Notes on
1.63 Advanced Environmental Fluid Mechanics
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chapter

7.2 Vorticity in inviscid rotating fluids
— Taylor -Proudman theorem

Ignoring viscosity, let \( \zeta = \nabla \times \mathbf{q} \) and use the identity

\[ \zeta \times \mathbf{q} = \mathbf{q} \cdot \nabla \mathbf{q} - \nabla \left| \mathbf{q} \right|^2 / 2 \]

The momentum equation can be written:

\[ \frac{\partial \mathbf{q}}{\partial t} + \zeta \times \mathbf{q} + 2 \mathbf{\Omega} \times \mathbf{q} = -\frac{\nabla p}{\rho} + \nabla \left( \phi - \frac{|\mathbf{q}|^2}{2} \right) \]  \hspace{1cm} (7.2.1)

Taking the curl of the above equation:

\[ \frac{\partial \zeta}{\partial t} + \nabla \times \left( (2 \mathbf{\Omega} + \zeta) \times \mathbf{q} \right) = \frac{\nabla \rho \times \nabla p}{\rho^2} \]

Using the identity

\[ \nabla \times \left( \mathbf{A} \times \mathbf{B} \right) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{A} \]

we get

\[ \nabla \times \left( (2 \mathbf{\Omega} + \zeta) \times \mathbf{q} \right) = -\mathbf{q} \nabla \cdot (2 \mathbf{\Omega} + \zeta) + (2 \mathbf{\Omega} + \zeta) \nabla \cdot \mathbf{q} + \mathbf{q} \cdot \nabla (2 \mathbf{\Omega} + \zeta) - (2 \mathbf{\Omega} + \zeta) \cdot \nabla \mathbf{q} \]

The first term on the right vanishes because \( \mathbf{\Omega} = \) constant and the divergence of curl is zero; the second vanishes for incompressible fluids. Let \( \zeta_a = \zeta + 2 \mathbf{\Omega} = \) absolute vorticity

\[ \frac{D \zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \mathbf{q} \cdot \nabla \zeta = \zeta_a \cdot \nabla \mathbf{q} + \frac{\nabla \rho \times \nabla p}{\rho^2} \]  \hspace{1cm} (7.2.2)

If the Rossby number is small (slow flow)

\[ \epsilon = \text{Rossby No.} = \frac{u}{2\Omega L} \ll 1 \]
and if $\rho = \text{constant}$, then
\[
\frac{\zeta}{2\Omega} \sim \frac{u}{2\Omega_L} \ll 1
\]
and
\[
2\vec{\Omega} \cdot \nabla \vec{q} = 0 \quad (7.2.3)
\]

The flow is two dimensional and does not vary along the rotation vector. Let $\Omega$ be parallel to the $z$ axis, then
\[
\frac{\partial \vec{q}}{\partial z} = 0 \quad (7.2.4)
\]
This has been proven experimentally by Taylor, see sketch in Figure 7.2.1.

**Theorem 1** *Taylor-Proudman theorem*: A steady and slow flow in a rotating fluid is two-dimensional in the plane perpendicular to the vector of angular velocity.

Figure 7.2.1: Taylor’s experiment showing the Taylor column above a truncated cylinder in a rotating fluid. The large container with water rotates but the cylinder is fixed in space. From Kundu.