

Propagation of Uncertainty through Mathematical Operations

Since the quantity of interest in an experiment is rarely obtained by measuring that quantity directly, we must understand how error propagates when mathematical operations are performed on measured quantities. Suppose we have a simple experiment where we want to measure velocity, V , by measuring distance, d , and time, t . We take measurements and come up with measured quantities $d \pm \delta d$ and $t \pm \delta t$. We can easily estimate V by dividing d by t , but we also need to know how to find δV . Below we investigate how error propagates when mathematical operations are performed on two quantities x and y that comprise the desired quantity q .

Addition and Subtraction

If we are trying to find the uncertainty, δq , associated with $q = x + y$, we can look at what the highest and lowest probable values would be. The highest value would be obtained by adding the best estimates for x and y to the total uncertainty for both values. Similarly, the lowest probable value would be obtained by adding the best estimates for x and y and subtracting both associated uncertainties:

$$(\text{highest probable value of } q) = x_{best} + y_{best} + (\delta x + \delta y) \quad (2)$$

$$(\text{lowest probable value of } q) = x_{best} + y_{best} - (\delta x + \delta y) \quad (3)$$

Since $q_{best} = x_{best} + y_{best}$, it is easy to see δq is equal to $\delta x + \delta y$ and q can be expressed as:

$$q_{best} = x_{best} + y_{best} \pm (\delta x + \delta y) \quad (4)$$

Similarly, for subtraction, we can write that:

$$q_{best} = x_{best} - y_{best} \pm (\delta x + \delta y) \quad (5)$$

A similar analysis as shown above can be applied to any number of quantities that are added or subtracted, and we can state a general rule:

Rule 1: Uncertainty in Sums and Differences

$$\text{For } q = x + \dots + z - (u + \dots + w), \\ \delta q = \delta x + \dots + \delta z - (\delta u + \dots + \delta w)$$

Or more simply put, when adding or subtracting quantities, their uncertainties add.

Uncertainty of a Product

We can perform a similar analysis as above for multiplication, but first we must define fractional uncertainty of a quantity:

$$(\text{fractional uncertainty in } x) = \frac{\delta x}{|x_{best}|}. \quad (6)$$

The fractional uncertainty (or, as it is also known, percentage uncertainty) is a normalized, dimensionless way of presenting uncertainty, which is necessary when multiplying or dividing. A measurement and its fractional uncertainty can be expressed as:

$$(\text{value of } x) = x_{best} \left(1 + \frac{\delta x}{|x_{best}|} \right). \quad (7)$$

For simplicity, hereafter the subscript ‘best’ will be omitted in the denominator of the fractional uncertainty, but it is assumed. For $q = xy$, we have measured values for x and y of the form:

$$(\text{measured } x) = x_{best} \left(1 + \frac{\delta x}{|x|} \right) \quad (8)$$

$$(\text{measured } y) = y_{best} \left(1 + \frac{\delta y}{|y|} \right) \quad (9)$$

Once again we look at the largest and smallest probable values of q :

$$(\text{largest probable value of } q) = x_{best} y_{best} \left(1 + \frac{\delta x}{|x|} \right) \left(1 + \frac{\delta y}{|y|} \right) \quad (10)$$

$$(\text{smallest probable value of } q) = x_{best} y_{best} \left(1 - \frac{\delta x}{|x|} \right) \left(1 - \frac{\delta y}{|y|} \right) \quad (11)$$

We expand (10) to get:

$$(\text{largest probable value of } q) = x_{best} y_{best} \left(1 + \frac{\delta x}{|x|} + \frac{\delta y}{|y|} + \frac{\delta x \delta y}{|x| |y|} \right) \quad (12)$$

We now must make the assumption that the fractional uncertainties are small (that is, they are less than 1). This is a reasonable assumption since most well-designed experiments will have fractional uncertainties that are 0.2 or less. If we can say that the fractional uncertainties are less than one, then their product will be much less than one and their contribution to the uncertainty negligible. From this argument, we neglect the last term in (12) and simplify the equation to:

$$(\text{largest probable value of } q) = x_{best} y_{best} \left(1 + \left(\frac{\delta x}{|x|} + \frac{\delta y}{|y|} \right) \right) \quad (13)$$

Similarly, we simplify the equation for the lowest probable value to:

$$(\text{smallest probable value of } q) = x_{best}y_{best} \left(1 - \left(\frac{\delta x}{|x|} + \frac{\delta y}{|y|} \right) \right) \quad (14)$$

This gives a probable range of values for q of:

$$q = x_{best}y_{best} \left(1 \pm \left(\frac{\delta x}{|x|} + \frac{\delta y}{|y|} \right) \right) \quad (15)$$

The equation for q is:

$$(\text{value of } q) = q_{best} \left(1 + \frac{\delta q}{|q|} \right) \quad (16)$$

and since $q_{best} = x_{best}y_{best}$, we conclude from (15) and (16) that:

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|} \quad (17)$$

Therefore, to find the uncertainty of two multiplied quantities, we add the *fractional* uncertainties.

Uncertainty in a Quotient

To estimate the uncertainty associated with the quotient $q=x/y$, we once again look at the largest value of q we could expect:

$$(\text{largest value of } q) = \frac{x_{best}}{y_{best}} \frac{1 + \frac{\delta x}{|x|}}{1 - \frac{\delta y}{|y|}} \quad (18)$$

We use a little bit of mathematical manipulation to simplify (18). The term on the right has the form:

$$\frac{1+a}{1-b}, \text{ where } a \text{ and } b \text{ are less than } 1 \quad (19)$$

The binomial theorem allows us to express the denominator term as an infinite series:

$$\frac{1}{1-b} = 1 + b + b^2 + \dots \quad (20)$$

Since b is assumed less than 1, b^2 and all of the higher order terms will all be $\ll 1$. These can be neglected and we can say that:

$$\frac{1}{1-b} \approx 1+b. \quad (21)$$

Then, (19) becomes

$$\frac{1+a}{1-b} \approx (1+a)(1+b) = 1+a+b+ab$$

Once again we eliminate ab because it is the product of two small numbers. We substitute the fractional uncertainties for a and b and simplify (18) to:

$$(\text{largest value of } q) = \frac{x_{best}}{y_{best}} \left(1 + \left(\frac{\delta x}{|x|} + \frac{\delta y}{|y|} \right) \right) \quad (22)$$

A similar procedure applied to the equation for the smallest value shows that:

$$(\text{smallest value of } q) = \frac{x_{best}}{y_{best}} \left(1 - \left(\frac{\delta x}{|x|} + \frac{\delta y}{|y|} \right) \right) \quad (23)$$

Knowing that $q_{best} = x_{best}/y_{best}$, we find once again that:

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$

Now we extend the separate results for multiplication and division to apply to any number of quantities that are multiplied or divided:

Rule 2: Uncertainty in Products and Quotients

Quantities $x \dots w$ are measured with small uncertainties $\delta x \dots \delta w$ and

$$q = \frac{x \times \dots \times z}{u \times \dots \times w}.$$

Then the fractional uncertainty in the computed value of q is:

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \dots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \dots + \frac{\delta w}{|w|}$$

In summary, when any number of quantities are multiplied or divided, their *fractional* uncertainties add.

Summing errors in quadrature

The two rules we have defined so far state that for addition or subtraction, we sum the uncertainties, and for multiplication and division, we sum the fractional uncertainties. However, if the original uncertainties associated with the measured quantities are independent and random, these rules will produce values of uncertainty that are unnecessarily large. Thus we consider another method for computing propagated error.

We previously stated that the highest value we would expect for the quantity $q = x + y$ is $x_{best} + y_{best} + (\delta x + \delta y)$. For this to be the actual value of q , though, we would have had to underestimate both x and y by their full amounts δx and δy . If x and y are independent errors and random in nature, we have a 50% chance that an underestimate of x is accompanied by an overestimate of y . The probability we would underestimate both by the full amount is very small and an uncertainty of $\delta q = \delta x + \delta y$ would be too large.

Without detailed justification, we present an alternative way of calculating uncertainty, assuming errors in measurement are governed by the normal (or Gaussian) distribution and that measured quantities are independent of each other. This method, called *adding in quadrature*, provides the following rules for computing uncertainty in q .

Rule 3: Uncertainty in Sums and Differences for Random and Independent Uncertainties

If x, \dots, w are measured with independent and random uncertainties $\delta x, \dots, \delta w$, and are used to compute $q = x + \dots + z - (u + \dots + w)$, then the uncertainty in q is the quadratic sum:

$$\delta q = \sqrt{(\delta x)^2 + \dots + (\delta z)^2 + (\delta u)^2 + \dots + (\delta w)^2}$$

Rule 4: Uncertainty in Products and Quotients for Random and Independent Uncertainties

If x, \dots, w are measured with independent and random uncertainties $\delta x, \dots, \delta w$, and are used to compute $q = \frac{x \times \dots \times z}{u \times \dots \times w}$ then the uncertainty in q is the quadratic sum:

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \dots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \dots + \left(\frac{\delta w}{w}\right)^2}$$

Notice that the procedure of adding in quadrature will always produce values smaller than using ordinary addition to sum uncertainties. Indeed, rules 1 and 2 are upper bounds for the uncertainty of a measured quantity. The uncertainty of q will be no larger than the values produced by rules 1 and 2. Rules 3 and 4 are a way uncertainty can be reduced under certain conditions.

Another important consequence of using rules 3 and 4 is that small uncertainties are made smaller when they are squared, making their contribution to the overall uncertainty negligible. Say, for example, we are trying to measure the density of a gas using the ideal gas law: $\rho = P/RT$, where ρ is the density of the gas, P is the pressure, R is the gas constant, and T is the temperature. R is a constant known with much precision, so we do not even consider its contribution to uncertainty. For temperature, we're using a high-precision thermometer and we estimate that $\delta T/T$ is 2%. Our pressure gauge, though, doesn't have very fine markings, and we estimate $\delta P/P$ is 10%. We then compute:

$$\frac{\delta\rho}{|\rho|} = \sqrt{\left(\frac{\delta P}{P}\right)^2 + \left(\frac{\delta T}{T}\right)^2} = \sqrt{(0.1)^2 + (0.02)^2} \quad (24)$$

and find $\delta\rho/|\rho| = 0.1$ or 10%. The small error associated with temperature made no contribution to the overall error. This tells us that the best way to reduce the overall uncertainty in an experiment is refine the measurement procedures for the quantities with the highest individual uncertainties.

A note of caution on assuming random and independent uncertainties: If we use one instrument to measure multiple quantities, we cannot be sure that the errors in the quantities are independent. If instrument calibration is the cause of the error, the errors are not independent and the total error should not be computed by summing in quadrature.

General Formula for Error Propagation

Very often, the quantity being investigated in an experiment cannot be computed through the simple mathematical operations discussed above. In many experiments, we must understand how error propagates through more complex functions, such as those that contain exponents, logarithms, and trigonometric functions.

Let us examine a quantity q that is some function of a variable x (such as $q=\sin x$ or $q=x^3$). We also assume, as usual, that our measurement of x has the form $x_{best} \pm \delta x$. Then our best estimate of q is $q_{best}=q(x_{best})$. As we have done before, we estimate the largest probable value of q . In this case, the maximum value of q is $q_{max}=q(x_{best} \pm \delta x)$. To determine what this value is, we must evaluate the function q at $x_{best} \pm \delta x$. We can do this by determining the slope of the function. From calculus, we know that for a continuous function $q(x)$ and a small increment u ,

$$q(x+u) - q(x) = \frac{dq}{dx} u . \quad (25)$$

Since δx is always small, we can say that:

$$q_{max} - q_{best} = q(x_{best} + \delta x) - q(x) = \frac{dq}{dx} \delta x$$

The uncertainty in q is simply the difference $q_{max} - q_{best}$ and we conclude that:

$$\delta q = \frac{dq}{dx} \delta x. \quad (26)$$

More generally, we can write a formula for computing the propagated error in a function of several variables. The only major change from the single-variable formula shown in (26) is that we must compute partial derivatives (i.e. $\partial q / \partial x$) due to the fact that q can be a function of multiple variables.

Rule 5: Uncertainty in a Function of Several Variables

If x, \dots, z are measured with independent and random uncertainties $\delta x, \dots, \delta z$ and are used to compute $q(x, \dots, z)$ then the uncertainty in q is

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x} \delta x\right)^2 + \dots + \left(\frac{\partial q}{\partial z} \delta z\right)^2}$$