)^\top].$$

The risk of returns is summarized by the non-degenerate variance-covariance matrix $\Omega_{\mathbf{r}}$. We assume that investors have precise knowledge of $\Omega_{\mathbf{r}}$. However, they do not know exactly the mean return vector μ. This is motivated by the fact that it is much easier to obtain accurate estimates of the variance and covariance of returns than their expected values, e.g., Merton (1992). The imperfect knowledge of the asset return distribution gives rise to model uncertainty.

2.2 The Preferences

Each agent in the economy has a state-independent utility function $u(W)$.

However, due to lack of perfect knowledge of the probability law of asset returns, the agent's preference is not represented by the standard expected utility, but instead by a multi-prior expected utility

$$U(W, \mathcal{P}(P, \phi))) = \min_{Q \in \mathcal{P}(P, \phi)} \left\{\mathrm{E}^Q[u(W)]\right\}, \tag{1}$$

where E^Q denotes the expectation under the probability measure Q, $\mathcal{P}(P, \phi)$ is a set of probability measures that depends on the probability measure P, called the reference prior, and the parameter $\phi \geq 0$, which is called the uncertainty aversion parameter. The set $\mathcal{P}(P, \phi)$ captures both the degree of model uncertainty and the agent's degree of uncertainty aversion. In particular, we assume that the larger the uncertainty aversion parameter ϕ, the larger the set $\mathcal{P}(P, \phi)$. The multi-prior expected utility preferences exhibits uncertainty aversion. The general nature and the axiomatic foundation of these preferences has been well studied in the literature (Gilboa and Schmeidler (1989)). What is specific to this paper

is the structure of $\mathcal{P}(P, \phi)$, in particular, the use of ϕ as the uncertainty aversion parameter. Since the exact structure of $\mathcal{P}(P, \phi)$ is important to our analysis and to the understanding of our results, we now turn to the description of the exact dependence of $\mathcal{P}(P, \phi)$ on P and ϕ.

A Single Source of Information

We begin with the case of a single source of information about the distribution of stock returns,

$$\mathcal{P}(P, \phi) = \{Q : \mathrm{E}[\xi \ln \xi] \leq \phi^2 \eta\},$$

where ξ is the density of Q with respect to P and η is a parameter to be described shortly.

The intuition behind this formulation of $\mathcal{P}(P, \phi)$ can be explained as follows. Since the investor lacks a perfect knowledge of the probability law of the returns, an econometrician is asked to estimate a model of the asset returns for the investor. After a slew of econometric analysis, typically including specification analysis and parameter estimation, the econometrician comes up with a model described by the probability measure P. However, the econometrician is not completely confident that this is the true model, due to not having enough data in the specification analysis and the parameter estimation, or due to simplifying assumptions made for tractability. On the other hand, the econometric analysis does provide more information than just the probability measure P. The true model can be narrowed down to a set $\mathcal{P}$ of probability measures. At the end of the analysis, the investor is presented with a probability measure P, called the reference prior, and a set $\mathcal{P}$ that summarize the precision of the econometric analysis.

Since both the econometrician and the investor are not completely sure of the reference prior P, each element in $\mathcal{P}$ is a possible alternative to the reference prior P. Let Q be an element in $\mathcal{P}$ and let its density be denoted by ξ, so that

$$dQ = \xi dP. \tag{2}$$

Knowing that the reference measure P is subject to misspecification and that the possible alternative is Q, the problem is how to evaluate the alternative. One way is to use the relative entropy index, $\mathrm{E}[\xi \ln \xi]$. One interpretation of the index is that it is an approximation to

5

the empirical log-likelihood ratio.[3] To elaborate, suppose that the data set available to the investor has T observations. Then the empirical log-likelihood ratio of the two models is

$$\frac{1}{T} \sum_{t=1}^{T} \ln \xi(X_t).$$

Now suppose that X_t, $t = 1, \ldots, T$, takes finitely many values, $x_1, \ldots, x_k$ in the data series. Then

$$\frac{1}{T} \sum_{t=1}^{T} \ln \xi(X_t) = \frac{1}{T} \sum_{i=1}^{k} \sum_{X_t = x_i} \ln \xi(X_t) = \sum_{i=1}^{k} \frac{T_i}{T} \ln \xi(x_i),$$

where T_i is the number of t such that $X_t = x_i$. By the law of large numbers, under the alternative model Q, T_i/T converges to $Q(x) = \xi(x)P(x)$ and hence $\frac{1}{T} \sum_{t=1}^{T} \ln \xi(X_t)$ converges to $\mathrm{E}[\xi \ln \xi]$. Thus, if Q is the true probability law, $\mathrm{E}[\xi \ln \xi]$ is a good approximation to the empirical log-likelihood when T is large. According to the traditional likelihood ratio theory, if the above sum is large, then the two alternatives, Q and P, can be clearly distinguished.[4] Therefore the set of possible alternative models according to the econometrician is given by

$$\mathcal{P}(P) = \{Q : \mathrm{E}[\xi \ln \xi] \leq \eta\}$$

where η is the parameter the econometrician uses to describe how much uncertainty there is about the reference probability P. For example, η could be chosen to define a rejection region for a test of the reference model P with a 95% confidence level.

The investor's uncertainty aversion is introduced through the parameter ϕ, so that the set provided by the econometrician is scaled up or down by ϕ:

$$\mathcal{P}(P, \phi) = \{Q : \mathrm{E}[\xi \ln \xi] \leq \phi^2 \eta\}.$$

Larger values of ϕ allow for a larger set of alternative models. Thus, more uncertainty averse agents are willing to entertain alternative models that are relatively far from the reference

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[3]See Anderson, Hansen and Sargent (1999) and Hansen and Sargent (2000) for other interpretations of the index.

[4]It is worth emphasizing that large $\frac{1}{T} \sum_{t=1}^{T} \xi(X_t) \ln \xi(X_t)$ should not be interpreted as evidence for *rejecting* the reference model P, as in the usual likelihood test: as explained above, the very fact that P is the reference prior implies that the econometrician has already gone through the preliminary analysis and picked P. The issue at this stage is only to find an index that summarize the information available.

model P, as measured by their relative entropy. An investor more averse to uncertainty would require a higher level of confidence, say 99%, than the one used by the econometrician, and vice versa.

For analytical tractability, we assume that stock returns are jointly normally distributed under the alternative models. Furthermore, we assume that the variance-covariance matrix of the returns is the same under all measures in $\mathcal{P}$, reflecting the fact that the investor knows the variance-covariance matrix $\Omega_{\mathbf{r}}$ precisely. Let Q be a measure in $\mathcal{P}$ with the density given by

$$(2\pi)^{-n/2}|\Omega_{\mathbf{r}}|^{-1/2}\exp\left\{-\frac{1}{2}(R-\hat{\mu})^{\top}\Omega_{\mathbf{r}}^{-1}(R-\hat{\mu})\right\},$$

which can be written as

$$(2\pi)^{-n/2}|\Omega_{\mathbf{r}}|^{-1/2}\exp\left\{-\frac{1}{2}(R-\mu)^{\top}\Omega_{\mathbf{r}}^{-1}(R-\mu)\right\}$$

$$\times\exp\left\{-\frac{1}{2}(\mu-\hat{\mu})^{\top}\Omega_{\mathbf{r}}^{-1}(\mu-\hat{\mu})-(\mu-\hat{\mu})^{\top}\Omega_{\mathbf{r}}^{-1}(R-\mu)\right\}.$$

Thus, the likelihood ratio of Q over P is given by

$$\xi(R)=\exp\left\{\frac{1}{2}(\mu-\hat{\mu})^{\top}\Omega_{\mathbf{r}}^{-1}(\mu-\hat{\mu})-(\mu-\hat{\mu})^{\top}\Omega_{\mathbf{r}}^{-1}(R-\hat{\mu})\right\}. \tag{3}$$

Given this particular structure of the set $\mathcal{P}(P,\phi)$, the representative investor's utility function can be written as

$$\min_{v\in\mathcal{V}(\phi)}\mathrm{E}\left[\xi u(W)\right], \tag{4}$$

where ξ is given by (3), $v=\mu-\hat{\mu}$ and the set $\mathcal{V}$ corresponds to $\mathcal{P}$:

$$\mathcal{V}(\phi)=\{v:\mathrm{E}[\xi\ln\xi]=1/2v^{\top}\Omega_{\mathbf{r}}^{-1}v\leq\phi^{2}\eta\}.$$

Multiple Sources of Information

In reality, the investor's knowledge about the distribution of asset returns often comes from different sources and it is often about a subset of the assets, as opposed to the joint distribution of all assets as in the previous subsection. To accommodate this, let J_k, $k=1,\ldots,$

K, be subsets of $\{1, \ldots, N\}$, each set J_k having N_k elements. Sets J_k are not necessarily disjoint. But we will assume that $\cup_k J_k = \{1, \ldots, N\}$, so that the investor has at least some information about each asset. Each k is an index of a source of information about assets in the set J_k. Let $J_k = \{j_1, \ldots, j_{N_k}\}$, so that the information is about the distribution of $R_{J_k} = (R_{J_1}, \ldots, R_{J_{N_k}})$. We assume that the reference probability distributions implied by the various sources of information for the corresponding subsets of assets coincide with the marginal distributions of the reference model P. Consider the density function of the distribution of R_{J_k},

$$(2\pi)^{-1} |\Omega_{\mathbf{r} J_k}|^{-1/2} \exp\left\{-\frac{1}{2}(R_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (R_{J_k} - \hat{\mu}_{J_k})\right\},$$

where $\hat{\mu}_{J_k} = (\hat{\mu}_{J_1}, \ldots, \hat{\mu}_{J_{N_k}})$, and $\Omega_{\mathbf{r} J_k}$ is the variance-covariance matrix of R_{J_k}, which is a sub-matrix of $\Omega_{\mathbf{r}}$. This density function can be written as

$$\exp\left\{-\frac{1}{2}(\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (\mu_{J_k} - \hat{\mu}_{J_k}) - (\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (R_{J_k} - \mu_{J_k})\right\}$$

$$\times (2\pi)^{-1} |\Omega_{\mathbf{r} J_k}|^{-1/2} \exp\left\{-\frac{1}{2}(R_{J_k} - \mu_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (R_{J_k} - \mu_{J_k})\right\}.$$

Thus, the likelihood ratio of the marginal distribution Q_{J_k} over P_{J_k} is

$$\xi_{J_k} = \exp\left\{\frac{1}{2}(\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (\mu_{J_k} - \hat{\mu}_{J_k}) - (\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (R_{J_k} - \hat{\mu}_{J_k})\right\}.$$

To relate to the probability measure Q, suppose its density function is

$$(2\pi)^{-n/2} |\Omega_{\mathbf{r}}|^{-1/2} \exp\left\{-\frac{1}{2}(R - \hat{\mu})^\top \Omega_{\mathbf{r}}^{-1} (R - \hat{\mu})\right\}.$$

Then

$$(2\pi)^{-1} |\Omega_{\mathbf{r} J_k}|^{-1/2} \exp\left\{-\frac{1}{2}(R_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r} J_k}^{-1} (R_{J_k} - \hat{\mu}_{J_k})\right\}$$

$$= \int (2\pi)^{-n/2} |\Omega_{\mathbf{r}}|^{-1/2} \exp\left\{-\frac{1}{2}(R - \hat{\mu})^\top \Omega_{\mathbf{r}}^{-1} (R - \hat{\mu})\right\} dR_{J_k-},$$

8

where $J_k- = \{1, \ldots, N\} - J_k$. Thus, ξ_{J_k} is the likelihood ratio of the marginal distribution of Q over that of P.

For notational convenience, let $\hat{\Omega}_{\mathbf{r}J_k}^{-1}$ denote the $N \times N$-matrix whose element in the j_mth row and j_nth column, for j_m and j_n in J_k, is equal to the element in the mth row and nth column of the matrix $\Omega_{\mathbf{r}J_k}^{-1}$; otherwise it is zero. Then

$$(\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{\mathbf{r}J_k}^{-1}(\mu_{J_k} - \hat{\mu}_{J_k}) = (\mu - \hat{\mu})^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1}(\mu - \hat{\mu}) = v^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v$$

In the case where there are multiple sources of information, the representative investor's utility function is given by

$$\min_{v \in \mathcal{V}(\phi)} \mathrm{E}\left[\xi u(W)\right], \tag{5}$$

where ξ is given by (3), and similar to the single source information case,

$$\mathcal{V}(\phi) = \{v : \mathrm{E}[\xi_{J_k} \ln \xi_{J_k}] = \frac{1}{2}v^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v \leq \phi^2 \eta_k, \ k = 1, \ldots, K\}. \tag{6}$$

2.3 A Measure of Uncertainty

To understand how the investor trades off uncertainty and expected return, it is useful to introduce a metric for uncertainty of various random variables. Let x be a random variable whose distribution is normal and whose variance is the same under P and all measures $Q \in \mathcal{P}$. An example of such a random variable would be the return on a portfolio of N risky assets. Define

$$\Delta_{\mathbf{u}}(x) = \sup_{\xi : Q \in \mathcal{P}} \mathrm{E}[\xi x]. \tag{7}$$

to be the uncertainty of a random variable x.

Applying the definition of uncertainty to the case of portfolio returns, $x = \theta R$, in the general case where the investor has multiple sources of information, for the portfolio θ,

$$\Delta_{\mathbf{u}}(\theta) = \sup_v \theta^\top v \tag{8}$$

9

subject to

$$\mathrm{E}[\xi_{J_k} \ln \xi_{J_k}] = \frac{1}{2} v^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v \le \eta_k, \quad k = 1, \ldots, K. \tag{9}$$

The value function $\Delta_{\mathbf{u}}(\theta)$ is independent of ϕ. Thus, our definition of uncertainty reflects the properties of the set $\mathcal{P}$ of candidate probability measures, not the preferences of the decision maker. Moreover, only the "shape" of the set $\mathcal{P}$ is important in determining the relative uncertainty of various portfolios. Scaling all constraints η_k by the same constant, thus preserving the shape of the set $\mathcal{P}$, has no effect on the measure of uncertainty.

We will denote a solution of (8) by $v_{\mathbf{u}}(\theta)$. Note that the solution may not be unique in general, with multiple values of v corresponding to the same value of the objective function. The following lemma shows that when all portfolio weights are non-zero, which is the case for the market portfolio in equilibrium, the solution of (8) is indeed unique.[5]

Lemma 1 *For θ such that all of its components are non-zero, the solution of (8) is unique. There exists a set of nonnegative coefficients $\phi_k(\theta)$ depending on θ such that*

$$v_{\mathbf{u}}(\theta) = \Omega_{\mathbf{u}}(\theta)\theta, \tag{10}$$

where

$$\Omega_{\mathbf{u}}(\theta) = \left(\sum_{k=1}^{K} \phi_k(\theta) \hat{\Omega}_{rJ_k}^{-1} \right)^{-1}.$$

A coefficient $\phi_k(\theta)$ is equal to zero if the kth constraint is not binding, but at least one of the coefficients is strictly positive.

2.4 Diversification of Uncertainty

In this section we summarize some of the properties of our measure of uncertainty, drawing a parallel with the variance as a measure of risk (return variance is the appropriate measure

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[5]One of the typical features of the multi-prior expected utility model is that the solution of the utility maximization problem is often not unique. The analytical feature of our formulation of the set $\mathcal{P}(P, \phi)$ is that, due to Lemma 1, the minimizer for the equilibrium situation we are considering is always unique. The crucial property of the set $\mathcal{P}(P, \phi)$ that gives rise to this uniqueness is the strict convexity of the relative entropy function, as can be seen in the proof of lemma 5.

of risk in our model, since asset returns are jointly normally distributed).

The definition of portfolio uncertainty $\Delta_u(\theta)$ given in (8) implies that it is a convex and symmetric function of the portfolio composition, $\Delta_u(-\theta) = \Delta_u(\theta)$, just as the variance of portfolio returns.

As with risk, one can draw a distinction between the total uncertainty of an asset (or a portfolio) and its systematic uncertainty. The total uncertainty of asset i is defined as

$$\Delta_u(e_i) = \max_{v \in \mathcal{V}(\phi)} e_i^\top v$$

where $e_i = (0 \dots 0 \;\; \underbrace{1}_{i\text{th position}} \;\; 0 \dots 0)^\top$. The systematic uncertainty of the asset i with respect to a portfolio θ is defined as its marginal contribution to the total portfolio uncertainty, in analogy with the definition of systematic risk:

$$\Delta_u^{\text{sys}}(e_i) = e_i^\top \frac{\partial \Delta_u(\theta)}{\partial \theta}.$$

More generally, systematic uncertainty of a portfolio $\hat{\theta}$ is given by

$$\Delta_u^{\text{sys}}(\hat{\theta}) = \hat{\theta}^\top \frac{\partial \Delta_u(\theta)}{\partial \theta} = \hat{\theta}^\top \Delta_u^{\text{sys}}(e_i).$$

The following lemma shows that $\Delta_u^{\text{sys}}(e_i)$ is well defined, as long as all components of the portfolio θ are non-zero and characterizes the sensitivity of the portfolio uncertainty to its composition.

Lemma 2 *Assuming that all components of the portfolio weights vector θ are non-zero, the sensitivity of the uncertainty of a portfolio to a change in its composition is given by*

$$\frac{\partial \ln \Delta_u(\theta)}{\partial \theta} = \frac{1}{\Delta_u(\theta)} v_u(\theta) = \frac{\Omega_u(\theta)\theta}{\theta^\top \Omega_u(\theta)\theta}. \tag{11}$$

Thus, $\Delta_u^{\text{sys}}(e_i) = e_i^\top v_u(\theta)$. This implies that systematic uncertainty of the market portfolio is equal to its total uncertainty. Also, since $v_u(\theta) \in \mathcal{V}(\phi)$, it is immediate that the total

11

uncertainty of an asset exceeds its systematic uncertainty, i.e.,

$$\Delta_{\mathtt{u}}(e_i) = \max_{v \in \mathcal{V}(\phi)} e_i^\top v \geq e_i^\top v_{\mathtt{u}}(\theta).$$

In the above, we have considered the sensitivity of portfolio uncertainty to a change in the composition of the portfolio when the portfolio weights are non-zero. This corresponds to the case when the portfolio already has a loading of all the assets. The other interesting case is when an asset is not in the portfolio to begin with, but is to be added to the portfolio. As the following lemma shows, this case is not as simple as the other case and the reason is that $\Delta_u(\theta)$ is in general no longer differentiable.

Lemma 3 *Let θ be a portfolio with $\theta_j = 0$. Let $\mathcal{K} = \{k : j \in J_k\}$. If there exists a solution $\bar{v}$ of (8) such that for all $k \in \mathcal{K}$,*

$$\frac{1}{2}\bar{v}^\top \hat{\Omega}_{rJ_k}^{-1} \bar{v} = \frac{1}{2}\bar{v}_{J_k}^\top \Omega_{rJ_k}^{-1} \bar{v}_{J_k} < \eta_k, \tag{12}$$

then $\Delta_u(\theta)$ is not differentiable in θ_j at $\theta_j = 0$. Otherwise $\Delta_u(\theta)$ is differentiable in θ_j at $\theta_j = 0$ and $\partial \Delta_u(\theta)/\partial \theta_j = \bar{v}_j$, where $\bar{v}$ is any solution of (8).

The intuition of this lemma is best seen in the following example. There are two assets and two sources of information, one for each asset,

$$\frac{1}{2}v_j^2 \sigma_j^2 \leq \eta_j, \quad j = 1, 2.$$

Let $\theta = (\theta_1, \theta_2)$ be a portfolio where $\theta_1 > 0$ and $\theta_2 = 0$. In this case,

$$\Delta_u(\theta) = \sqrt{2\eta_1}\sigma_1\theta_1.$$

and the solutions of (8) are of the form, $v^* = (\sqrt{2\eta_1}\sigma_1, v_2)$ where v_2 is arbitrary as long as it satisfies the constraint above. In other words, when $\theta_2 = 0$, the second source of information about the second asset is irrelevant for the uncertainty of the portfolio. However, the moment when θ_2 becomes positive, the second source of information starts to contribute

to the uncertainty of the portfolio and the rate at which it adds to the uncertainty of the portfolio is given by $\sqrt{2\eta_2}\sigma_2$. This rate is $-\sqrt{2\eta_2}\sigma_2$ when θ_2 becomes negative so that the uncertainty of the portfolio increases by $\sqrt{2\eta_2}\sigma_2$. As a result, $\partial\Delta_u(\theta)/\partial\theta_0$ at $\theta_2 = 0$ does not exists.

In general, when the information about a particular asset has not be fully reflected, which is what (12) says, the rates at which an asset contributes to the uncertainty of the portfolio when the asset is added in a long or short position differ, resulting non-differentiability.

Interestingly, this potential non-differentiability has equilibrium implication for the bid and ask spread of an asset price. See Routledge and Zin (2002) and also the discussion in Section 4.

3   Portfolio Choice

Using the utility function introduced above, the investor's utility maximization problem is

$$\sup_{\theta} \inf_{v \in \mathcal{V}(\phi)} \mathrm{E}\left[\xi u(W)\right], \tag{13}$$

subject to the wealth constraint

$$W = W_0[\theta^\top(R - r1) + 1 + r].$$

Without loss of generality we set $W_0 = 1$. The following lemma shows that the optimal choice of v is given by the solution $v_u(\theta)$ of (8).

Lemma 4 *Problem (13) is equivalent to*

$$\max_{\theta} \min_{|y| \leq \phi\Delta_u(\theta)} \left\{\mathrm{E}\left[\xi_{(\theta,y)}(R_\theta)u(W)\right]\right\}, \tag{14}$$

where

$$\xi_{(\theta,y)}(R_\theta) = \exp\left\{-\frac{y^2}{2\theta^\top\Omega_r\theta} - \frac{y(R_\theta - \theta^\top\mu + y)}{\theta^\top\Omega_r\theta}\right\}.$$

is the density of the return on the portfolio θ, $R_\theta = \theta^\top R$. Furthermore, if (θ, v) is the solution of (13), then

$$v = \phi\, v_u(\theta). \tag{15}$$

Moreover, the optimal portfolio policy θ satisfies

$$\mathrm{E}\left[U'(W - \phi\Delta_u(\theta))\left(R_\theta - r1 - \phi v_u(\theta)\right)\right] = 0.$$

The following figure illustrates the trade-off between expected return and uncertainty implicit in the lemma above.

4 The Equilibrium

The definition of equilibrium for our economy is that of the standard rational expectations equilibrium slightly extended to account for the fact that the objective probability law is not known. Specifically, the econometricians provide an estimate of the probability law of the (exogenous) dividend vector and a set of possible alternatives (at certain confidence level). Through the equation

$$R_j = \frac{D_j}{p_j} - 1, \quad j = 1, \ldots, N,$$

this translates, for a fixed price vector $p = (p_1, \ldots, p_N)$, to an estimated law for the returns and a set of possible alternatives. Taking these as given,[6] the investors determine their asset demands. The equilibrium arises if the price vector $p = (p_1, \ldots, p_N)$ is such that the markets for all assets clear.

4.1 Risk Premium and Uncertainty Premium

Define

$$\zeta = \frac{U'(W - \phi\Delta_u(\theta))}{\mathrm{E}[U'(W - \phi\Delta_u(\theta))]}$$

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[6]Earlier, for expositional convenience, we expressed everything in terms of returns.

Then, according to Lemma 4,

$$E[\zeta R] = r1 + \phi v_{\mathtt{u}}(\theta), \tag{16}$$

and hence

$$E[\zeta R_m] = r + \phi \Delta_{\mathtt{u}}(\theta), \tag{17}$$

where the subscript m denotes the market portfolio. By applying Stein's Lemma to (16) and (17), we find that the expected return premia on the individual stocks and on the market are given by

$$\mu - r = \frac{E[U''(W - \phi\Delta_{\mathtt{u}}(\theta))]}{E[U'(W - \phi\Delta_{\mathtt{u}}(\theta))]} \mathrm{cov}(R_m, R) + \phi\, v_{\mathtt{u}}(\theta) \tag{18}$$

$$\mu_m - r = \frac{E[U''(W - \phi\Delta_{\mathtt{u}}(\theta))]}{E[U'(W - \phi\Delta_{\mathtt{u}}(\theta))]} \underbrace{\sigma_{\mathtt{r}}^2(\theta)}_{\text{risk}} + \phi \underbrace{\Delta_{\mathtt{u}}(\theta)}_{\text{uncertainty}} \tag{19}$$

The first term in (19) may be viewed as the market risk premium, being proportional to the variance of the market portfolio. The proportionality coefficient depends on the preferences of the representative agent. For a special case of the CARA utility function, $U(W) = -\exp(-\gamma W)$, it equals the absolute risk aversion coefficient of the agent, γ. In general, however, this term is affected by the agent's uncertainty aversion as well, since it depends on $\phi\Delta_{\mathtt{u}}(\theta)$. With this reservation in mind, we will denote the entire first term by $\lambda_{\mathtt{r}}$. The second term, $\phi\Delta_{\mathtt{u}}(\theta)$, has a natural interpretation of the market uncertainty premium, given by the product of the uncertainty aversion parameter and the degree of uncertainty of the market portfolio. We will denote it by $\lambda_{\mathtt{u}}$.

Equations (16,17) imply a relation between expected excess returns on individual assets, which we state as the following theorem.

Theorem 1 *The equilibrium vector of expected excess returns is given by*

$$\mu - r1 = \lambda_r \beta_r + \lambda_u \beta_u, \tag{20}$$

where λ_r and λ_u are the market risk and uncertainty premia and β_r and β_u are the risk and

uncertainty betas with respect to the market:

$$\beta_r = \frac{\partial \ln \sigma_r^2(\theta)}{\partial \theta} = \frac{1}{\sigma_r^2(\theta)} \Omega_r \theta,$$

$$\beta_u = \frac{\partial \ln \Delta_u(\theta)}{\partial \theta} = \frac{1}{\Delta_u(\theta)} \Omega_u(\theta) \theta.$$

β_r defines the vector of market risk betas of stocks, i.e., their betas with respect to the market portfolio. As stated in the theorem, an equivalent definition of the market risk beta is as sensitivity of the total risk of the market portfolio to a change in its composition, i.e., $\beta_r = \partial \ln \sigma_r^2(\theta)/\partial \theta$. The definition of the market uncertainty betas β_u is analogous. According to Lemma 2, β_u defines the sensitivity of the uncertainty of the market portfolio to a change in its composition. Note that the uncertainty betas depend only on the relative uncertainty of various portfolios and not on the uncertainty aversion of the representative agent. Scaling the constraint set $\mathcal{P}$ by multiplying η_k's by the same constant has no effect on β_u. We also find that, like risk, uncertainty is partially "diversifiable" in a sense that for a particular asset only its contribution the total market uncertainty is compensated in equilibrium by higher expected return.

In equilibrium, the investor is compensated for bearing both risk and uncertainty. Thus, two assets with the same beta with respect to the market risk can have different equilibrium expected returns. This not only sets our model apart conceptually from the standard CAPM, but also points to the empirical relevance of our model. To elaborate, consider first the case where there is a single source of information. In this case,[7]

$$\Omega_u = \frac{\sqrt{2\eta}}{\sigma_\theta} \Omega_r,$$

where θ is the equilibrium market portfolio, and hence

$$\mu - r\mathbf{1} = \frac{E[U''(W - \phi \Delta_u(\theta))]}{E[U'(W - \phi \Delta_u(\theta))]} \Omega_r \theta + \phi \frac{\sqrt{2\eta}}{\sigma_\theta} \Omega_r \theta = \left(\frac{E[U''(W - \phi \Delta_u(\theta))]}{E[U'(W - \phi \Delta_u(\theta))]} + \phi \frac{\sqrt{2\eta}}{\sigma_\theta} \right) \sigma_\theta^2 \beta.$$

Since the utility-dependent coefficient

$$\frac{E[U''(W - \phi \Delta_u(\theta))]}{E[U'(W - \phi \Delta_u(\theta))]}$$

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[7]See Section 4.2.

16

is not observable, the cross-sectional distribution of expected asset returns in a world with a single source of information will be observationally indistinguishable from that in a world where there is no model uncertainty.

Note that in the case of a single source of information, the reason that the uncertainty premium is observationally indistinguishable from the risk premium is that the two are proportional to each other in the cross-section. When there is more than one source of information, this is no longer the case (Section 4.2 contains an example) and hence the observational equivalence no longer holds. Therefore, by studying the cross-section of asset returns, one can potentially test for the existence of uncertainty premia.

4.2 Independent Sources of Information

To help derive testable implications of theorem (1), we consider a special case where the N risky assets can be divided into K groups, with investor having a separate source of information about each group. Without loss of generality, assume that the first N_1 assets are in the first group, the next N_2 in the second group, and so on.

Lemma 5 *If the K sources of information are independent, then*

$$\phi_k(\theta) = \frac{1}{\sqrt{2\eta_k}}\sigma_k, \quad k = 1, \ldots, K, \tag{21}$$

where σ_k is the standard deviation of returns on the portfolio P_k of assets in group k combined with their market portfolio weights.

As a result of this lemma, the model uncertainty matrix simplifies to a block diagonal matrix,

$$\Omega_{\mathbf{u}} = \begin{bmatrix} \frac{\sqrt{2\eta_1}}{\sigma_1}\Omega_{\mathbf{r}J_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\sqrt{2\eta_K}}{\sigma_K}\Omega_{\mathbf{r}J_K} \end{bmatrix} \tag{22}$$

Intuitively, the block-diagonal form of $\Omega_{\mathbf{u}}$ could arise if the agent had separate models for returns on each group of assets, e.g., different models for returns on fixed income securities,

stocks, and commodities or a different model of returns on equity in the United States, Japan and Europe. After all, it is common practice in academic research to specify, estimate, and test the models of individual classes of assets independently of each other. This would imply that if the uncertainty faced by the agent about the model of returns on the group of assets k were to change, it would have no effect on the amount of uncertainty remaining about the model of returns on any other group of assets. Then Theorem 1 implies that the market uncertainty beta of an asset j from the asset group k is given by

$$\beta_{\mathbf{u},j} = \frac{\sqrt{2\eta_k}\sigma_k}{\sum_{n=1}^{K} \sqrt{2\eta_n}\sigma_n}\beta_{\mathbf{r},P_k,j},$$

where σ_k^2 is the variance of returns on the portfolio P_k of assets in group k combined with their market portfolio weights and $\beta_{P_k,j}$ is the beta of returns on asset j with respect to such portfolio. The market price of uncertainty is given by

$$\lambda_{\mathbf{u}} = \phi^2 \sum_{k=1}^{K} \sqrt{2\eta_k}\sigma_k$$

As a result, we have the following.

Corollary 2 *If the uncertainty matrix has a block-diagonal form (22), the cross-section of expected returns on the assets in group k is characterized by*

$$\mu_j = r + \lambda_r \beta_{r,j} + \phi^2 \sqrt{2\eta_k}\sigma_k \beta_{r,P_k,j}. \tag{23}$$

Thus, the cross-sectional differences in returns within each group of assets can be described by the assets' loadings on two "factors" – the aggregate market portfolio and the value-weighted portfolio of assets within the corresponding group.

The relation (23) could be tested empirically using the standard cross-sectional methodology. Note that (23) implies that the second factor used in addition to the market is specific to the group of assets under consideration. The presence of the second factor distinguishes our model from the standard static CAPM. The pricing relation (23) is also distinct from dynamic, multi-factor models, in which *all* assets earn risk premium as compensation for their

covariation with the systematic risk factors. Under model uncertainty, the asset's expected return is affected by its correlation with the portfolio P_k only if such asset is subject to the same source of model uncertainty as other assets in that portfolio. Moreover, factors in the standard intertemporal pricing model earn excess return because they could be used to hedge against changes in the investment opportunity set. This does not have to be the case under model uncertainty. In our static model, the investment opportunity set cannot change by assumption, yet portfolios P_k appear to serve as pricing factors within the corresponding group of assets.

Using (23), one can also identify a number of restrictions across the asset groups. For instance, a within-the-group cross-sectional regression of returns on the market betas and the group-portfolio betas should recover the two coefficients: $\lambda_{\mathbf{r}}$ and $\phi^2 \sqrt{2\eta_k}\sigma_k$. One can then test whether the estimates of the market risk premium $\lambda_{\mathbf{r}}$ are identical across the groups. Moreover, since $\sum_{k=1}^{K} \phi^2 \sqrt{2\eta_k}\sigma_k = \lambda_{\mathbf{u}}$, one could compare the resulting estimate of $\lambda_{\mathbf{r}} + \lambda_{\mathbf{u}}$ with the direct estimate of the expected return on the market portfolio.

5   The Effect of Changes in Uncertainty

Another way to highlight the effect of uncertainty on asset prices is by performing a comparative statics experiment of increasing the degree of uncertainty in the model. To simplify the exposition, we will assume that the representative agent has a CARA utility function, $U(W) = -\exp(-\gamma W)$.

All securities in the model, as well as zero net supply derivative assets that do not follow a normal distribution, can be priced using the risk-neutral probability density. The risk-neutral density is given by

$$\frac{f(R)\xi(R)u'(W)}{\int f(R)\xi(R)u'(W)\,dR},$$

where R is the vector of equilibrium returns on the primitive securities, W is the end-of-period wealth, and $\xi(R)$ is the probability density corresponding to the equilibrium value of

$v_{\mathbf{u}}(\theta)$. Then the price of any security with payoff $X(R)$ is given by

$$\frac{1}{1+r} \frac{\int f(R)\xi(R)u'(W)X(R)\,dR}{\int f(R)\xi(R)u'(W)\,dR}.$$

Let $\Omega_{\mathbf{r}D}$ denote the variance-covariance matrix of dividends. Similarly, define $\Omega_{\mathbf{u}D} = \left(\sum_{k=1}^{K} \phi_k \hat{\Omega}_{\mathbf{r}D,J_k}^{-1}\right)^{-1}$. Then the risk-neutral probability density can be expressed as

$$(2\pi)^{-n/2}|\Omega_{\mathbf{r}D}|^{-1/2} \exp\left\{-\frac{1}{2}(D + v_D - \mathrm{E}[D])^{\top}\Omega_{\mathbf{r}D}^{-1}(D + v_D - \mathrm{E}[D])\right\}$$

$$\times \exp\left\{-\gamma\left(D + v_D - \mathrm{E}[D] - \frac{1}{2}\gamma^2 1^{\top}\Omega_{\mathbf{r}D}1\right)\right\},$$

where

$$v_D = \exp\left\{-\gamma(1+r) - \frac{\gamma^2}{2}1^{\top}\Omega_{\mathbf{r}D}1\right\}\Omega_{\mathbf{u}D}1$$

This expression for the risk-neutral density implies that an increase in uncertainty, i.e., an increase in $\Omega_{\mathbf{u}D}$, leads to a shift in the mean of the risk-neutral distribution. The effect is particularly easy to visualize in a one risky asset case. An increase in uncertainty results in a downward shift in the mean of the risk-neutral distribution, as illustrated in Figure 1.

It is instructive to compare this behavior with an increase in prior uncertainty in the standard Bayesian framework. The Bayesian approach assumes uncertainty neutrality, whereas our approach assumes uncertainty aversion. This is best illustrated using Ellsberg experiment and its following variant: in the second urn the number of red ball is between 10 and 90 so that the probability of drawing red is between 0.1 and 0.9. In this case, the Bayesian approach would still assign 0.5 to drawing red and hence be indifferent between a bet on the second urn and that on the first urn, just as in the original Ellsberg experiment. Thus even though the amount of uncertainty is different in these two experiment, the Bayesian approach makes no distinction. Note that in this experiment the mean of the posterior distribution remains unchanged in the Bayesian approach.

Increased prior variance in the Bayesian framework results in an increase in the variance of the risk-neutral distribution, while in our model an increase in uncertainty would shift the
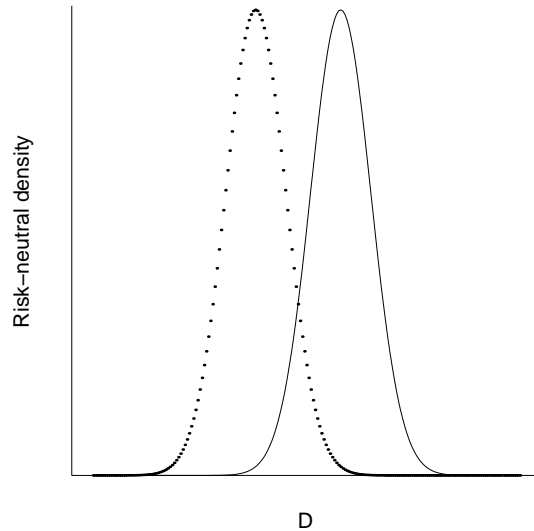
20

Figure 1: Risk-neutral distributions. The dotted line corresponds to the economy with a higher degree of uncertainty.

mean of the risk-neutral distribution downward, as illustrated in Figure 2. Thus, an increase is model uncertainty would have different implications for the prices of derivative securities relative to an increase in prior variance in a Bayesian model. Prices of out-of-the-money call options fall as model uncertainty increases. However, this may not be the case in the Bayesian framework.

6   Conclusion

We have developed a single-period equilibrium model incorporating, not only risk, but also uncertainty and uncertainty aversion. We have shown that there is an uncertainty premium in equilibrium expected returns on financial assets. In particular, the cross-sectional distribution of expected returns can be formally described by a two-factor model, where expected returns are derived as compensation for the asset's contribution to the equilibrium risk and uncertainty of the portfolio held by the agent. We were able to derive several empirically testable implications of this result. While prior research on model uncertainty has been concerned with its implications for the time-series of asset prices, by characterizing the cross-section of returns we were able to address some of the observational equivalence issues raised in the literature. In particular, we demonstrated that the effect of model uncertainty
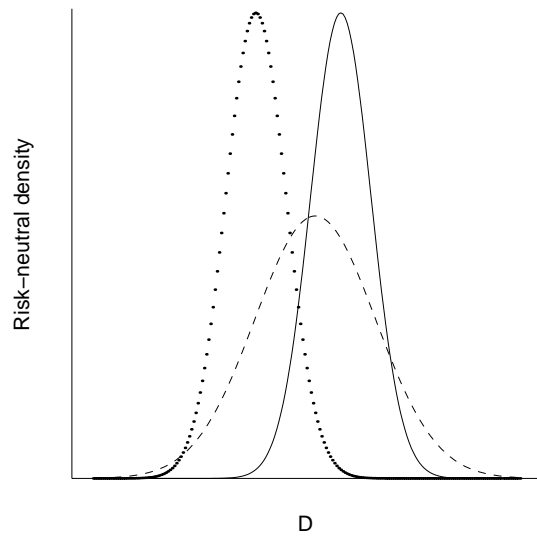
Figure 2: Risk-neutral distributions. The dotted line corresponds to the economy with uncertainty aversion and a higher degree of uncertainty. The dashed line corresponds to the Bayesian economy with increased prior uncertainty.

in our framework is distinct from risk aversion and cannot be captured by any specification of the risk aversion parameter.

Appendix

Proof of Lemma 1

Suppose to the contrary that $\bar{v}$ and $\underline{v}$ are two distinct solutions. Let $v(a) = a\bar{v} + (1-a)\underline{v}$. The strict convexity of all the functions defining the choice set implies that for $a \in (0,1)$,

$$\frac{1}{2}v(a)^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v(a) \leq \eta_k, \quad k = 1, \ldots, K.$$

Now let k, if exists, be such that

$$\frac{1}{2}v(a)^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v(a) = \eta_k$$

holds for $a = 0$, $a = 1$, and for some $a \in (0,1)$. Then it must be the case that $\bar{v}_{J_k} = \underline{v}_{J_k}$. Denote by A the set of such k. If

$$J_A = \cup_{k \in A} J_k = \{1, \ldots, n\},$$

then $\bar{v} = \underline{v}$, a contradiction to assumption. So, $J_A \neq \{1, \ldots, n\}$. Without loss of generality, we assume that $J_A = \{2, \ldots, n\}$. Then for all v of the form $v = (v_1, \bar{v}_2, \ldots, \bar{v}_n)$ with $v_1 \in R$,

$$\frac{1}{2}v^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v = \eta_k, \quad k \in A.$$

Note that $v(a)$ is of the form $(a\bar{v}_1 + (1-a)\underline{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$. Thus for $v = (0.5\bar{v}_1 + 0.5\underline{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$,

$$\frac{1}{2}v^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v < \eta_k, \quad k \notin A.$$

Combining the two cases, $k \in A$ and $k \notin A$, together, by continuity, there is a $\epsilon > 0$ such that for all $v = (v_1, \bar{v}_2, \ldots, \bar{v}_n)$ with $v_1 \in (0.5\bar{v}_1 + 0.5\underline{v}_2 - \epsilon, 0.5\bar{v}_1 + 0.5\underline{v}_2 + \epsilon)$,

$$\frac{1}{2}v^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v \leq \eta_k, \quad k = 1, \ldots, K.$$

But, given the linearity of the objective function, this means $\bar{v}$ and $\underline{v}$ cannot be the solution of (8). This is a contradiction.

The second statement of the lemma is a straightforward application of the Lagrangian duality approach.

Proof of Lemma 2

Since the constraint set $\mathcal{P}$ is convex and compact, $\Delta_{\mathbf{u}}(\theta)$ is a convex function. Optimality conditions imply that $\phi^{-1}v_{\mathbf{u}}(\theta)$ is a subgradient of the value function $\Delta_{\mathbf{u}}(\theta)$ at θ. The solution $v_{\mathbf{u}}$ of is unique, according to lemma 1. Thus, the function $\Delta_{\mathbf{u}}(\theta)$ has a unique subgradient, therefore it is in fact differentiable, and $\phi^{-1}v_{\mathbf{u}}$ is equal to the gradient of $\Delta_{\mathbf{u}}(\theta)$. This establishes the statement of the lemma.

Proof of Lemma 3

Without loss of generality, let $j = 1$. If the condition of the first claim of the lemma is satisfied, there exists a $\epsilon > 0$ such that for any $|x| < \epsilon$, $v_x = \bar{v} + (x, 0, \ldots, 0)$ satisfies all constraints of (8). Since $\theta_1 = 0$, v_x is also a solution of (8). The claim follows.

For the second part, let $\bar{v}$ be a solution of (8). If it is the unique solution of (8), then there is nothing to prove. Suppose $\bar{v}$ and $\underline{v}$ are two distinct solutions of (8). Let $v(a) = a\bar{v} + (1-a)\underline{v}$. We claim that there exists a $k \in \mathcal{K}$ such that

$$\frac{1}{2}v(a)_{J_k}^\top \Omega_{\mathbf{r}J_k}^{-1} v(a)_{J_k} = \eta_k$$

holds for $a = 0$, $a = 1$ and some $a \in (0,1)$. Suppose the contrary. By strict convexity,

$$\frac{1}{2}v(a)_{J_k}^\top \Omega_{\mathbf{r}J_k}^{-1} v(a)_{J_k} < \eta_k, \quad k \in \mathcal{K}$$

for $a \in (0,1)$. Also the convexity of all the functions defining the choice set implies that for $a \in (0,1)$,

$$\frac{1}{2}v(a)^\top \hat{\Omega}_{\mathbf{r}J_k}^{-1} v(a) = \frac{1}{2}v(a)_{J_k}^\top \Omega_{\mathbf{r}J_k}^{-1} v(a)_{J_k} \leq \eta_k, \quad k = 1, \ldots, K.$$

Since the objective function of (8) is linear, $v(a)$ is a solution of (8) for all $a \in (0,1)$. But this is a contradiction to assumption of the second part of the lemma. Thus the claim is shown. It then follows from the claim that $\bar{v}_{J_k} = \underline{v}_{J_k}$ and hence $\bar{v}_j = \underline{v}_j$. Since $\bar{v}$ and $\underline{v}$ are arbitrary, we have $\bar{v}_j = \underline{v}_j$ for all solutions of (8). The differentiability follows. ∎

Proof of Lemma 4

Since the distribution of W depends only on the distribution of $\theta^\top R$, for each fixed θ, $E[\xi u(W)]$ depends only on $y = \theta^\top v$, and it is given by

$$E[\xi u(W)] = E[\xi_{(\theta,y)} u(W)],$$

where

$$\xi_{(\theta,y)} = \exp\left\{-\frac{y^2}{2\theta^\top \Omega_{\mathrm{r}} \theta} - \frac{y(R_\theta - \theta^\top \mu + y)}{\theta^\top \Omega_{\mathrm{r}} \theta}\right\}.$$

Thus the original utility function can be written as

$$\max_\theta \min_{|y| \leq \phi \Delta_{\mathrm{u}}(\theta)} \left(E[\xi_{(\theta,y)} u(W)]\right) \tag{A1}$$

which is (14). The characterization for v follows immediately.

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