4.4  Buoyant plume from a steady heat source

[Reference]:
Gebhart, et. al. (Jalluria, Maharjan, Saammakia),

Let $\tilde{T} = T - T_\infty =$ temperature variation where $T_{\infty}$ is a constant (no ambient stratification). For a strong enough heat source, we expect boundary layer behavior,

$$ \frac{\partial}{\partial r} \gg \frac{\partial}{\partial x}, \ u \gg v, \ \frac{\partial p}{\partial r} \approx 0 $$

The boundary layer equations are

$$ \frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial r} = 0 \quad (4.4.1) $$

$$ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = g \beta (T - T_\infty) + \nu \frac{\partial}{\partial r} \left( \frac{r \frac{\partial u}{\partial r}}{r} \right) \quad (4.4.2) $$

$$ u \frac{\partial \tilde{T}}{\partial x} + v \frac{\partial \tilde{T}}{\partial r} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{T}}{\partial r} \right) \quad (4.4.3) $$

The centerline $r = 0$ is an axis of symmetry,

$$ v = \frac{\partial u}{\partial r} = \frac{\partial \tilde{T}}{\partial r} = 0 \quad (4.4.4) $$

Far outside the plume $r \to \infty$

$$ u \to 0 \text{ and } T \to T_\infty, \ (\tilde{T} \to 0) \quad (4.4.5) $$

Rewrite (4.4.3) as

$$ \frac{\partial (ru \tilde{T})}{\partial x} + \frac{\partial (rv \tilde{T})}{\partial r} - \tilde{T} \left( \frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial r} \right) $$

$$ = \frac{\partial (ru \tilde{T})}{\partial x} + \frac{\partial (rv \tilde{T})}{\partial r} = k \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{T}}{\partial r} \right) \quad (4.4.6) $$
after using continuity. Now integrating the last equation from \( r = 0 \) to \( r = \infty \)

\[
\frac{\partial}{\partial x} \int_{0}^{\infty} 2\pi ru\tilde{T}dr + 2\pi \int_{0}^{\infty} \frac{\partial (rv\tilde{T})}{\partial r} dr = k2\pi \int_{0}^{\infty} dr \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{T}}{\partial r} \right)
\]

therefore

\[
\frac{\partial}{\partial x} \int_{0}^{\infty} 2\pi ru\tilde{T}dr + 2\pi rv\tilde{T}\bigg|^{\infty}_{0} = 2\pi k \left( r \frac{\partial \tilde{T}}{\partial r} \right)_{r=\infty}^{r=0}
\]

(4.4.7)

Using the boundary conditions, we get or

\[
\int_{0}^{\infty} 2\pi ru\tilde{T}dr = \text{constant}
\]

Note that

\[
\int_{0}^{\infty} 2\pi rdr \ u\rho C\tilde{T} = \text{rate of buoyancy flux}
\]

= rate of heat flux

= \( Q \) (given rate of heat release at \( x = 0 \))

therefore,

\[
Q = \int_{0}^{\infty} 2\pi rdr\rho uC\tilde{T}
\]

(4.4.8)

This is a boundary condition.

Let the stream function \( \psi \) be defined by

\[
ru = \frac{\partial \psi}{\partial r}, \quad rv = -\frac{\partial \psi}{\partial x}
\]

(4.4.9)

(4.4.1) is automatically satisfied. From the momentum equation:

\[
\left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \frac{1}{r} \frac{\partial^2 \psi}{\partial x \partial r} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) = g\beta \tilde{T} + \nu \frac{\partial}{\partial r} \left[ \frac{r}{\partial \psi} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right]
\]

(4.4.10)

From the energy equation

\[
\frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial \tilde{T}}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial \tilde{T}}{\partial r} = k \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{T}}{\partial r} \right)
\]

(4.4.11)

and from the buoyancy flux condition

\[
Q = 2\pi \rho C \int_{0}^{\infty} rdr \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \tilde{T}
\]

(4.4.12)
Try a similarity solution with the one-parameter transformation
\[ x - \lambda^a x^*, \quad r = \lambda^b r^*, \quad \psi = \lambda^c \psi^*, \quad \tilde{T} = \lambda^d \tilde{T}^* \]

From (4.4.10),
\[ \lambda^{2c-4b-a} = \lambda^{2c-4b-a} = \lambda^d = \lambda^{c-4b} \]  
(4.4.13)
from (4.4.11)
\[ \lambda^{c+d-2b-a} = \lambda^{d-2b} \]  
(4.4.14)
and from (4.4.12)
\[ \lambda^{c+d} = 1 \]  
(4.4.15)

From these three equations we get
\[ \frac{c}{a} = 1, \quad \frac{b}{a} = \frac{1}{2}, \quad \frac{d}{a} = -1. \]

We leave it as an exercise to show that the similarity variable can be taken to be
\[ \eta = \frac{r}{x^{1/2}} \]  
(4.4.16)
and the similarity solutions to be
\[ \psi = x F(\eta), \quad \text{and} \quad \tilde{T} = x^{-1} G(\eta) \]  
(4.4.17)

After much algebra, and noting
\[ \frac{\partial \eta}{\partial r} = \frac{1}{x^{1/2}}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{2} \frac{r}{x^{3/2}} = -\frac{1}{2} \frac{r}{x^{1/2}} \frac{1}{x} = -\frac{\eta}{2x} \]
we get from (4.4.10)
\[ \nu F''' + \left( \frac{F'}{\eta} \right)' (F - \nu) + g \beta \eta G = 0 \]  
(4.4.18)
and from (4.4.11)
\[ k (\eta G')' + (FG)' = 0 \]  
(4.4.19)

Before integrating, let us normalize :
\[ \eta = \alpha \bar{\eta}, \quad F = \gamma \bar{F}, \quad G = \sigma \bar{G}. \]  
(4.4.20)

It follows from (4.4.18) that
\[ \frac{\nu \gamma}{\alpha^3} \bar{F}''' + \frac{\gamma}{\alpha^3} \left( \frac{\bar{F}'}{\bar{\eta}} \right)' (\gamma \bar{F} - \nu) + g \beta \alpha \sigma \bar{\eta} \bar{G} = 0 \]  
(4.4.21)
where prime denotes $d/d\bar{\eta}$. Setting $\gamma = \nu$ and

$$\frac{\nu^2}{\alpha^3} = g\beta\sigma$$

which relates $\sigma$ and $\alpha$,

$$\sigma = \frac{\nu^2}{g\beta\alpha^4} \quad (4.4.22)$$

we get

$$\bar{F}''' + \left( \frac{\bar{F}'}{\bar{\eta}} \right)' (\bar{F} - 1) + \bar{\eta}\bar{G} = 0 \quad (4.4.23)$$

Similar normalization of (4.4.19) gives

$$\frac{k\alpha\sigma}{\alpha^2} (\bar{\eta}\bar{G}')' + \frac{\gamma\sigma}{\alpha}(\bar{F}\bar{G})' = 0 \quad (4.4.24)$$

which can be simplified to

$$(\bar{\eta}\bar{G}')' + Pr(\bar{F}\bar{G})' = 0 \quad (4.4.25)$$

where

$$Pr = \frac{\nu}{k} = \text{Prandtl Number} \quad (4.4.26)$$

For water $\nu = 10^{-2}cm^2/s, k = 1.42cm^2/s$, hence $Pr = 7$. For air $\nu = 0.145cm^2/s, k = 0.202cm^2/s$, hence $Pr = 0.75$.

We now integrate (4.4.25) to give

$$\bar{\eta}\bar{G}' + Pr\bar{F}\bar{G} = \text{constant}$$

Since $\psi(x, 0) = 0$, we must have $F(0) = 0$; the constant above is zero.

$$\bar{\eta}\bar{G}' + Pr\bar{F}\bar{G} = 0 \quad (4.4.27)$$

Equation (4.4.27) can be written

$$\frac{\bar{G}'}{\bar{G}} = -Pr\frac{\bar{F}}{\bar{\eta}} \quad \text{or} \quad \frac{d \ln \bar{G}}{d\bar{\eta}} = -Pr\frac{\bar{F}}{\bar{\eta}}$$

$$\ln \bar{G} = -Pr\int_0^{\bar{\eta}} \frac{\bar{F}}{\bar{\eta}}d\bar{\eta} + \text{constant}$$

$$\bar{G}(\bar{\eta}) = \bar{G}(0) \exp \left( -Pr\int_0^{\bar{\eta}} \frac{\bar{F}}{\bar{\eta}}d\bar{\eta} \right) \quad (4.4.28)$$

Substituting Eqn. (4.4.28) into Eqn. (4.4.23), the resulting equation for $\bar{F}$ must be integrated numerically.

Now let us find the boundary conditions for $F$ or $\bar{F}$. 
Eqn. (4.4.8) becomes

\[
\frac{Q}{2\pi \rho C} = \int_0^\infty dr \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \frac{G(\eta)}{x} = \int_0^\infty dr \frac{r}{x} x^{1/2} F' \frac{G'}{x} = \int_0^\infty d\eta (F' G) = \nu \sigma \int_0^\infty d\bar{\eta} \bar{F}' \bar{G}
\]

Therefore,

\[
\int_0^\infty d\bar{\eta} \bar{F}' \bar{G} = \frac{Q}{2\pi \rho C \nu \sigma}
\]

Let us choose

\[
\frac{Q}{2\pi \rho C \nu \sigma} = 1
\]

so that

\[
\int_0^\infty d\bar{\eta} \bar{F}' \bar{G} = 1
\]

is the boundary condition for \( \bar{F} \) and \( \bar{G} \). Now (4.4.31) defines \( \sigma \), the scale of \( G \). Note that larger \( Q \) implies larger \( \sigma \) and smaller \( \alpha \). Thus a stronger heat source leads to a greater centerline temperature and a thinner plume. Also,

\[
u \to 0 \quad \text{as} \quad r \to \infty
\]

hence

\[
u = \frac{1}{r} \frac{\psi_x}{\eta} = \frac{F'}{\eta} = \frac{\nu}{\alpha^2} \frac{\bar{F}'}{\bar{\eta}} \to 0, \quad \text{as} \quad \eta \sim \bar{\eta} \to \infty
\]

The radial velocity is, in general

\[
v = \frac{1}{r} \frac{\psi_x}{\eta} = \frac{1}{r} \left( \frac{x}{x^{1/2}} \right)
\]

Since

\[
v \to 0 \quad \text{as} \quad \eta \to 0,
\]

we must have,

\[F(0) = 0.\]

Clearly

\[F(\bar{\eta}) = 0 \quad \text{as} \quad \bar{\eta} \to 0 \quad (4.4.33)\]

The numerical results by Mollendorf & Gelhart, 1974, are shown in Figs. 4.4.1, for various Prandtl numbers. A schlierian photograph due to Gebhart (copied from Van Dyke An Album of Fluid Motion) is shown in Figure fig:plumeVD.

Remark:

\[u = \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{F'}{\eta} \left( = \frac{x}{x^{1/2}} \right) \]

Along the centerline \( u(x,0) = \left( \frac{x}{\eta} \right)_0 \) = constant depending on \( Pr \). Why? Buoyancy acceleration is counteracted by entrainment.
Remark: Let the radius of the plume be \( a \) which varies as
\[
a \sim x^{1/2}
\]
This is consistent with the behavior that \( u \sim x^0 \), and \( \bar{T} \sim x^{-1} \), since
\[
a^2u\bar{T} = Q
\]
On the other hand the mass flux rate is
\[
ua^2 \sim x
\]
and the momentum flux rate is
\[
u^2a^2 \sim x
\]
hence both approach zero at the source. Thus a plume is the result of energy source, not of mass or momentum.
Figure 4.4.1: Velocity profiles in an axisymmetric plume. (From Mollendorf and Gebhart, 1974.)

Figure 4.4.2: Temperature profiles in an axisymmetric plume. (From Mollendorf and Gebhart, 1974.)
203. Plane convection plume rising from a heated horizontal wire. A thin wire 0.1 in. long is heated electrically in atmospheric air. Each fringe in this interferogram represents a temperature difference of 0.5°C. The reference grid wires are spaced 1/4 by 1/4 in. In good accord with self-similar solutions of the boundary-layer equations, the width of the plume grows as the \( t^{1/3} \)-power of height. Gebhart, Pera & Schoor 1970

Figure 4.4.2: A 2D thermal plume from a line heat source. From Van Dyke, photo by Gebhart, Pera and Schoor 1970,