

INFERENCE AND ARBITRAGE:  
THE IMPACT OF STATISTICAL ARBITRAGE ON  
STOCK PRICES

[Preliminary]

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*Abstract:*

This paper models the impact of statistical arbitrageurs on stock prices and trading volume when the drift of the dividend process is unknown to the hedge fund. The learning process of statistical arbitrageurs leads to an optimal trading strategy that can be upwardsloping in prices. The presence of privately informed investors makes the equilibrium price dependent the history of trading volume and prices, and the optimal trading strategy of statistical arbitrageurs can be a positive feedback strategy for certain parameters and histories.

# 1 Introduction

## 1.1 Motivation

Statistical arbitrageurs employ a variety of investment strategies to take advantage of mispriced assets. The common feature of these strategies is that temporary deviations of prices from their fundamental value are exploited. The principal difficulty is to distinguish permanent movements in prices due to fundamental changes from temporary fluctuations in prices due to supply and demand disturbances. In this paper, the inference problem of statistical arbitrageurs is modeled explicitly in the context of the stock market.

The arbitrageurs trade against two classes of investors: noise traders and fundamental traders. The noise traders are causing deviations of prices from their fundamental value that the statistical arbitrageurs exploit. The fundamental traders obtain private information about the drift rate of the dividend. This set-up is realistic for small stocks, where information gathering is costly, and information asymmetries are likely to be large. The set-up can also be used to analyze the conditions under which statistical arbitrageurs will find it profitable to acquire fundamental information, as opposed to

Statistical arbitrageurs are assumed to be rational, risk neutral and have a long-term investment horizon. They condition their trading strategies on all publicly available information, which is the history of dividends, trading volume and prices. In order to assess the fundamental value of an asset, the arbitrageur needs to estimate a model, i.e. needs to learn from past observations of publicly observable data. This model guides the hedge fund in distinguishing price changes of an asset due to fundamentals or demand and supply disturbances.

Arbitrageurs are faced with nonstandard (i.e. not normally distributed) uncertainty about the underlying economic environment. The informed investors learn the true drift of the dividend that can take two values. In order to take advantage of the mispricing induced by the noise traders, the statistical arbitrageurs must assess to what extent the current price reflects the demands from the noise traders or the informed investors.

The presence of uncertainty about the drift rate of dividends makes the inference problem of the arbitrageurs nonlinear. As prices move, arbitrageurs update their belief both about the next realization of dividends, and the true drift rate. Holding the belief about the growth rate fixed yields a linear pricing schedule. However, in certain ranges of prices, the hedge fund will strongly revise beliefs about the growth rate of dividends. This leads to an amplification of new information in a certain range of prices.

Arbitrageurs face a trade-off between an inference and an arbitrage effect<sup>1</sup>. As prices go up, the arbitrageurs have an incentive to sell the asset, as it becomes more expensive given a belief about the growth rate of dividends. However, a higher price also makes it more likely that the true expected payoff is high,

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<sup>1</sup>The term arbitrage is used in a loose sense, denoting risky arbitrage opportunities.

which leads to a strong updating. This is the inference effect, which makes the hedge fund's trades upwardsloping in the price of the risky for a certain range of parameters and prices. The information structure is similar to the model of Wang (1993), where the drift of the dividend process is changing according to a mean reverting process. In the model presented here, the drift is constant, but can take only two values, which leads to a nonlinear pricing function.

Intuitively, the trading strategy of the arbitrageurs is upwardsloping for the following reasons. As prices increase, statistical arbitrageurs infer that the fundamentals must be better than previously thought and increase their asset holdings. In other regions of the price, a drop in the price represents a buy opportunity for statistical arbitrageurs. The arrival of new fundamental information has different impacts on the fund's trading strategy depending on the level of prices. In the range of prices where the arbitrage effect dominates the inference effect, the statistical arbitrageurs learn a lot about the relative likelihood of the high or the low state, which makes the price move drastically. Small disturbances due to noise or fundamental information makes prices move very strongly in these ranges. When prices are very low or very high, not much is learned from new information, and the price reacts very little to either noise or news.

The absence of a simple relationship between the arrival of information and movements in prices has lead many to question the relevance of informational sources for movements in asset prices. Instead, it is often argued that noise traders or irrational speculators are causing movements without news and are mitigating the impact of new information. In the framework presented here, the stochastic structure leads to a nonlinear relationship between new information and prices that has such pricing behavior as a consequence. Little fundamental news moves prices dramatically at times, whereas big pieces of news have little impact on prices at other times. Such nonlinearities are very important for empirical work. Boudoukh et al. (2001) find that a nonlinear regression can explain much of the variation on OJ future prices uncovered by Roll (1984).

In the remainder of the paper, both a static and a dynamic version of the model is analyzed. The static version in section 2 is very simple to solve. The static version gives, however, no insight as to the impact of the learning of statistical arbitrageurs on the autocorrelations of prices and trading positions. The dynamic model is developed. In section 3.

## 1.2 Outline:

Section 2: The static model

Sequential trade equilibrium and rational expectations equilibrium.

Section 3: Continuous Time 1

Appendix I: Proofs

### 1.3 Related Literature

The share of assets under management of statistical arbitrageurs has increased dramatically over the past ten years. At the same time, the academic community as well as institutional investors and policymakers have become more and more aware of the limits to arbitrage. The near collapse of LTCM in 1998 has prompted extensive research into the impact of statistical arbitrageurs on equilibrium asset prices. Many of these recent papers focus on the importance of margin constraints and other imperfections in order to understand the limitations to the arbitrage activities conducted by statistical arbitrageurs.

This paper takes a different approach. The focus is not the constraints that are imposed on statistical arbitrageurs, but rather the importance of learning. In the model, none of the traders are constraint.

⟨Incomplete⟩

## 2 The Static Model

This section analyzes the simple static model. It is a useful benchmark for the dynamic model that is developed in the next section. Many of the comparative statics of the one period model are directly inherited by the dynamic model. However, the static model also has limitations, in particular, nothing can be inferred about the autocorrelations of prices and trading strategies.

The static model is solved as sequential equilibrium, where the informed agents submit their demand schedule first. After observing the net demand - total demand less the supply of the informed - the uninformed statistical arbitrageurs conduct their inference and submit their demand. Finally, the price clears the market. This sequential equilibrium is demonstrated to be equivalent to a REE, where informed and uninformed submit their demand schedules simultaneously.

### 2.1 The Informed Investor's Demand

Among a total of  $N$  agents, there are  $N^I$  informed investors and  $N^A$  uninformed statistical arbitrageurs. This section is concerned with the derivation of the optimization and inference problem for the informed investors. Informed agents know the true mean  $\mu$  of the distribution of dividends  $D$ , which are normal conditional on  $\mu$ :

$$D|\mu \sim N(\mu, \sigma_D^2)$$

The only asset beside the stock is cash that is not interest bearing. The informed investors maximize expected utility:

$$\begin{aligned} & \max_{y^I} E[U(w^I)|\mu] \\ \text{st. } w^I &= w_0 + y^I(D - p) \end{aligned}$$

where the utility for the informed agents is assumed to be exponential with CARA  $\alpha$ . The FOC for maximization is then:

$$E[U'(w^I)(D - p)|\mu] = 0$$

This gives the demand of the informed investors:

$$y^I = \frac{\mu - p}{\alpha \sigma_D^2} \tag{1}$$

The total supply of the asset is assumed to be

$$\begin{aligned} & S + u \\ u & \sim N(0, \sigma_u^2) \end{aligned}$$

The net demand of the informed agents denoted  $\hat{y}^I(p, x)$  is linear in  $p$  and a newly defined variable  $x$ :

$$\begin{aligned}\hat{y}^I &= N^I y^I - S - u \\ &= \frac{N^I}{\alpha \sigma_D^2} (x - p) - S \\ &\equiv \hat{y}^I(p, x)\end{aligned}\tag{2}$$

where the variable  $x$  is defined as:

$$x = \mu - \frac{\alpha \sigma_D^2}{N^I} u$$

The distribution of  $x$  conditional on  $\mu$  is then:

$$x|\mu \sim N(\mu, \sigma_x^2)$$

where  $\sigma_x^2 \equiv \left(\frac{\alpha \sigma_D^2}{N^I}\right)^2 \sigma_u^2$ .

## 2.2 The Arbitrageur's Inference

The statistical arbitrageurs observe the net demand of the informed  $\hat{y}^I(p, x)$ . They submit a demand schedule that is a function of price and  $x$ . Recall from the previous section that dividend  $D$  is distributed normally conditional on the mean  $\mu$ . The uninformed have the following prior beliefs about the distribution of  $\mu$ :

$$\begin{aligned}\Pr(\mu = 0) &= 1 - \pi_0 \\ \Pr(\mu = \overline{D}) &= \pi_0\end{aligned}$$

The uninformed believe that the dividend is distributed either with a high  $\mu = \overline{D}$  or a low mean  $\mu = 0$ . The distributional assumption that there are only two states for  $\mu$  is not critical for the arguments that are developed in the remaining paper. In the static model, it is straightforward to extend the results to cases when  $\mu$  is either exponentially or uniformly distributed. The assumption that there are only 2 states makes the dynamic model much more tractable. All uninformed investors are identical. They are endowed with initial wealth  $w_0$  and have the choice of investing their wealth in two assets: money and a risky asset. There are no borrowing constraints. Uninformed investors maximize the following program:

$$\begin{aligned}\max_{y^A} & E[w^A | \hat{y}^I(p, x)] \\ \text{st. } w^A &= w_0 + y^A(D - p)\end{aligned}$$

Due to the risk neutrality of the uninformed statistical arbitrageurs the demand schedule is the following:

$$y^A(p, x) \begin{cases} = & +\infty & \text{if } p < E[D|\hat{y}^I(p, x)] \\ \in & (-\infty, +\infty) & \text{if } p = E[D|\hat{y}^I(p, x)] \\ = & -\infty & \text{if } p > E[D|\hat{y}^I(p, x)] \end{cases} \quad (4)$$

Because of the linearity of  $\hat{y}^I(x, p)$ , the uninformed statistical arbitrageurs can infer  $x$  for any price  $p$ . To solve for their demand schedule, the statistical arbitrageurs therefore only need to compute  $E[D|x]$ . By the law of iterated expectations:

$$E[D|x] = \sum_{\mu \in \{0, \bar{D}\}} E[D|x, \mu] \Pr[\mu|x] \quad (5)$$

There are two elements to the expectation. The first is the expected dividend conditional on  $x$  and a particular state  $\mu$ . The second is the probability distribution of  $\mu$  conditional on  $x$ . Much of the economic intuition of the model will be driven by this second factor. The inference problem of the statistical arbitrageurs can be interpreted as consisting of a standard, linear inference problem holding fix the state of the world  $\mu$ . The net demand of the informed leads the hedge fund to additionally update their belief about the true state, which is expressed by the change in the probability distribution of  $\mu|x$ . As the uninformed investors observe a movements in  $x$ , they will update their assessment as to what the relative likelihood of being in the high dividend state is. The distribution of  $\mu$  conditional on  $x$  can be computed using Bayes rule:

$$\Pr[\mu|x] = \frac{f(x|\mu) \Pr[\mu]}{\sum_{\tilde{\mu} \in \{0, \bar{D}\}} f(x|\tilde{\mu}) \Pr[\tilde{\mu}]}$$

Replacing for the conditional densities of being in the high versus the low state:

$$\begin{aligned} \pi(x) &\equiv \Pr[\mu = \bar{D}|p] \\ &= \frac{\pi_0 \phi(x)}{1 - \pi_0 + \pi_0 \phi(x)} \\ \phi(x) &= \exp[\bar{D} \sigma_x^{-2} (x - \bar{D}/2)] \end{aligned} \quad (6)$$

Note that the likelihood ratio  $\phi(x)$  is simply the Radon-Nikodym derivative that makes the distribution of  $x|\mu = \bar{D}$  equivalent to the distribution of  $x|\mu = 0$ . The statistical arbitrageurs use the net demand schedule to infer  $x$  and in turn to update their best guess of the likelihood of the high dividend state. The likelihood that higher mean dividend will occur is clearly an increasing function of the price  $x$  and the prior distribution over the two states,  $\pi_0$ . Figure 1 displays the typical shape of the distribution of  $\pi(x)$ .

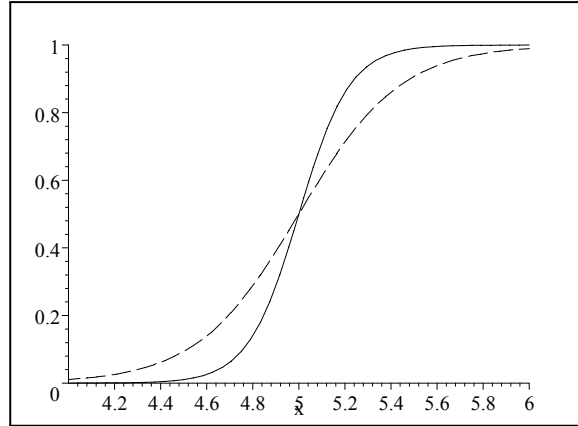


Figure 1: The shape of  $\pi(x)$

The figure is drawn with equal weighted prior ( $\pi_0 = .5$ ) and a high-dividend mean of  $\bar{D} = 10$ . The solid line corresponds to a case of a low variance of  $x$ , and the dotted line to a relatively higher variance of  $x$ . Higher variance of the  $x$  makes the inference about the mean  $\mu$  more difficult, so that the curve is flatter around the unconditional mean (which is 5 in this example). What matters for the slope of the curve is the variance of the signal and not the signal to noise ratio. A lower variance of the dividend enhances the precision of the inference about the true mean of the distribution.

Using the results derived thus far allows us to compute the expected value of the dividend conditional on  $p$ :

$$E[D|x] = \bar{D}\pi(x) \tag{7}$$

The demand schedule for the risky asset by the uninformed investors is therefore:

$$y^A(p, x) = \begin{cases} = & +\infty & \text{if } p < \bar{D}\pi(x) \\ \in & (-\infty, +\infty) & \text{if } p = \bar{D}\pi(x) \\ = & -\infty & \text{if } p > \bar{D}\pi(x) \end{cases} \tag{8}$$



## 2.3 General Equilibrium

**Definition 1** A *Sequential Trade Equilibrium (STE)* is a price  $p$  and an allocation  $y = (y^I, y^A)$  such that:

1. Informed investors maximize expected utility conditional on their observation of  $\mu$  and submit a demand schedule  $y^I(\mu, p)$
2. Statistical arbitrageurs maximize expected utility having observed the net demand  $\hat{y}^I(x, p)$  and submit a demand schedule  $y^A(x, p)$
3. Markets clear

The general equilibrium occurs when the demand of the statistical arbitrageurs plus the net demand of the informed investors equal 0:

$$\hat{y}^I(x, p) + N^A y^A(x, p) = 0$$

This equation determines the net trades of the statistical arbitrageurs. From the demand schedule of the statistical arbitrageurs, the equilibrium price can be determined:

$$p = \bar{D}\pi(x) \tag{9}$$

The Figure 2 is a graphical representation of the equilibrium price as a function of  $x$ .<sup>2</sup> The price function reflects the quasi-linearity that the learning of the statistical arbitrageurs imposes. For very low and very high levels of  $x$ , the price is approximately a linear function of  $x$ . In an intermediate range, the price is very sensitive to  $x$ . This steep part of the pricing function occurs when  $x$  is around 10. This range of the pricing function corresponds to an area where movements in the fundamental  $s$  or the noise  $u$  are amplified. A small demand shock by the noise traders leads to strong updating by the statistical arbitrageurs.

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<sup>2</sup>All the figures are drawn for the following parameter values:

$$\begin{aligned} \sigma_D &= 1, \sigma_u = 1 \\ \alpha &= 1, \bar{D} = 10, \mu = 10 \\ N &= 10, \delta = .1 \\ S &= 0, u = 0 \\ \pi_0 &= .5 \end{aligned}$$

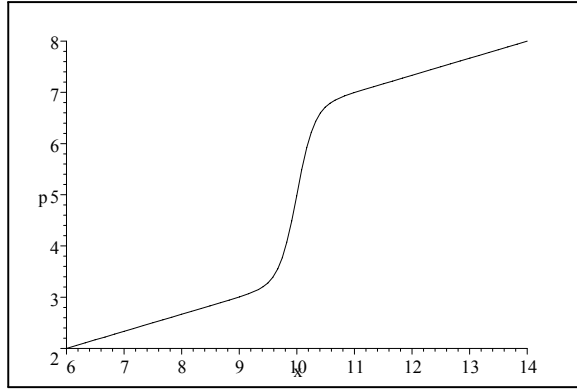


Figure 2: Equilibrium price

In figure 3, the equilibrium asset holdings of the uninformed statistical arbitrageurs are plotted against  $x$ . Whereas the demand schedule is a downwardsloping function of  $x$  when  $x$  is small, it turns upwardsloping once  $x$  is approaching a value of 10. This is when the "inference effect" starts to dominate the "arbitrage effect": even though prices move up sharply around  $x = 10$  - as can be seen in figure 2 - the asset holdings of the uninformed is increasing.

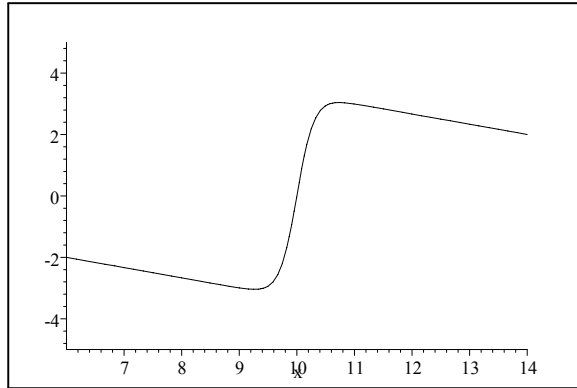


Figure 3: Hedge Fund's Asset Holdings

From the market clearing condition, the equilibrium asset holdings of the statistical arbitrageurs can be derived:

$$y^A(x, p) = S/N^A - \frac{N^I/N^A}{\alpha\sigma_D^2} (x - \bar{D}\pi(x)) \quad (10)$$

The uninformed have a downwardsloping asset holdings in terms of price, except in the range where the inference effect dominates the price effect.

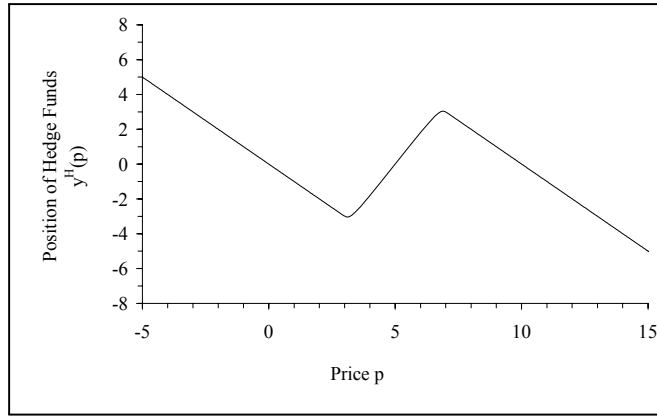


Figure 1: Figure 4: Asset Holdings as Function of Price

Figure 4 shows very clearly the trade-off that the statistical arbitrageurs face. For relatively low prices, an increase in the price makes the asset relatively cheap, which is the arbitrage effect. statistical arbitrageurs take advantage of the fact that in a low range of prices, it is very unlikely that the high state  $\mu = \bar{D}$  is true, and the optimal investment strategy is mainly determined by the relative variances of noise traders and the dividend. The lower the variance of the noise traders relative to the variance of the dividend, the more downwardsloping is the demand as a function of price. However, in an intermediate range of prices, the inference effect dominates the arbitrage effect. As prices increase for a level of prices around 5, higher prices lead the statistical arbitrageurs to infer that the high state  $\mu$  is more likely, which makes them increase their position in the risky asset.

Finally, figure 5 demonstrates the role of different levels of  $\bar{D}$  for the response of the price to a noise trader shock. In the example, the solid line corresponds to  $\bar{D} = 10$ , and the dotted line corresponds to  $\bar{D} = 10.2$ . For a noise shock of 0, the difference in the high dividend state is amplified: a mean dividend that is higher by .2 in the good state leads to a price that is 1 higher. This reflects the fact that in this range, very strong learning is taking place. The slope of the high expected payoff state (the dotted line) is steeper than the slope of the lower expected payoff state. The price reacts the more strongly, the higher the mean dividend. However, for very large shocks, the difference between the two lines narrows, and approaches .2. This reflects the fact that large shocks are mistaken as a very bad signal, and the  $\pi$  approaches 1 in both cases  $\bar{D} = 10$  and  $\bar{D} = 10.2$ .

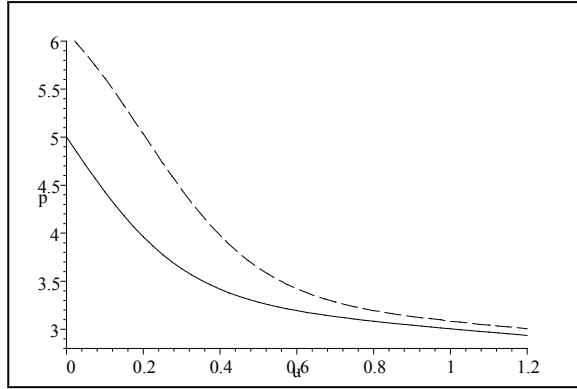


Figure 5: Response of Price to Noise Shock

The following proposition summarizes the results obtained so far:

**Proposition 2** *The unique STE has price:*

$$p = \overline{D}\pi(x)$$

with  $\pi(x)$  as defined in 6 and  $x$  defined as in ?? The position in the risky asset by the statistical arbitrageurs at the equilibrium are:

$$y^A(p(x)) = S/N^A - \frac{N^I/N^A}{\alpha\sigma_D^2} (x - \overline{D}\pi(x))$$

where the constants are the same as in 10

### 2.3.1 Comparison to REE

**Definition 3** *A REE is a price  $p$  and an allocation  $y = (y^I, y^A)$  such that:*

1. *informed investors maximize utility conditional on  $(\mu, p)$  and submit demand schedules  $y^I(p, \mu)$*
2. *uninformed statistical arbitrageurs maximize utility conditional  $(p)$  and submit demand schedules  $y^A(p)$*
3. *markets clear*

**Proposition 4** *The STE is a REE.*

### 2.3.2 Conditions for upwardsloping equilibrium asset holdings

Conditions for upwardsloping asset holdings of the uninformed can also be derived more formally:

**Proposition 5**  $dy^A/dp > 0$  in some range iff

$$\bar{D}^2 > (1 + (1 - \pi_0)/\pi_0)^2 \alpha^2 \sigma_D^4 \sigma_u^2 / (N\delta)^2 \quad (11)$$

The proof of the proposition is in appendix I. When the uninformed agents observe prices, they use them for two reasons. On the one hand, the day traders want to invest when assets are relatively cheap. This is the **"price effect"**. On the other hand, statistical arbitrageurs infer from higher prices that the dividend state is more likely. This is the **"inference effect"**. Which one of the two effects dominates depends on the level of prices. When the price is close to the unconditional average  $\pi_0 \bar{D}$ , the inference is the strongest, and given a low enough coefficient of risk aversion, the demand of the uninformed is upwardsloping around that region.

### 3 The Continuous Time Infinite Horizon Model

The static model of the previous section delivered many results and intuitions that carry over to the dynamic model. The stochastic structure of the infinite horizon model presented here has the same structure as the static model. Dividends are paid out continuously. The informational advantage that the informed agents possess is the knowledge of the drift of the dividend process, that can be either high or low.

#### 3.1 The Set-up

The economy has finite horizon  $t \in [0, \infty]$ . There is one risky asset with price  $P$ , and a riskless bond with Price  $B$  with continuous return  $r$ . The risky asset pays out dividends continuously. The dividend is accumulated according to the following process:

$$\begin{aligned} dD_t &= \mu dt + \sigma_D dZ_D \\ D_0 &= 0 \end{aligned} \tag{12}$$

where  $Z$  is a Wiener Process. The informed investor learn  $\mu$  at time 0, which can take two possible values: 0 and  $\bar{D}$ . The uninformed statistical arbitrageurs do not know  $\mu$ . They have the following priors over the distribution of  $\mu$  :

$$\begin{aligned} \Pr[\mu = 0 \text{ at } t = 0] &= 1 - \pi_0 \\ \Pr[\mu = \bar{D} \text{ at } t = 0] &= \pi_0 \end{aligned} \tag{13}$$

There are  $N^I$  informed investors,  $N^A$  statistical arbitrageurs, and  $N^U$  noise traders in the economy. The information set of the informed investors is:

$$F_t^I = \{D_\tau, P_\tau, \mu, y_\tau : \tau \leq t\} \tag{14}$$

where  $y_\tau$  is the vector of demands of the informed and uninformed agents. The information set of the uninformed statistical arbitrageurs is:

$$F_t^A = \{D_\tau, P_\tau, y_\tau^{ns} : \tau \leq t\} \tag{15}$$

where  $y_\tau^{ns}$  is the net demand of the informed agents. The informed investors have exponential utility with coefficient 1. The uninformed are risk neutral. There are no borrowing constraints. The riskless bond has instantaneous return  $r$ .

The simplest specification of noise traders is that they believe that the true drift of the dividend process evolves according to an Ornstein-Uhlenbeck process with random initial condition:

$$\begin{aligned} du_t &= -\theta u_t dt + \sigma_u dZ_u \\ u_0 &\sim N(0, \sigma_u) \text{ independent of } Z_u \end{aligned} \tag{16}$$

where  $Z_u$  is a Brownian Motion independent of  $Z_D$ .

## 3.2 The Statistical arbitrageurs Filtering and the Equilibrium Price Process

### 3.2.1 Derivation of the Pricing Function

Assume for now that the observation of the dividend is the only source of information for the Statistical arbitrageurs. The uninformed also observe the demand of the informed, but assume temporarily that no information is revealed from the demand. The price is then given by the following no-arbitrage condition:

$$P_t = E \left[ \int_t^\infty \left( e^{-r(s-t)} D_s \right) ds | F_t^A \right] \quad (17)$$

By the law of total probabilities, the expectation can always be rewritten as:

$$\pi_t E \left[ \int_t^\infty \left( e^{-r(s-t)} D_s \right) ds | F_t^A, \mu = \bar{D} \right] + (1 - \pi_t) E \left[ \int_t^\infty \left( e^{-r(s-t)} D_s \right) ds | F_t^A, \mu = 0 \right]$$

where  $\pi_t = \Pr(\mu = \bar{D} | F_t^A)$ . Conditional on  $\mu$ , the process of  $D$  is a standard continuous time random walk with constant coefficients, so that the expectation is readily computed:

$$E \left[ \int_t^\infty \left( e^{-r(s-t)} D_s \right) ds | F_t^A, \mu \right] = \frac{D_t + \mu/r}{r}$$

The price is therefore:

$$P_t = \frac{D_t + \pi_t \bar{D}/r}{r} \quad (18)$$

The variables observable to the statistical arbitrageurs are the dividend, price, and demands of the other market participants. It will turn out that in equilibrium, the statistical arbitrageurs can back out level of noise plus the drift from the demand of the other agents, that will be denoted by  $x$ :

$$x_t \equiv \alpha \mu + u_t \quad (19)$$

$$\equiv \tilde{\mu} + u_t \quad (20)$$

where  $\alpha$  is a constant that is to be determined.

The process that  $x$  follows is therefore O-U:

$$dx_t = -\theta x_t + \sigma_u dZ_u$$

$$x_0 = \tilde{\mu} + u_0$$

From Küchler and Sørensen (1997), the likelihood function of  $x_0$  conditional on  $\mu$  is:

$$f_x(x_t|x^t, \mu) = \exp\left[\frac{\theta}{2\sigma_u^2}\left(-(x_t - \tilde{\mu})^2 + (x_0 - \tilde{\mu})^2 - \theta \int_0^t (x_s - \tilde{\mu})^2 ds + t/\sigma_u^2\right)\right]$$

where  $x^t$  denotes the history of  $x$  from 0 to  $t$ . The likelihood function of the dividends conditional on  $\mu$  is simply

$$f_D(D_t|D^t, \mu) = \frac{1}{\sqrt{2\pi\sigma_D^2 t}} \exp\left[-\frac{1}{2\sigma_D^2 t}(D_t - \mu)\right] \quad (21)$$

The Radon-Nikodym derivatives of changing the measures of  $D$  from  $\mu = \bar{D}$  to  $\mu = 0$  is:

$$\psi_t \equiv \frac{f_D(D_t|D^t, \mu = \bar{D})}{f_D(D_t|D^t, \mu = 0)} = \exp\left[\frac{\bar{D}}{\sigma_D^2}(D_t - t\bar{D}/2)\right] \quad (22)$$

Similarly, the Radon-Nikodym derivatives of changing the measures of  $D$  from  $\mu = \bar{D}$  to  $\mu = 0$  is:

$$\phi_t \equiv \frac{f_x(x_t|x^t, \mu = \bar{D}) f_x(x_0|x_0, \mu = \bar{D})}{f_x(x_t|x^t, \mu = 0) f_x(x_0|x_0, \mu = 0)} \quad (23)$$

$$= \exp\left[\frac{\theta\bar{D}}{\sigma_u^2}\left(x_t - x_0 + \theta \int_0^t x_s ds - \theta\bar{D}t/2\right) + \frac{\bar{D}}{\sigma_u^2}(x_0 - \bar{D})\right] \quad (24)$$

By Bayes rule, the independence of dividends and noise gives a simple formula for  $\pi_t$ :

$$\pi_t = \frac{\phi_t \psi_t}{\phi_t \psi_t + 1/\pi_0 - 1} \quad (25)$$

Taking time derivatives by application of Ito's Lemma gives:

$$d\pi_t = \pi_t(1 - \pi_t) \left[ \frac{d\psi_t}{\psi_t} + \frac{d\phi_t}{\phi_t} - \pi_t \left( \frac{\langle d\psi_t^2 \rangle}{\psi_t^2} + \frac{\langle d\phi_t^2 \rangle}{\phi_t^2} \right) \right] \quad (26)$$

where  $\langle d\phi_t^2 \rangle$  and  $\langle d\psi_t^2 \rangle$  denote the quadratic variations of  $\phi$  and  $\psi$  respectively. Differentiating  $\psi$  and  $\phi$  gives the diffusion equation for  $\pi_t$

$$d\pi_t = \pi_t(1 - \pi_t) \bar{D} \sigma_D^{-2} \left( (\mu - \pi_t \bar{D} (1 + \tilde{\theta})) dt + \sigma_D dZ_D + \tilde{\theta} dZ_u \right) \quad (27)$$



where  $\tilde{\theta} = \theta\sigma_D/\sigma_u$ . This diffusion can alternatively be obtained by extending Theorem 9.2. of Liptser and Shiryaev (2000) to a set-up with multiple brownian motions.

The price process is then:

$$dP_t = \eta_P(\pi_t) dt + \sigma_P(\pi_t) dZ \quad (28)$$

where

$$\begin{aligned} \eta_P(\pi_t) &= r^{-1} \left( \mu + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2} \left( \mu - \pi_t \bar{D} (1 + \tilde{\theta}) \right) \right) \\ \sigma_P(\pi_t) &= \left[ r^{-1} (1 + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2}) \sigma_D, r^{-2} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2} \tilde{\theta} \right] \\ dZ &= [dZ_D, dZ_u]' \end{aligned}$$

### 3.3 The Problem of the Informed and the Equilibrium Allocation

The informed investors have to solve the following program to maximize their expected utility:

$$\begin{aligned} &\max E \left[ \int_0^\infty e^{-\delta t - \alpha c_t} dt \right] \\ \text{s.t. } dW^I &= (rW + y^I (D - rP) - c) dt + y^I dP \end{aligned}$$

In order to optimize, the informed investors solve the Bellmann equation. Denoting the value function by  $J$ , the optimization problem can be written as:

$$\begin{aligned} &\max_{c, y^I} u(c_t^I) + E [dJ(W_t^I, D_t, \mu, t) | F_t^I] / dt \\ \text{subject to} & dW_t^I = (rW - c^I + y^I (\eta_P + D - rP)) dt + y^I \sigma_P dZ \\ \text{Transversality Condition} & \lim_{t \rightarrow \infty} E e^{-rt} J(W_t^I, D_t, \mu, t) = 0 \end{aligned} \quad (29)$$

The guess for the value function is:

$$J(W, D, \mu, t) = \exp(-rt - rW - z(\pi, D, \mu) - \ln r) \quad (30)$$

Note that  $P$  only depends on  $D$  and  $t$ , which allows the value function to be written in that form.

**Proposition 6 (Optimization)** *The Bellmann equation associated with the DP problem is:*

$$0 = rz - \frac{1}{2}\sigma_P^{-2}(\eta_P + D - rP)^2 + ((\eta_P + D - rP)\sigma_P^{-2}\sigma_{P\pi} - \eta_\pi)z'_\pi + \frac{1}{2}\sigma_\pi^2 z''_{\pi\pi}$$

*The optimal consumption and investment strategies are:*

$$\begin{aligned} y &= \frac{\eta_P + D - rP}{r\sigma_P^2} - \frac{\sigma_{P\pi}}{r\sigma_P^2}z_\pi \\ c &= rW + z \end{aligned} \tag{31}$$

The proof of this proposition is in the appendix and involves standard techniques.

### 3.3.1 Random Walk noise

The investors know  $\mu$  and observe  $D$ . Furthermore, they observe  $P$ , and they can compute  $\pi$ , which only depends on  $D$ , and not on  $u$  in the case  $\theta = 0$ . In this case, the change of measure from  $\mu = \bar{D}$  to  $\mu = 0$  simply becomes:

$$\phi_t = \exp \left[ \frac{\bar{D}}{\sigma_u^2} (x_0 - \bar{D}) \right] \tag{32}$$

Only the realization of  $x_0$  is informative due to the random walk nature of the noise. In the case of random walk noise, the optimization problem of the previous section simplifies considerably.

**Proposition 7** *With random walk noise,  $\theta = 0$ , the optimal investment and consumption are:*

$$\begin{aligned} y^I &= \frac{\mu - \pi\bar{D}}{\sigma_D^2} - r^{-1}\frac{\sigma_\pi}{\sigma_P}\tilde{z}'_\pi(\pi) \\ c^I &= rW + \tilde{z} + \frac{1}{2}\sigma_D^{-2}(\mu - \pi\bar{D})^2 \end{aligned}$$

where  $\tilde{z}$  solves the following ODE:

$$0 = r\tilde{z} + \frac{1}{2}\sigma_\pi^2\tilde{z}''_{\pi\pi} + \frac{1}{2}r^{-1}\sigma_\pi^2\sigma_D^{-2}\bar{D}$$

**Proposition 8** *Without noise traders, the demand is fully revealing  $\mu$ , and therefore, the only possible equilibrium is*

$$P = r^{-1}D + r^{-2}\mu$$

The result that the demand is fully revealing when no noise traders are present should not come as a surprise. As  $D$  is observable, and only  $\mu$  is private information, the demand of the informed agents must reveal  $\mu$ . The next section introduces noise traders.

In the case  $\theta = 0$ , noise traders demand follows the following process:

$$\begin{aligned} du &= \sigma_u dZ_u \\ u_0 &\sim N(0, \sigma_u^2) \end{aligned}$$

The aggregate demand of the noise traders and the informed traders is then

$$\mu N^I / \sigma_D^2 + u - (N^I + N^N) \left( \frac{\pi \bar{D}}{\sigma_D^2} + r^{-1} \frac{\sigma_\pi}{\sigma_P} \dot{z}'_\pi(\pi) \right)$$

The demand therefore reveals

$$x_t = u_t + \mu N^I / \sigma_D^2 \tag{33}$$

However, as  $\mu$  is not changing over time, the statistical arbitrageurs cannot learn anything from  $\mu + u$ , except at time 0. The presence of noise traders is therefore like a shift in initial beliefs of the statistical arbitrageurs,  $\pi_0$ , but does not affect the pricing function in any other way.

**Proposition 9** *When noise traders are present and  $\theta = 0$ , the equilibrium price is:*

$$P_t = r^{-1} (D_t + r^{-1} \bar{D} \pi_t)$$

### 3.3.2 Analysis of the prices and volume with random walk noise

[To be completed ...]

### 3.3.3 Myopic informed investors

When the informed investors are myopic, their demand does not have a hedging component. The amount invested in the risky asset is therefore simply the local price of risk divided by the coefficient of risk aversion:

$$y^I(\pi, D) = \frac{\eta_P + D_t - rP_t}{r\sigma_{PP}} \tag{34}$$

The total demand of noise traders plus informed traders can now be written as:

$$\frac{N^I \mu + N^u u + (N^I + N^u) (\varsigma_t \tilde{\theta} \pi \bar{D} - \bar{D} \pi_t)}{r [\sigma_D, \varsigma_t \tilde{\theta}] \sigma'_P}$$

where  $\varsigma_t \equiv \frac{r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2}}{(1 + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2})}$

As conjectured for the determination of the pricing function, it is therefore a function of  $x$ , so that the conjectured price function is indeed an equilibrium. {discuss multiplicity}.

### 3.3.4 Analysis history of volume and price behavior

[To be completed ...]

## 4 Appendix: Proofs

### Proofs for section 2

The following lemma 10 is needed to derive the REE equilibrium.

**Lemma 10** *Let  $z, x, y$  be random variables and  $h(\cdot)$  and  $g(\cdot)$  functions.  $g(\cdot)$  is continuously differentiable and weakly increasing. Then*

$$E[h(z)|g(x), \mu] = E[h(z)|x, \mu] \quad (35)$$

**Proof. Lemma 10:** Consider 3 random variables  $x, y, z$  and the monotone, differentiable functions  $g(\cdot)$  and  $h(\cdot)$ . Then:

$$E[h(z)|g(x), y] = E[h(z)|x, y] \quad (36)$$

Let's first consider the distribution  $f(z|x, y)$ . Per definition of the conditional expectation, we can write:

$$f(z|x, y) = \frac{f(z, x|y)}{f(x|y)} \quad (37)$$

Next, given the distribution  $f_{u,v|y}(u, v|y)$  with  $u = z$  and  $v = g(x)$ , what is the distribution  $f_{z,x|y}(z, x|y)$ ? This is a change of measure, and the following formula applies:

$$\begin{aligned} f_{z,x|y}(z, x|y) &= \left| \frac{\frac{du}{dz}}{\frac{dv}{dz}} \frac{\frac{du}{dx}}{\frac{dv}{dx}} \right| f_{u,v|y}(z, g(x)|y) \\ &= |g'(x)| f_{u,v|y}(z, g(x)|y) \end{aligned} \quad (38)$$

Similarly, consider the marginal density  $f_{x|y}(x|y)$ . Given the distribution  $f_{v|y}(v|y)$  with  $v = g(x)$ , what is the distribution  $f_{x|y}(z, x|y)$ ?

$$f_{x|y}(x|y) = |g'(x)| f_{v|y}(z, g(x)|y)$$

Therefore, we can rewrite:

$$\begin{aligned} f(z|x, y) &= \frac{f(z, x|y)}{f(x|y)} \\ &= \frac{|g'(x)| f_{u,v|y}(z, g(x)|y)}{|g'(x)| f_{v|y}(z, g(x)|y)} \\ &= \frac{f_{u,v|y}(z, g(x)|y)}{f_{v|y}(z, g(x)|y)} \\ &= f(z|g(x), y) \end{aligned} \quad (40)$$

Then the statement about the expectation follows immediately:

$$\begin{aligned}
& E[h(z)|g(x), y] && (41) \\
&= \int h(z)f(z|g(x), y)dz \\
&= \int h(z)f(z|x, y)dz \\
&= E[h(z)|x, y]
\end{aligned}$$

■

**Proof. Proposition 4**

Start by guessing that the equilibrium pricing function is a monoton function of  $x$  :

$$p = p(x)$$

The arbitrageurs condition their demand on the observation of  $p$ . Applying Lemma 10 shows that

$$\begin{aligned}
E [D|p(x)] &= \sum_{\mu} E [D|p(x), \mu] \Pr(\mu) \\
&= \sum_{\mu} E [D|x, \mu] \Pr(\mu) \\
&= E [D|x]
\end{aligned}$$

The computation of  $E [D|x]$  is the same as in 7, so that the no-arbitrage condition pins down:

$$p = E [D|x] = \bar{D}\pi(x)$$

The last thing to check is that the demand of the arbitrageurs plus the demand of the noise traders reveals no more information than  $x$ . From the optimization of the informed investors,

$$\begin{aligned}
y^I(p, \mu) &= \frac{E[D|\mu] - p}{\alpha\sigma_D^2} \\
&= \frac{1}{\alpha\sigma_{D|s}^2} \left[ \beta^I s + (1 - \beta^I)\mu - p \right]
\end{aligned}$$

exactly as in ???. But then the total demand of the informed investors plus noise traders reveals the same information as the net demand in the STE (see 2). Markets clear, as the asset holdings of the arbitrageurs is undetermined once the price is pinned down at the expected value conditional on their information set, which is  $x$ . ■

**Proof. Proposition 5:**

In equilibrium, the position of the hedge fund depends only on  $x$ , so that the price only effects the position of the hedge fund indirectly:

$$\frac{dy^A}{dp} = \frac{\partial y^A}{\partial x} \frac{\partial x}{\partial p} > 0 \quad (42)$$

Using Proposition 2 yields

$$\frac{\partial y^A}{\partial x} = -\frac{B\alpha\sigma_\epsilon^2\sigma_u^2}{(1-\delta)\sigma_D^2\delta N^2} + \frac{F\delta(\sigma_D^2 + \sigma_\epsilon^2)\sigma_u^2}{(1-\delta)\sigma_D^2\alpha\sigma_\epsilon^2\sigma_u^2} \bar{D}\pi'_x(x)$$

And, for the price:

$$\frac{\partial p}{\partial x} = B + F\bar{D}\pi'_x(x) \quad (43)$$

So that the condition for an upwardsloping demand becomes:

$$\begin{aligned} \frac{dy^A}{dp} &= \frac{-\frac{B\alpha\sigma_\epsilon^2\sigma_u^2}{(1-\delta)\sigma_D^2\delta N^2} + \frac{F\delta(\sigma_D^2 + \sigma_\epsilon^2)\sigma_u^2}{(1-\delta)\sigma_D^2\alpha\sigma_\epsilon^2\sigma_u^2} \bar{D}\pi'_x(x)}{B + F\bar{D}\pi'_x(x)} > 0 \quad (44) \\ \bar{D}\pi'_x(x) &> \frac{\sigma_D^2}{(\sigma_D^2 + \sigma_\epsilon^2)} \end{aligned}$$

What is  $\pi'_x$ ? Recall first:

$$\bar{D}\pi_x = \frac{\bar{D}}{1 + \frac{1-\pi_0}{\pi_0} \exp((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x))}$$

$$\begin{aligned} \bar{D}\pi'_x &= \bar{D}(l\bar{D}/\sigma_x^2) \frac{1}{\left(1 + \frac{1-\pi_0}{\pi_0} \exp((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x))\right) \left(1 + \frac{1-\pi_0}{\pi_0} \exp -((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x))\right)} \\ &= \frac{l\bar{D}^2/\sigma_x^2}{1 + \left(\frac{1-\pi_0}{\pi_0}\right)^2 + \frac{1-\pi_0}{\pi_0} \exp((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x)) + \frac{1-\pi_0}{\pi_0} \exp -((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x))} \end{aligned}$$

Where is the slope maximal? The FOC is:

$$\pi''_x = 0$$

$$\pi''_x = \frac{l\bar{D}^2/\sigma_x^2 \left( \left( \frac{1-\pi_0}{\pi_0} \exp((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x)) - \frac{1-\pi_0}{\pi_0} \exp -((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x)) \right) \right)}{1 + \left(\frac{1-\pi_0}{\pi_0}\right)^2 + \frac{1-\pi_0}{\pi_0} \exp((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x)) + \frac{1-\pi_0}{\pi_0} \exp -((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x))} = 0$$

which is equivalent to:

$$\exp((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x)) = \exp -((l\bar{D}/\sigma_x^2)(l\bar{D}/2 - x))$$

$$l\bar{D}/2 = x$$

Replacing into the condition:

$$\bar{D}\pi'_x|_{s-Cu=\bar{D}/2} = \frac{l\bar{D}^2/\sigma_x^2}{\left(1 + \frac{1-\pi_0}{\pi_0}\right)^2}$$

Replacing back yields:

$$\frac{\bar{D}^2}{\left(1 + \frac{1-\pi_0}{\pi_0}\right)^2} > \sigma_D^4 \alpha^2 \sigma_u^2 / (N\delta)^2$$

■



## Proofs for section 3

### Proof. Proposition 6

Let's start by rewriting the differential term in the Bellman equation by applying Itô's Lemma:

$$\begin{aligned}
& \frac{E [dJ(W_t^I, D_t, \pi_t, \mu, t) | F_t^I]}{dt} \\
= & \frac{1}{dt} \{ u(c_t) + J_t + J_W E [dW^I | F_t^I] + J_\pi E [d\pi | F_t^I] + J_D E [dD | F_t^I] \\
& + \frac{1}{2} J_{WW} E [dW^{I2} | F_t^I] + \frac{1}{2} J_{\pi\pi} E [d\pi^2 | F_t^I] + \frac{1}{2} J_{DD} E [dD^2 | F_t^I] \\
& + J_{W\pi} E [dW^I d\pi | F_t^I] + J_{WD} E [dW^I dD | F_t^I] \} \\
d\pi_t = & \pi_t (1 - \pi_t) \bar{D} \sigma_D^{-2} \left( (\mu - \pi_t \bar{D} (1 + \tilde{\theta})) dt + \sigma_D dZ_D + \tilde{\theta} dZ_u \right) \\
dW^I = & (rW + y^I (D - rP) - c) dt + y^I dP \\
dP_t = & \eta_P (\pi_y) dt + \sigma_P (\pi_t) dZ \\
\eta_P (\pi_t) = & r^{-1} \left( \mu + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2} \left( \mu - \pi_t \bar{D} (1 + \tilde{\theta}) \right) \right) \\
\sigma_P (\pi_t) = & \left[ r^{-1} (1 + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2}) \sigma_D, r^{-2} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2} \tilde{\theta} \right] \\
dZ = & [dZ_u, dZ_D]' \\
0 = & u(c) + J_t + J_W (rW + y^I (\eta^P + D - rP) - c) + J_\pi \eta^\pi + J_D \mu \\
& + \frac{1}{2} J_{WW} y^2 \sigma_P^2 + \frac{1}{2} J_{\pi\pi} \sigma_\pi^2 + \frac{1}{2} J_{DD} \sigma_D^2 \\
& + J_{W\pi} y \sigma_{P\pi} + J_{WD} y \sigma_{PD} \\
0 = & u(c) + J_t + J_W \left( rW + y^I (D - rP) - c + yr^{-1} \left( \mu + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2} \left( \mu - \pi_t \bar{D} (1 + \tilde{\theta}) \right) \right) \right) \\
& + J_\pi \pi_t (1 - \pi_t) \bar{D} \sigma_D^{-2} \left( (\mu - \pi_t \bar{D} (1 + \tilde{\theta})) \right) + J_D \mu \\
& + \frac{1}{2} J_{WW} y^2 \left[ (r^{-1} + r^{-2} \pi (1 - \pi) \bar{D}^2 \sigma_D^{-2})^2 \sigma_D^2 + r^{-4} \pi^2 (1 - \pi)^2 \bar{D}^4 \sigma_D^{-4} \tilde{\theta}^2 \right] \\
& + \frac{1}{2} J_{\pi\pi} E [d\pi^2 | F_t^I] \pi^2 (1 - \pi)^2 \bar{D}^2 \sigma_D^{-4} (\sigma_D^2 + \tilde{\theta}^2) + \frac{1}{2} J_{DD} \sigma_D^2 \\
& + J_{W\pi} y \pi_t (1 - \pi_t) \bar{D} \sigma_D^{-2} \left( r^{-1} (1 + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2}) \sigma_D^2 + r^{-2} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2} \tilde{\theta}^2 \right) \\
& + J_{WD} y r^{-1} (1 + r^{-1} \pi_t (1 - \pi_t) \bar{D}^2 \sigma_D^{-2}) \sigma_D^2
\end{aligned}$$

Note that  $\mu$  is discrete, so that it the Bellman equation does not have terms in  $J_\mu$  or  $J_{\mu\cdot}$ .

Replacing the diffusion of  $\pi$  and  $D$  into the Bellman equation gives:

$$\begin{aligned} 0 &= u(c) + J_t + J_W(rW + y(\eta_P + D - rP) - c) + J_\pi\eta_\pi + J_D\mu \\ &\quad + \frac{1}{2}J_{WW}y^2\sigma_P^2 + \frac{1}{2}J_{\pi\pi}\sigma_\pi^2 + \frac{1}{2}J_{DD}\sigma_D^2 \\ &\quad + J_{W\pi}y\sigma_{P\pi} + J_{WD}y\sigma_{PD} + J_{\pi D}\sigma_{\pi D} \end{aligned}$$

The FOCs are:

$$y_t = -\frac{\eta_P + D - rP}{\sigma_P^2} \frac{J_W}{J_{WW}} - \frac{\sigma_{P\pi}}{\sigma_P^2} \frac{J_{W\pi}}{J_{WW}} - \frac{\sigma_{PD}}{\sigma_P^2} \frac{J_{WD}}{J_{WW}} \quad (45)$$

$$u'(c) = J_W \quad (46)$$

Let's guess the value function:

$$J = \exp(-rt - rW - z - \ln r) \quad (47)$$

So that:

$$\begin{aligned} y &= \frac{\eta_P + D - rP}{r\sigma_P^2} - \frac{\sigma_{P\pi}}{r\sigma_P^2} z_\pi - \frac{\sigma_{PD}}{r\sigma_P^2} z_D \\ c &= rW + z \end{aligned} \quad (48)$$

Note that utility becomes with optimal consumption  $u(c) = e^{-rt-c} = e^{-rt-rW-z} = rJ$ . Replacing this, optimal consumption and the value function gives:

$$\begin{aligned} 0 &= rz - ry(\eta_P + D - rP) - z_\pi\eta_\pi - z_D\mu \\ &\quad + \frac{1}{2}r^2y^2\sigma_P^2 + \frac{1}{2}z_{\pi\pi}\sigma_\pi^2 + \frac{1}{2}z_{DD}\sigma_D^2 \\ &\quad + z_\pi ry\sigma_{P\pi} + z_D ry\sigma_{PD} + z_{\pi D}\sigma_{\pi D} \end{aligned}$$

Replace  $y$  into the Bellman equation. Then:

$$\begin{aligned} 0 &= rz - z_\pi\eta_\pi - z_D\mu \\ &\quad + \frac{1}{2}z_{\pi\pi}\sigma_\pi^2 + \frac{1}{2}z_{DD}\sigma_D^2 + z_{\pi D}\sigma_{\pi D} \\ &\quad - \frac{1}{2}r^2y^2\sigma_P^2 \end{aligned}$$

In order to replace for  $y$ , do the following substitutions:

$$\begin{aligned} yr\sigma_P^2 &= m - \sigma_{P\pi}z_\pi - \sigma_{PD}z_D \\ m &= \eta_P + D - rP \\ -\frac{1}{2}y^2r^2\sigma_P^2 &= -\frac{1}{2}\sigma_P^{-2}(m - \sigma_{P\pi}z_\pi - \sigma_{PD}z_D)^2 \\ &= -\frac{1}{2}\sigma_P^{-2}m^2 - \frac{1}{2}\sigma_\pi^2z_\pi^2 - \frac{1}{2}\sigma_D^2z_D^2 + m\sigma_P^{-2}\sigma_{P\pi}z_\pi + m\sigma_P^{-2}\sigma_{PD}z_D - \sigma_{\pi D}z_Dz_\pi \end{aligned}$$

Then the Bellman equation reduces to:

$$\begin{aligned}
0 &= rz - \frac{1}{2}\sigma_P^{-2}m^2 \\
&\quad + (m\sigma_P^{-2}\sigma_{P\pi} - \eta_\pi)z_\pi + (m\sigma_P^{-2}\sigma_{PD} - \mu)z_D \\
&\quad + \frac{1}{2}\sigma_\pi^2z_{\pi\pi} + \frac{1}{2}\sigma_D^2z_{DD} \\
&\quad + \sigma_{\pi D}(z_{\pi D} - z_Dz_\pi)
\end{aligned}$$

The equation shows that one state variable is sufficient for optimization. Choosing  $\pi$  as state reduces the equation to:

$$0 = rz - \frac{1}{2}\sigma_P^{-2}m^2 + (m\sigma_P^{-2}\sigma_{P\pi} - \eta_\pi)z_\pi + \frac{1}{2}\sigma_\pi^2z_{\pi\pi}$$

■

**Proof. Proposition 7**

In the cas that  $\theta = 0$ , the price process and the updating processes are characterized by the following drifts and variances:

$$\begin{aligned}
\eta_P &= r^{-1}\mu + r^{-2}\pi_t(1 - \pi_t)\bar{D}^2\sigma_D^{-2}(\mu - \pi\bar{D}) \\
m &= \eta_P + D - rP = \eta_P - r^{-1}\pi\bar{D} = (r^{-1} + r^{-2}\pi_t(1 - \pi_t)\bar{D}^2\sigma_D^{-2})(\mu - \pi\bar{D}) \\
\sigma_P &= (r^{-1} + r^{-2}\pi_t(1 - \pi_t)\bar{D}^2\sigma_D^{-2})\sigma_D \\
\eta_\pi &= \pi_t(1 - \pi_t)\bar{D}\sigma_D^{-2}(\mu - \pi\bar{D}) \\
\sigma_\pi &= \pi_t(1 - \pi_t)\bar{D}\sigma_D^{-2}\sigma_D
\end{aligned}$$

Under these processes, note the following:

$$(m\sigma_P^{-2}\sigma_{P\pi} - \eta_\pi) = 0$$

The Bellman equation is therefore:

$$0 = rz - \frac{1}{2}\sigma_D^{-2}(\mu - \pi\bar{D})^2 + \frac{1}{2}\sigma_\pi^2z_{\pi\pi}$$

Now use the following substitution:

$$r\tilde{z} = rz - \frac{1}{2}\sigma_D^{-2}(\mu - \pi\bar{D})^2$$

Then the Bellman equation becomes:

$$0 = r\tilde{z} + \frac{1}{2}\sigma_\pi^2\tilde{z}_{\pi\pi} + \frac{1}{2}r^{-1}\sigma_\pi^2\sigma_D^{-2}\bar{D}$$

The solution is:

$$z(x) = \rho_1 b \frac{-\int x^{\frac{1}{2}\frac{\rho_1+a}{a}}(1-\pi)^{\frac{1}{2}\frac{-\rho_1+a}{a}} dx x^{\frac{1}{2}\frac{-\rho_1+a}{a}}(1-\pi)^{\frac{1}{2}\frac{\rho_1+a}{a}} + \int x^{\frac{1}{2}\frac{-\rho_1+a}{a}}(1-\pi)^{\frac{1}{2}\frac{\rho_1+a}{a}} dx x^{\frac{1}{2}\frac{\rho_1+a}{a}}(1-\pi)^{\frac{1}{2}\frac{-\rho_1+a}{a}}}{a(a-4)}$$

where  $\rho_1$  is a root of

$$\hat{Z}^2 - a^2 + 4a = 0$$

and

$$\begin{aligned} a &= .5\bar{D}^2\sigma_D^{-2} \\ b &= .5r^{-2}\bar{D}^3\sigma_D^{-4} \end{aligned}$$

The optimal demand and consumption simplifies as well:

$$\begin{aligned} y &= \frac{\mu - \pi\bar{D}}{\sigma_D^2} - r^{-1}\frac{\sigma_\pi}{\sigma_P}\hat{z}'_\pi(\pi) \\ c &= rW + z \end{aligned} \tag{49}$$

■

## 5 Literature

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Liptser, R. S. and A. N. Shiryaev (2000) "Statistics of Random Processes", Vol. I & II, 2nd Edition, Berlin: Springer Verlag

[To be completed ...]

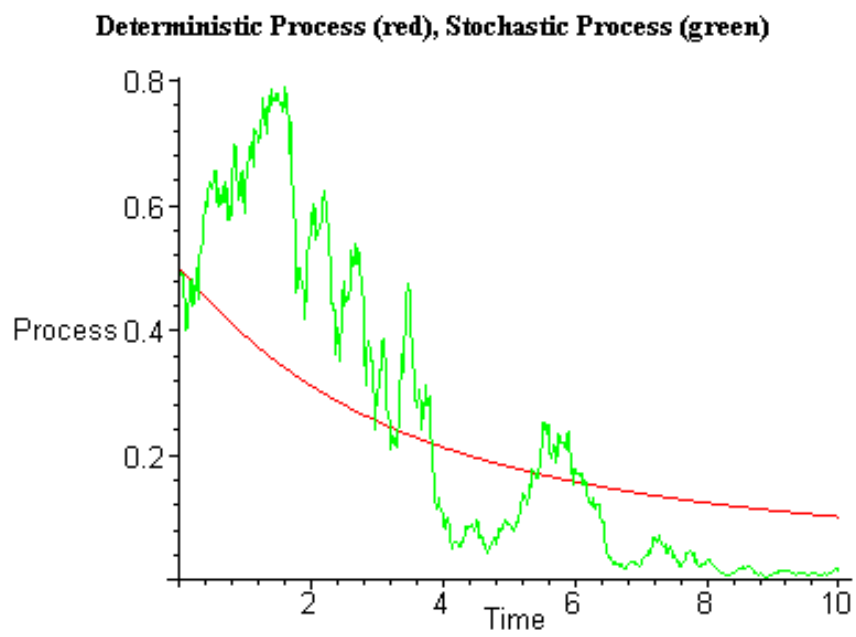


Figure 2: Simulation 1

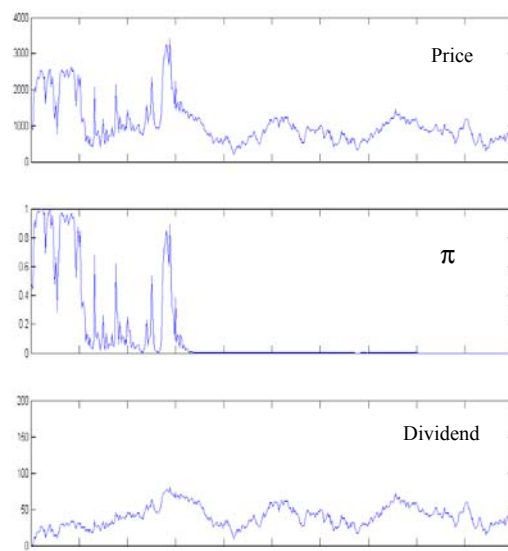


Figure 3: Simulation 2