The Survival and Price Impact of Irrational Traders

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Motivation

- Large empirical literature devoted to asset pricing anomalies; Many anomalies are easily explained by behavioral models/assumptions. For example:
  - Under/Over-reaction $\rightarrow$ Momentum
  - Loss Aversion $\rightarrow$ Equity Premium
  - Mental Categories $\rightarrow$ Co-movement
- **Theoretical Issue:** How can behavioral or irrational traders bring about anomalies in a robust marketplace? In particular, why do the anomalies survive exploitation?

Conceptual Issues in the Literature

- Evolutionary Selection (Friedman (1953))
  vs. Limited Arbitrage (Shleifer and Vishny (1997))

- Competitive Markets (Sandroni (2000))
  vs. Optimal Portfolio Growth (De Long, Shleifer, Summers, and Waldman (1991) – DSSW)

- Irrational Traders: Long Run Survival vs. Long Run Impact

Preview of Results

- Irrational traders can survive and dominate in a competitive market, even with no limits on arbitrage
  - Intuition: Wrong beliefs do not necessarily generate arbitrage opportunities, and an irrational portfolio might be closer to the optimal growth portfolio.

- Irrational traders can have a large price impact even when their wealth is very small
  - Intuition: Small irrational traders can change stock price moments and induce a large hedging demand in rational traders. This indirect effect may change stock prices.

The Model – Lucas Exchange Economy

- Single share of stock that is a claim on a dividend process

  $$dD_t = D_t\left(\mu dt + \sigma dB_t\right)$$

  and a risk free asset in zero net supply

- Terminal consumption only (time T) – focus on portfolio effects rather than the savings decision.

- Two traders, equal endowments

- CRRA preferences: $U(C) = \frac{1}{1-\gamma}C_T^{1-\gamma}, \quad \gamma \geq 1$

  – Direct market impact (through demand) scales with wealth.

The Model – Irrationality

- Rational trader believes P, maximizes $\mathsf{E}^P\left[\ \frac{1}{1-\gamma}C_{R,T}^{1-\gamma}\ \right]$

$$dD_t^P = D_t\left[(\mu)\,dt + \sigma dB_t^P\right]$$

- Irrational trader believes Q, maximizes $\mathsf{E}^Q\left[\ \frac{1}{1-\gamma}C_{N,T}^{1-\gamma}\ \right]$

$$dD_t^Q = D_t\left[\left(\mu + \sigma^2\eta\right)dt + \sigma dB_t^Q\right]$$

- η parametrizes irrationality: $\eta > 0$ is optimistic, $\eta < 0$ pessimistic; no learning

- $\xi_t = (dQ/dP)_t$, so that

$$\mathsf{E}^Q\left[\ \frac{1}{1-\gamma}C_{N,T}^{1-\gamma}\ \right] = \mathsf{E}^P\left[\ \xi_T\frac{1}{1-\gamma}C_{N,T}^{1-\gamma}\ \right]$$

State Dependent Utility

- Different Beliefs and State Dependent Utility:

$$\sum_{s\in\{states\}} q_s U(C_s) = \sum_{s\in\{states\}} p_s \frac{q_s}{p_s} U(C_s)$$

- $\frac{q_s}{p_s}$ is the ratio of probabilities (analogous to $\xi_t = (dQ/dP)_t$).
  State dependent utility: $\frac{q_s}{p_s}U(C_s)$ – it depends on s.

- Irrational trader has utility $\frac{1}{1-\gamma}C_{N,T}^{1-\gamma}$ under Q,
  and utility $\xi_T \frac{1}{1-\gamma}C_{N,T}^{1-\gamma}$ under P.
  We can write down a utility function for N under P, so there is no riskless arbitrage.

Solution Method 1

- Complete Markets $\rightarrow$ Central Planner's Problem: maximize a weighted sum of utilities state by state

$$\max \left[\frac{1}{1-\gamma} C_{R,T}^{1-\gamma} + b\,\xi_T \frac{1}{1-\gamma} C_{N,T}^{1-\gamma} \right]$$

$$\text{s.t.} \quad C_{R,T} + C_{N,T} = D_T$$

- Solving yields pareto optimal consumption:

$$C_{R,T} = \frac{1}{1 + (b\,\xi_T)^{1/\gamma}} D_T \quad \text{and} \quad C_{N,T} = \frac{(b\,\xi_T)^{1/\gamma}}{1 + (b\,\xi_T)^{1/\gamma}} D_T$$

- And a pricing formula where we have normalized R_f=0:

$$P_t(Z_T) = \frac{\mathsf{E_t}\left[\left(1 + (b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma} Z_T\right]}{\mathsf{E_t}\left[\left(1 + (b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}\right]}$$

Solution Method 2

- Equal endowments $\Rightarrow W_{R,0} = W_{N,0}$. Using the pricing formula for consumption, solve for the social weights.

- Solving, $b = e^{(\gamma-1)\eta\sigma^2 T}$ goes to infinity with the length of the economy. Contrast to $\xi_T = e^{-\frac{1}{2}\eta^2\sigma^2 T + \eta\sigma B_T}$, which goes to zero. We only see $b\,\xi_T$ together (as in consumption).

- We now know all quantities and prices – it is just a matter of substituting and evaluating

- Limit of Economies ($\lim_{T\to\infty}$) for asymptotic solutions.

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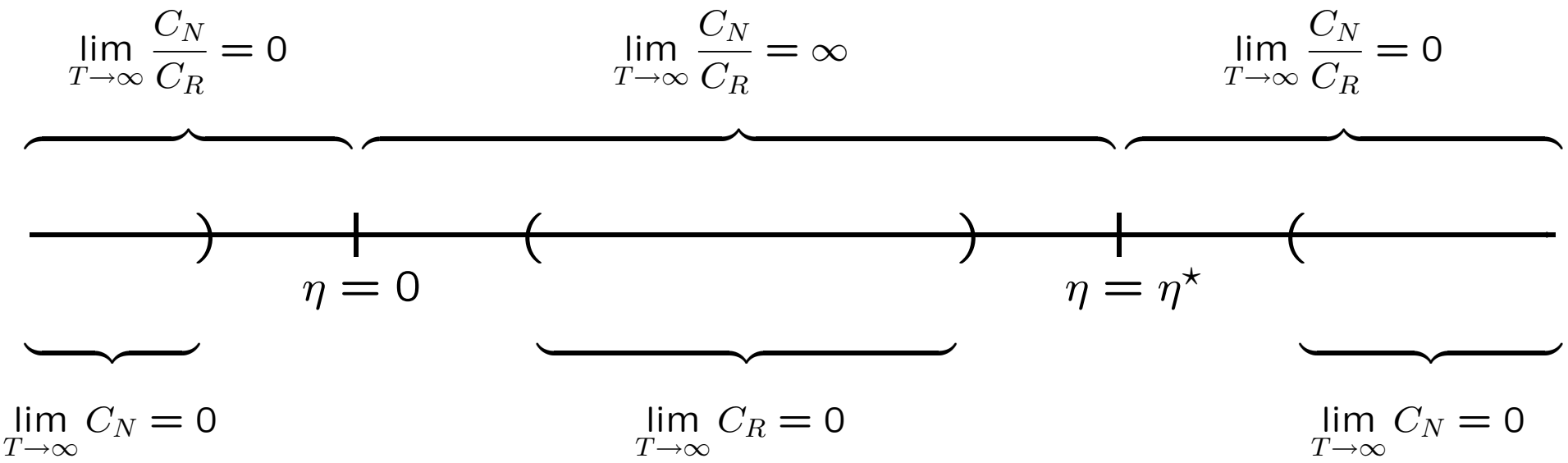
Survival 1

- Relative survival of R: if $\lim_{T\to\infty} \frac{C_{R,T}}{C_{N,T}}$ is not zero.

- Absolute survival of R: if $\lim_{T\to\infty} C_{R,T}$ is not zero.
  Absolute survival is weaker than relative survival.

- Compute limits from consumption:

$$C_{R,T} = \frac{1}{1 + (b\,\xi_T)^{1/\gamma}} D_T \quad \text{and} \quad C_{N,T} = \frac{(b\,\xi_T)^{1/\gamma}}{1 + (b\,\xi_T)^{1/\gamma}} D_T$$


$\lim_{T\to\infty} \frac{C_N}{C_R} = 0$
$\lim_{T\to\infty} \frac{C_N}{C_R} = \infty$
$\lim_{T\to\infty} \frac{C_N}{C_R} = 0$
$\eta = 0$
$\eta = \eta^\star$
$\lim_{T\to\infty} C_N = 0$
$\lim_{T\to\infty} C_R = 0$
$\lim_{T\to\infty} C_N = 0$

Survival 3

- No Coexistence – the two probability measures diverge completely. Differs from both limited arbitrage and partial equilibrium results.

- Symmetric Outcome – either trader can end with zero consumption or a zero share of consumption. The game is not "Can the rational trader drive the irrational trader out?"

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Solution Method 3

- Very difficult to evaluate wealth and stock price integrals. Use asymptotics:

$$X_T \sim Y_T \quad \Leftrightarrow \quad \lim_{T\to\infty} \frac{X_T}{Y_T} = 1$$

- Regime shifts: set $t = \lambda T$, take limit as $T \to \infty$.

  – So we look across a sequence of economies as the length of the economies goes to infinity. λ is the fraction of economy's duration that has already passed.

- Compare volatilities to obtain portfolios. Everything besides portfolio breakdown would be the same in a discrete time version of the model.

Stock and Wealth Processes $-\ \eta < 0$

$$S_t \sim \begin{cases} \overbrace{e^{\left(\frac{\mu}{\sigma^2}-\gamma+\eta\right)\sigma^2 T+\frac{1}{2}\left[(2\gamma-1)-2\gamma\eta+\eta^2\right]\sigma^2 t+(1-\eta)\sigma B_t}}^{\eta \text{ has an effect here}} & 0<\lambda<\lambda_S \\ \underbrace{e^{\left(\frac{\mu}{\sigma^2}-\gamma\right)\sigma^2 T+\frac{1}{2}(2\gamma-1)\sigma^2 t+\sigma B_t}}_{\text{partial equilibrium stock price}} & \lambda_S<\lambda\leq 1 \end{cases}$$

$$\frac{W_{R,t}}{W_{N,t}} \sim \begin{cases} \approx \overbrace{\infty \text{ for large } t=\lambda T}^{} \\ e^{\frac{1}{2}\eta\sigma^2[\eta-\eta^\star]\,t-\eta\sigma B_t} & 0<\lambda<\lambda_N \\ \underbrace{e^{\frac{1}{2}\frac{\eta^2}{\gamma^2}(\gamma-1)\sigma^2 T+\frac{1}{2}\frac{\sigma^2}{\gamma^2}\eta[\eta-\gamma\eta^\star]\,t-\frac{\eta}{\gamma}\sigma B_t}}_{\text{partial equilibrium wealth growth}} & \lambda_N<\lambda\leq 1 \end{cases}$$

λ_S, λ_N ordering ambiguous; $\eta^\star = 2\,(\gamma-1)$

Portfolio Policies $-\ \eta < 0$

$$\theta_{R,t} \sim \begin{cases} \underset{\text{(myopic)}}{\frac{\gamma-\eta}{\gamma(1-\eta)}} - \underset{\text{(hedging)}}{\frac{(\gamma-1)\eta}{\gamma(1-\eta)}} = \underset{\text{(total)}}{1} & 0 < \lambda < \lambda_S \\ 1 + 0 = 1 & \lambda_S < \lambda \leq 1 \end{cases}$$

$$\theta_{N,t} \sim \begin{cases} \underset{\text{(myopic)}}{\frac{1}{1-\eta}} + \underset{\text{(hedging)}}{0} = \underset{\text{(total)}}{\frac{1}{1-\eta}} & 0 < \lambda < \min(\lambda_N, \lambda_S) \\ 1 + \frac{\eta}{\gamma} + 0 = 1 + \frac{\eta}{\gamma} & \max(\lambda_N, \lambda_S) < \lambda \leq 1 \end{cases}$$

λ_S, λ_N ordering ambiguous

Stock and Wealth Processes 3

- Small traders can have a large impact on prices, and this is coincident with a large hedging demand for the large trader

- The story is one of indirect effect: Small trader (rational or irrational) changes the stock prices moments over time; induces a large hedging demand from the other trader; creates a large price impact.

- Partial equilibrium limits are reached, but at different times for the stock and the wealths. P.E. process is unreliable.

- Decreasing wealth/size does not mean decreasing influence

Intuition: Stock Price Moments

- Example: $\eta < 0$, λ small, $t = \lambda T$,

$$S_t = W_{R,t} + W_{N,t} \approx e^{\left(\frac{\mu}{\sigma^2} - \gamma + \eta\right)\sigma^2 T + \frac{1}{2}\left[(2\gamma - 1) - 2\gamma\eta + \eta^2\right]\sigma^2 t + (1-\eta)\sigma B_t}$$

$$* \left[1 + e^{\frac{1}{2}\eta\sigma^2[\eta^\star - \eta]\, t + \eta\sigma B_t}\right]$$

- $e^{\frac{1}{2}\eta\sigma^2[\eta^\star - \eta]\, t + \eta\sigma B_t} \approx 0$ for large $t = \lambda T$ – asymptotically zero.

- The moments of the stock price change in t for any finite T.

- For very negative B_t (extreme states), price process changes.

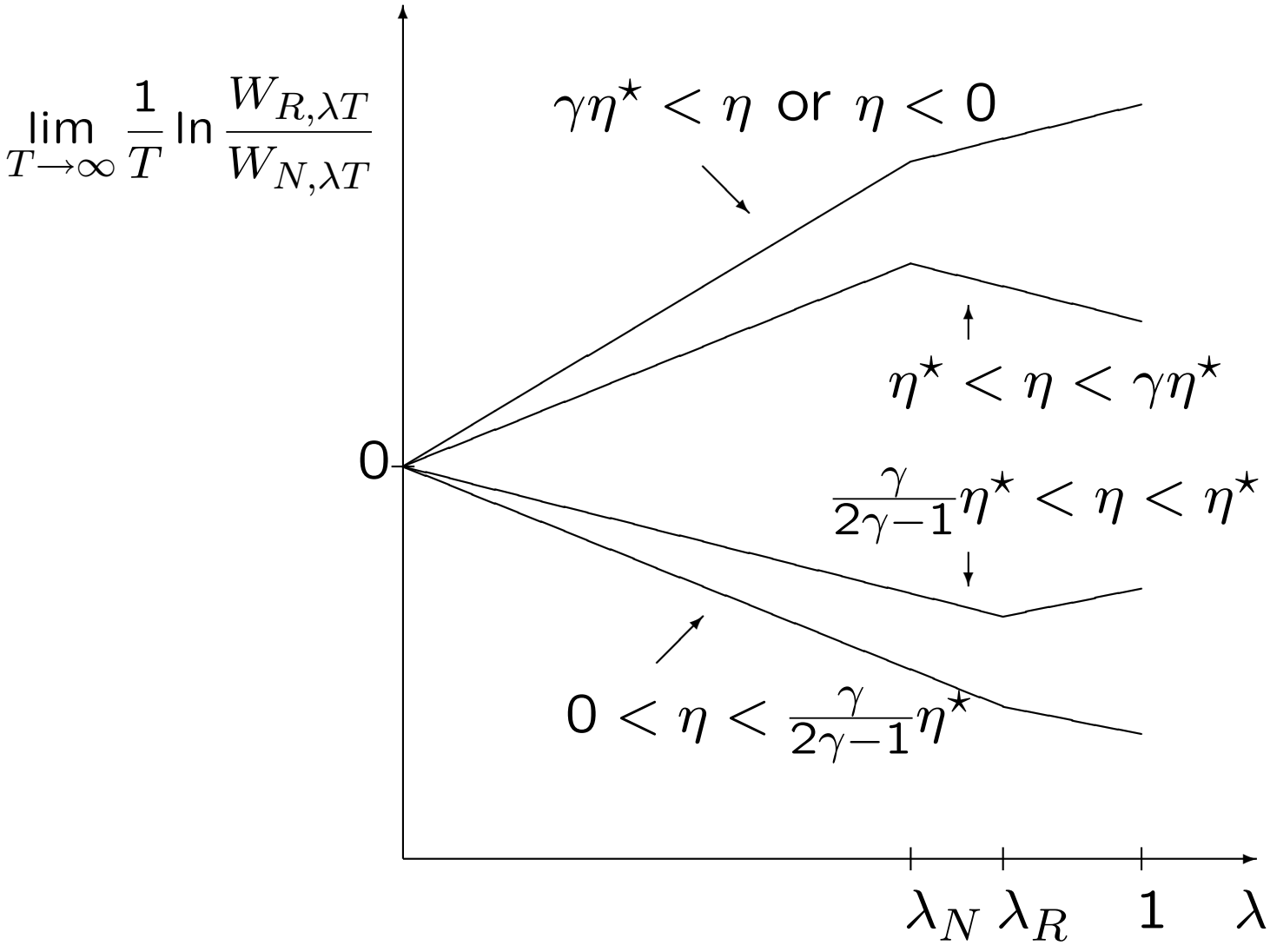
Intuition: Key Results

- **Key Result:** The changing moments are relevant because in extreme states the price follows a very different process

- **Key Result:** If traders have heterogeneous beliefs and one is losing wealth over time, he will affect the stock price moments. So any survival arguments hinge on the price impact of the outgoing trader.

- The larger impact on an economy from a small trader may come indirectly: by what actions does he or she induce in larger traders.

Asymptotic Wealth Growth 1



Asymptotic Wealth Growth 2

- Traders who grow faster near the end of the economy do not necessarily survive at the end

- Increasing wealth does not mean increasing influence, or even increasing survival prospects.

Welfare

- Rational trader must increase his expected utility from the autarky case (certainty of $\frac{1}{2}D_T$). So rational trader's expected utility is high even when $C_{R,T} \overset{a.s.}{\to} 0$.

- In fact, $\mathsf{E}^Q\left[\ \frac{1}{1-\gamma}C_{N,T}^{1-\gamma}\ \right] = \mathsf{E}^P\left[\ \frac{1}{1-\gamma}C_{R,T}^{1-\gamma}\ \right] = \mathsf{E}^P\left[\ \frac{1}{1-\gamma}D_T^{1-\gamma}\ \right]$. Consequence of diverging probability measures.

- Certainty equivalent of the rational trader for the consumption bundle $C_{N,T}$ is zero. Survival and utility are very different.

Conclusions

- Long-run survival and long-run impact are very different: diminishing wealth does not mean diminishing influence. Nor does increasing wealth mean increasing influence.

- Competitive markets do not necessarily eliminate irrational traders; irrationality may even be protected