4.3 Convection driven by buoyancy - Mountain Wind

ref: Prandtl: Fluid Dynamics.

Due to solar heating during the day, a mountain slope may be warmer than the surrounding air in a summer night. Let the air near a mountain slope be stably stratified

\[ T_o = T_0 + Ny', \]  

(4.3.1)

where \( T_0 = \) constant, and \( N > 0 \). Let the slope temperature be:

\[ T_s = T_1 + Ny', \]  

(4.3.2)

where \( T_1 > T_0 \). See the left of Figure 4.3.2. Consider first the static equilibrium:

\[ 0 = -\frac{dp_o}{dz} - \rho_o g \]
hence
\[ p_o = p_o(\infty) + \int_z^\infty \rho_o g \, dz \]

Let \( A \) and \( B \) be two points with the same elevation but \( A \) is on the slope and \( B \) is in the air. Since \( p_A < p_B \),
\[ \frac{\partial p_o}{\partial x} < 0 \]
implying
\[ \frac{\partial p_o}{\partial x'} < 0 \]

The pressure gradient must drive an upward flow along the slope.

Let us consider the dynamics. Let
\[ T(x, y) = T_o + \theta(y) \quad (4.3.3) \]
and
\[ \rho(x, y) = \rho_o + S(y) = \text{static density } + \text{dynamic density} \quad (4.3.4) \]

By the equation of state,
\[ \rho = \rho_o \left[ 1 - \beta (T - T_0) \right] = \rho_o \left[ 1 - \beta (T_o - T_0) \right] - \rho_o \beta \theta. \]

Therefore
\[ \rho_o = \rho_o \left[ 1 - \beta (T_o - T_0) \right] = \rho_o (1 - \beta N y') \quad (4.3.5) \]
and
\[ S(x, y) = -\rho_o \beta \theta(x, y) \quad (4.3.6) \]

Note by rotation of coordinates,
\[ T_o - T_0 = N y' = N (x \sin \alpha + y \cos \alpha). \quad (4.3.7) \]

The flow equations are:
\[ u_x + v_y = 0 \quad (4.3.8) \]
\[ \rho \left( u_u + v_u \right) = -p_{dx} + \mu \left( u_{xx} + u_{yy} \right) - (\rho - \rho_u) g \sin \alpha \quad (4.3.9) \]
\[ \rho \left( u_v + v_u \right) = -p_{dy} + \mu \left( v_{xx} + v_{yy} \right) - (\rho - \rho_u) g \cos \alpha \quad (4.3.10) \]
\[ uT_x + vT_y = k \left( T_{xx} + T_{yy} \right), \quad (4.3.11) \]

where \( T \) is the total temperature and
\[ k = \frac{K}{\rho_o c_p}, \]
is the thermal diffusivity. Since \( \partial / \partial x = 0, v = 0 \) from continuity. From Eqn. (4.3.9)
\[ \nu u_{yy} + (\beta g \sin \alpha) \theta = 0. \quad (4.3.12) \]
after invoking Boussinesq approximation. In Eqn. (4.3.11),
\[
\frac{\partial T}{\partial x} = \frac{\partial T_0}{\partial x} = N \sin \alpha.
\]

Therefore,
\[
u N \sin \alpha = k\theta_y.
\]

Combining Eqns. (4.3.12) and (4.3.13), we get
\[
d^4u \frac{dy^4}{dy^4} + \left( \frac{\beta gN \sin^2 \alpha}{\nu k} \right) u = 0
\]

and
\[
d^4\theta \frac{dy^4}{dy^4} + \left( \frac{\beta gN \sin^2 \alpha}{\nu k} \right) \theta = 0
\]

Let
\[
l^4 = \frac{4\nu k}{\beta gA \sin^2 \alpha} \quad \text{and} \quad y = \ell \eta
\]
then
\[
d^4u \frac{d\eta^4}{d\eta^4} + 4u = 0; \quad \text{and} \quad d^4\theta \frac{d\eta^4}{d\eta^4} + 4\theta = 0
\]

The velocity is
\[
u = U e^{-\eta} \sin \eta \quad \text{so that} \quad u(0) = 0
\]

The temperature is
\[
\theta = \theta_0 e^{-\eta} \cos \eta
\]

The boundary conditions at \( \eta \sim \infty \) are satisfied. In order that \( \theta(0) = T_1 - T_0 \) on \( \eta = 0 \) we choose
\[
\theta_0 = T_1 - T_0
\]

Note that the boundary layer thickness is
\[
\delta \sim O(\ell) \sim \left( \frac{4\nu k}{\rho gN \sin^2 \alpha} \right)^{1/4}
\]
Thus if \( \alpha \downarrow, \delta \uparrow \) as \( 1/\sin^2 \alpha \).

Using Eqn. (4.3.13), we get
\[
N \sin \alpha U e^{-\eta} \sin \eta = k \left( \frac{\beta gN \sin^2 \alpha}{4\nu k} \right)^{1/2} 2\theta_0 e^{-\eta} \sin \eta.
\]

Hence,
\[
U = \theta_0 \left( \frac{\beta gk}{N\nu} \right)^{1/2}
\]
Finally
\[ u = (T_1 - T_0) \left( \frac{\beta g k}{N \nu} \right)^{1/2} e^{-\eta} \sin \eta. \]  

(4.3.23)

and
\[ \theta = (T_1 - T_0) e^{-\eta} \cos \eta \]  

(4.3.24)

It is easy to show from (4.3.13) that the total mass flux rate is
\[ M \left. \frac{d\theta}{dy} \right|_0 \]  

(4.3.25)

Note from (4.3.22) that \( U \) is independent of \( \alpha \). If \( \alpha \downarrow \), the buoyancy force is weaker, but the shear rate \( \partial u / \partial y \) is smaller, hence the wall resistance is smaller. \( U \) is not reduced!

Figure 4.3.2: Wind along a valley due to feeding from mountains

On a warm slope (due to solar heating during the day), air rises at night. If there are two slopes forming a valley, fluid must be supplied from the bottom of the valley; this is the reason for valley wind blowing from low altitude to high.

On a cold slope (due to radiation loss at night) air sinks at high noon. Valley wind must flow from high to low.