6.3 Saffman-Taylor instability in porous layer- Viscous fingering

Refs:

In petroleum recovery water is often used to drive oil from the reservoir. An oil reservoir can also be covered by a layer of water from above. Phenomenon of fingering often occurs when oil is extracted from beneath the water layer. Although known to mining engineers, Saffman & Taylor (1958) gave the first theory and performed simulated experiments in a Hele-Shaw cell.

Consider a moving interface in a stationary coordinate system. Let the initial seepage velocity $V$ be vertical and the interface be a plane, then

$$y = \frac{Vt}{n}$$  \hspace{1cm} (6.3.1)

where $n$ is the porosity. If the interface is disturbed then its position is at

$$y = \frac{Vt}{n} + \eta(x,t)$$  \hspace{1cm} (6.3.2)

At any interior point, $\phi$ is the velocity potential,

$$\phi = -\frac{k}{\mu} (p + \rho gy)$$  \hspace{1cm} (6.3.3)

where $k$ is the permeability related to conductivity $K$ by

$$K = \frac{\rho g k}{\mu}$$  \hspace{1cm} (6.3.4)
The pressure is
\[ p = -\frac{\mu}{k}\phi - \rho g y \]
(6.3.5)

Thus in fluid 1 (upper fluid)
\[ p_1 = -\frac{\mu_1}{k_1}g\phi_1 - \rho_1 g y \]
(6.3.6)

By continuity,
\[ \nabla^2 \phi_1 = 0, \quad y > \frac{Vt}{n} + \eta(x,t), \]  
(6.3.7)

In the lower fluid (2),
\[ p_2 = -\frac{\mu_2}{k_2}g\phi_2 - \rho_2 g y \]
(6.3.8)

and
\[ \nabla^2 \phi_2 = 0, \quad y < \frac{Vt}{n} + \eta(x,t) \]  
(6.3.9)

Let us first examine the basic uniform flow where the interface is plane \((\eta = 0)\). The potentials are
\[ \phi_1^o = Vy + f_1(t) = -\frac{k_1}{\mu_1}(p_1^o + \rho_1 g y) \]
(6.3.10)
\[ \phi_2 = Vy + f_2(t) = -\frac{k_2}{\mu_2}(p_2^o + \rho_2 g y) \]
(6.3.11)

Note that an arbitrary function of \(f(t)\) is added to the potential without affecting the velocity field. The pressures are
\[ p_1^o = -\left(\frac{\mu_1 V}{k_1} + \rho_1 g\right)y - \frac{\mu_1 f_1(t)}{k_1}, \quad y > \frac{Vt}{n} \]  
(6.3.12)

and in the lower fluid (2),
\[ p_2^o = -\left(\frac{\mu_2 V}{k_2} + \rho_2 g\right)y - \frac{\mu_2 f_2(t)}{k_2}, \quad y < \frac{Vt}{n} \]  
(6.3.13)

In order that pressure is continuous at \(y = Vt/n\) for all \(t\), we must have
\[ f_1(t) = F_1 t, \quad f_2(t) = F_2 t \]
(6.3.14)

where \(F_1, F_2\) are constants and
\[ -\left(\frac{\mu_1 V}{k_1} + \rho_1 g\right)\frac{V}{n} - \frac{\mu_1 F_1}{k_1} = -\left(\frac{\mu_2 V}{k_2} + \rho_2 g\right)\frac{V}{n} + \frac{\mu_2 F_2}{k_2} \]

Thus
\[ \frac{\mu_2 F_2}{k_2} - \frac{\mu_1 F_1}{k_1} = -\frac{V}{n}\left\{\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)\left(\frac{V}{n} + (\rho_2 - \rho_1 g)\right)\right\} \]
(6.3.15)
Note that it is only the difference that matters.

We now consider a small disturbance on the interface.

\[
y = \frac{Vt}{n} + \eta(x, t)
\]  
(6.3.16)

where

\[
\eta = ae^{i\alpha x - i\omega t}
\]  
(6.3.17)

is small. The total solution is

\[
\phi_1 = Vy + F_1t + B_1e^{i\alpha x - \alpha(y - Vt/n) - i\omega t}, \quad y > \frac{Vt}{n} + \eta(x, t)
\]  
(6.3.18)

\[
\phi_2 = Vy + F_2t + B_2e^{i\alpha x + \alpha(y - Vt/n) - i\omega t}, \quad y < \frac{Vt}{n} + \eta(x, t)
\]  
(6.3.19)

The linearized kinematic boundary condition is that velocities must be continuous.

\[
n \frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} \bigg|_{y=Vt/n} = \frac{\partial \phi_2}{\partial y} \bigg|_{y=Vt/n}
\]  
(6.3.20)

Thus

\[
-\omega nae^{i\alpha x - i\omega t} = -\alpha aB_1e^{i\alpha x - i\omega t} = \alpha aB_2e^{i\alpha x - i\omega t}
\]  
(6.3.21)

hence,

\[
B_1 = -B_2 = \frac{i\omega na}{\alpha}
\]  
(6.3.22)

Now we require continuity of pressure on \( y = Vt/n + \eta \),

\[
-\frac{\mu_1}{k_1} \left( V\eta + \frac{i\omega \eta}{\alpha} \right) - \rho_1 g \eta = -\frac{\mu_2}{k_2} \left( V\eta - \frac{i\omega \eta}{\alpha} \right) - \rho_2 g \eta
\]  
(6.3.23)

Eliminating \( \eta \) we get

\[
\omega = \frac{\alpha (\rho_2 - \rho_1) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right)}{n \left( \frac{\mu_2}{k_2} + \frac{\mu_1}{k_1} \right)}
\]  
(6.3.24)

Clearly \( \omega \) is real. If \( \omega > 0 \), or

\[
(\rho_2 - \rho_1) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) > 0
\]  
(6.3.25)

the flow is stable. If \( \omega < 0 \), or

\[
(\rho_2 - \rho_1) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) < 0,
\]  
(6.3.26)

the flow is unstable.
From the simple model of a tubular porous medium,

\[ K = \frac{n \rho g R^2}{8 \mu} = \frac{\rho g k}{\mu} \]  
(6.3.27)

hence

\[ k = \frac{n R^2}{8} \]  
(6.3.28)

is independent of viscosity and depends only on \( n \) and the pore size. Assume therefore \( k_1 = k_2 \) and that oil (lighter more viscous) lies above water \( \rho_1 < \rho_2 \) and \( \mu_1 > \mu_2 \). If \( V < 0 \) (water pushed downward by oil) then the flow is always stable. Consider \( V > 0 \). The flow is unstable if

\[ V > V_c = \frac{(\rho_2 - \rho_1)g}{\left(\frac{\mu_1}{k_1} - \frac{\mu_2}{k_2}\right)} \]  
(6.3.29)

Too high an extraction rate causes instability which marks the onset of fingers.

If the water layer is on top of the oil layer, then \( \rho_2 - \rho_1 < 0 \); the flow is unstable even if \( V = 0 \). Since \( \mu_2/k_2 - \mu_1/k_1 > 0 \) a downward flow (water toward oil) is always unstable. A upward flow can be unstable if

\[ 0 < V < V_c = \frac{(\rho_1 - \rho_2)g}{\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)} \]  
(6.3.30)

Note also that the growth (decay) rate is higher for shorter waves.

A gallery of beautiful photographs of fingering taken from Hele-Shaw experiments can be found in the survey by Homsy. Here are two samples.
Figure 6.3.1: Fingering in a Hele-Shaw model. (Wooding) From Homsy.
Figure 6.3.2: Fingering in a Hele shaw model. From Homsy.