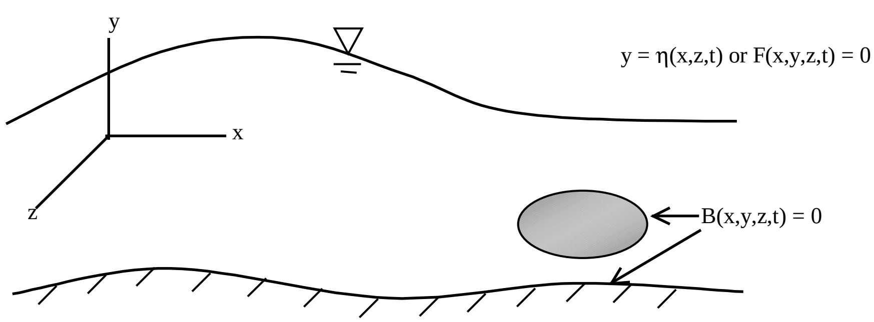
13.021 – Marine Hydrodynamics
Lecture 19

13.021 - Marine Hydrodynamics
Lecture 19

Water Waves

Exact (nonlinear) governing equations for surface gravity waves assuming potential theory



Unknown variables:

$$\vec{v}\,(x,y,z,t) = \nabla\phi\,(x,y,z,t)$$
$$\eta\,(x,z,t) \;\; \textbf{or} \;\; F\,(x,y,z,t)$$
$$p\,(x,y,z,t)$$

Field equation:

$$\textbf{Continuity:} \nabla^2\phi = 0 \quad y < \eta \textbf{ or } F < 0$$

$$\textbf{Given } \phi, \textbf{ can calculate } p: \frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p - p_a}{\rho} + gy = 0; \quad y < \eta \textbf{ or } F < 0$$

$$\textbf{Far way, no disturbance:} \partial\phi/\partial t, \nabla\phi \to 0 \textbf{ and } p = \underbrace{p_a}_{\text{atmospheric}} - \underbrace{\rho g y}_{\text{hydrostatic}}$$

Boundary Conditions:

1. **On an impervious boundary $B(x,y,z,t) = 0$, we have KBC:**

$$\vec{v} \cdot \hat{n} = \nabla\phi . \hat{n} = \frac{\partial \phi}{\partial n} = \vec{U}\left(\vec{x}, t\right) \cdot \hat{n}\left(\vec{x}, t\right) = U_n \textbf{ on } B = 0$$

   Alternatively: a particle P on B remains on B, i.e. B is a material surface; e.g. if P is on B at $t = t_0$, i.e.

$$B(\vec{x}_P, t_0) = 0, \textbf{ then } B(\vec{x}_P(t), t_0) = 0 \textbf{ for all } t,$$

   so that, following P, $B = 0$.

$$\therefore \frac{DB}{Dt} = \frac{\partial B}{\partial t} + \left(\nabla\phi \cdot \nabla\right) B = 0 \textbf{ on } B = 0$$

   For example, flat bottom at $y = -h$:

$$\partial\phi/\partial y = 0 \textbf{ on } y = -h \textbf{ or } B : y + h = 0$$

2. **On the free surface, $y = \eta$ or $F = y - \eta(x,z,t) = 0$.**

   KBC: free surface is a material surface, no perpendicular relative velocity to free surface, particle on free surface remains on free surface:

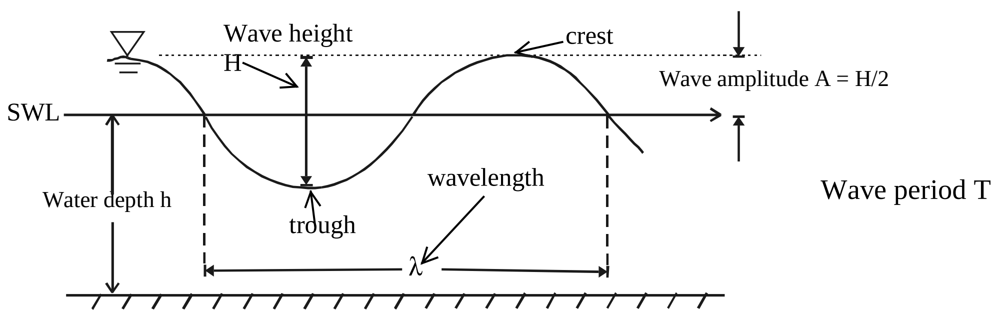
$$\frac{DF}{Dt} = 0 = \frac{D}{Dt}\left(y - \eta\right) = \underbrace{\frac{\partial \phi}{\partial y}}_{\substack{\text{vertical}\\ \text{velocity}}} - \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x} \underbrace{\frac{\partial \eta}{\partial x}}_{\substack{\text{slope}\\ \text{of f.s.}}} - \frac{\partial \phi}{\partial z} \underbrace{\frac{\partial \eta}{\partial z}}_{\substack{\text{slope}\\ \text{of f.s.}}} \textbf{ on } y = \underbrace{\eta}_{\substack{\text{still}\\ \text{unknown}}}$$

DBC: $p = p_a$ **on** $y = \eta$ **or** $F = 0$. **Apply Bernoulli equation at** $y = \eta$**:**

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla \phi\right|^2 + g\eta = 0 \textbf{ on } y = \eta \textbf{ (assuming} p_a = 0)$$

Linearized (Airy) Wave Theory

Consider <u>**small**</u> **amplitude waves: (small free surface slope)**



Assume amplitude small compared to wavelength, i.e., $\frac{A}{\lambda} << 1$. **Consequently:** $\frac{\phi}{\lambda^2/T}, \frac{\eta}{\lambda} << 1$, **and we keep only linear terms in** ϕ,η**:**

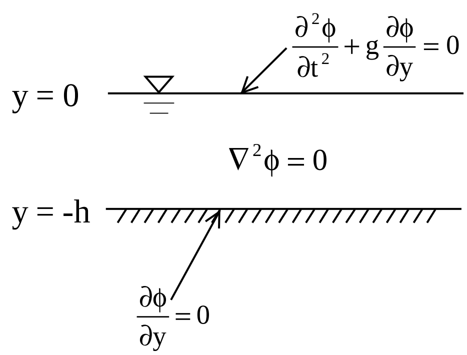
$$\textbf{For example: } ()|_{y=\eta} = \underbrace{()_{y=0}}_{\textbf{keep}} + \underbrace{\eta \frac{\partial}{\partial y}\, ()|_{y=0}}_{\textbf{discard}} + \ldots \textbf{ Taylor series}$$

Finally:

$$1)\nabla^2\phi = 0, -h < y < 0 \, \{y < 0\}$$

$$2)\frac{\partial\phi}{\partial y} = 0, y = -h \, \{\nabla\phi \to 0, y \to -\infty\}$$

$$\left.\begin{array}{l} 3a) FSKBC : \frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}, y = 0 \\[2ex] 3b) FSDBC : \frac{\partial\phi}{\partial t} + g\eta = 0, y = 0 \end{array}\right\} 3)\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0; y = 0$$



Linear (Airy) Waves in constant finite depth h {infinite depth}

$$\nabla^2\phi = 0, -h < y < 0; \{y < 0\} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 0, y = -h; \{\nabla\phi \to 0, y \to -\infty\} \tag{2}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0, y = 0; \{\textbf{same}\} \tag{3}$$

Given ϕ:

$$\eta\left(x,t\right)=-\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{y=0} \tag{4}$$

$$p-p_a=\underbrace{-\rho\frac{\partial\phi}{\partial t}}_{\textbf{dynamic}}-\underbrace{\rho gy}_{\textbf{hydrostatic}} \tag{5}$$

<u>Solution of 2D Periodic Plane Progressive Waves</u> (using separation of variables)

Math . . . (solve (1), (2), (3))

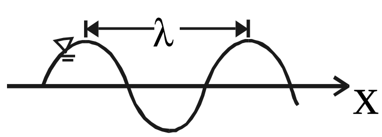
Answer:

$$\phi=\frac{gA}{\omega}\sin\left(kx-\omega t\right)\frac{\cosh k\left(y+h\right)}{\cosh kh};\left\{\frac{gA}{\omega}\sin\left(kx-\omega t\right)e^{ky}\right\}$$

$$\eta\underbrace{=}_{\substack{\textbf{using}\\ \textbf{(4)}}}A\cos\left(kx-\omega t\right);\left\{same\right\}$$

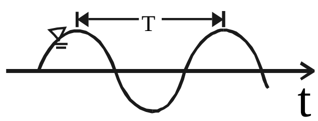
where A is the wave amplitude $=H/2$.

1. **At $t=0$ (say), $\eta=A\cos kx\rightarrow$ periodic in x with <u>wavelength</u>:** $\lambda=2\pi/k[L]$



k = wavenumber = 2π/λ [L^{-1}]

2. **At $x=0$ (say), $\eta=A\cos\omega t\rightarrow$ periodic in t with <u>period:</u>** $T=2\pi/\omega[T]$



ω = frequency = $2\pi/T$ $[T^{-1}]$, e.g. rad/sec

3. $\eta = A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right]$; $\frac{\omega}{k} = [V]$

Following a point with velocity $\frac{\omega}{k}$, i.e. $x_p = \left(\frac{\omega}{k}\right)t + const.$ the phase of η does not change: $\frac{\omega}{k} = \frac{\lambda}{T} \equiv V_p$ phase velocity.

Dispersion Relationship

So far, any ω, k combination is allowed. Must satisfy FSBC, substitute ϕ into equation (3):

$$-\omega^2 \cosh kh + gk \sinh kh = 0,$$

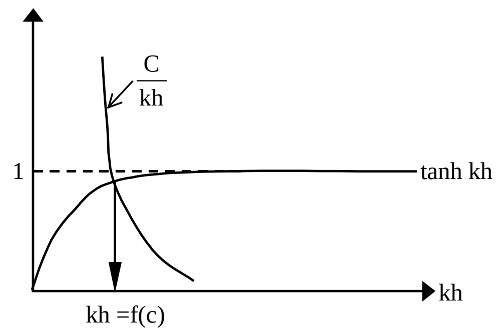
which gives

$$\underbrace{\omega^2 = gk \tanh kh; \left\{\omega^2 = gk\right\}}_{\textbf{Dispersion Relationship}} \tag{6}$$

Given k (and h) $\rightarrow \omega$.
Given ω (and h), ...
From (6):

$$C \equiv \frac{\omega^2 h}{g} = (kh)\tanh(kh)$$

$$\frac{C}{kh} = \tanh kh \rightarrow k\,(\textbf{only}1)$$



Dispersion Relationship (6) uniquely relates ω and k, given h.

$$\omega = \omega(k; h) \text{ or } k = k(\omega; h)$$

In general, $k \uparrow$ as $\omega \uparrow$ or $\lambda \uparrow$ as $T \uparrow$.

$$\frac{\lambda}{T} = V_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kh}; \left\{ V_p = \sqrt{\frac{g}{k}} \right\}$$

$V_p \uparrow$ **as** $T \uparrow$ **or** $\lambda \uparrow$. **For fixed** k **(or** λ**),** $V_p \uparrow$ **as** $h \uparrow$ **(from shallow water)**

$$V_p = V_p(k) \text{ or } V_p(\omega) \rightarrow \textbf{ frequency dispersion}$$

Solution of the Dispersion Relationship

$$\omega^2 = gk \tanh kh$$

Property of $\tanh kh$**:**

$$\tanh kh = \frac{\sinh kh}{\cosh kh} = \frac{1 - e^{-2kh}}{1 + e^{-2kh}} \cong \begin{cases} kh \textbf{ for } kh << 1, \textbf{ i.e. } h << \lambda \quad (\textbf{long waves or shallow water}) \\ 1 \textbf{ for } kh >\sim 3\,(\textbf{e.g.} \tanh 3 = 0.995)\, kh > \pi \rightarrow h > \frac{\lambda}{2} \\ (\textbf{ short waves or deep water}) \end{cases}$$

1. **Deep water or short waves,** $kh >> 1 (h >\sim \lambda/2)$

$$\omega^2 = gk,\ \ \lambda = \frac{g}{2\pi}T^2(\lambda(\textbf{in ft.}) \approx 5.12T^2 \ \ \textbf{(in sec.)})$$

$$V_p = \frac{\omega}{k} = \frac{g}{\omega} = \sqrt{\frac{g}{2\pi}\lambda};\ \ \lambda = \frac{2\pi}{g}V_p^2 \leftarrow \ \textbf{frequency dispersion}$$

2. **<u>Shallow water or long waves</u>,** $kh << 1,\ \ (h/\lambda <\sim 1/20$**in practice**$)$

$$\omega^2 \cong gk \cdot kh \rightarrow \omega = \sqrt{gh}\ k;\ \ \lambda = \sqrt{gh}\ T$$

$$V_p = \frac{\omega}{k} = \frac{\lambda}{T} = \sqrt{gh} \leftarrow \textbf{no frequency dispersion}$$

3. **<u>Intermediate depth or wavelength</u>. Need to solve**

$$\omega^2 = gk \tanh kh$$

given ω, h **for** k **(given** k, h **for** ω **- easy!)**

(a) Use tables or graphs (e.g. JNN fig. 6.3)

$$\omega^2 \quad = gk \tanh kh = gk_\infty,\ \ \textbf{then}\ \ \frac{k_\infty}{k} = \frac{\lambda}{\lambda_\infty} = \frac{V_p}{V_{p\infty}} = \tanh kh$$

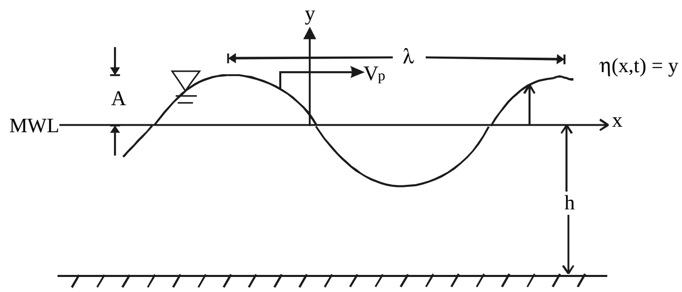
(b) Use numerical approximation (hand calculator)

i. calculate $C = \omega^2 h/g$**.**

ii.

$$\left.\begin{array}{lll} \textbf{If } C > 2, & kh \approx C(1 + 2e^{-2C} - 12e^{-4C} + \ldots) & \textbf{(deeper)} \\ \textbf{If } C < 2, & kh \approx \sqrt{C}(1 + 0.169C + 0.031C^2 + \ldots) & \textbf{(shallower)} \end{array}\right\}\ \textbf{about 4 decimals}$$

Characteristics of a Plane Linear Progressive Wave



$$k = \frac{2\pi}{\lambda},\ \omega = \frac{2\pi}{T},\ H = 2A$$

Define $U \equiv \omega A$.

Linear Solution:

$$\phi = \frac{Ag}{\omega}\frac{\cosh k\,(y+h)}{\cosh kh}\sin\left(kx - \omega t\right);\eta = A\cos\left(kx - \omega t\right)$$

with

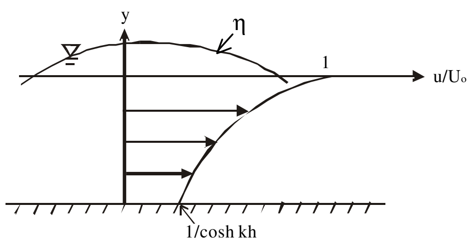
$$\omega^2 = gk\tanh kh$$

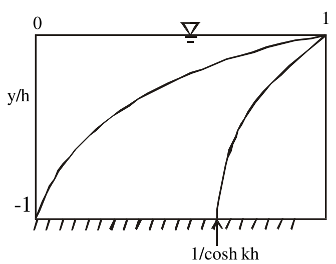
Velocity Field: On y = 0:

$u = \frac{\partial\phi}{\partial x} = \frac{Agk}{\omega}\frac{\cosh k(y+h)}{\cosh kh}\cos\left(kx-\omega t\right)$ $= \underbrace{A\omega}_{U}\frac{\cosh k(y+h)}{\sinh kh}\cos\left(kx-\omega t\right)$	$v = \frac{\partial\phi}{\partial y} = \frac{Agk}{\omega}\frac{\sinh k(y+h)}{\cosh kh}\sin\left(kx-\omega t\right)$ $= \underbrace{A\omega}_{U}\frac{\sinh k(y+h)}{\sinh kh}\sin\left(kx-\omega t\right)$

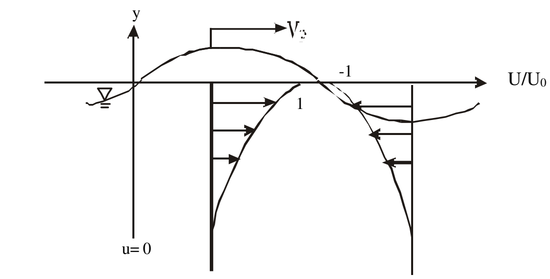
$u = U_o = A\omega\frac{1}{\tanh kh}\cos\left(kx-\omega t\right)$ $\frac{u}{U_o} = \frac{\cosh k(y+h)}{\cosh kh}\begin{cases}\sim e^{ky} & \text{deep water}\\ \sim 1 & \text{shallow water}\end{cases}$	$v = v_o = A\omega\sin\left(kx-\omega t\right) = \frac{\partial\eta}{\partial t}$ $\frac{v}{v_o} = \frac{\sinh k(y+h)}{\sinh kh}\begin{cases}\sim e^{ky} & \text{deep water}\\ \sim 1+\frac{y}{h} & \text{shallow water}\end{cases}$

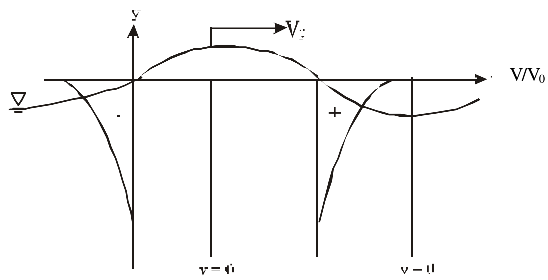
Finite Depth





Deep Water





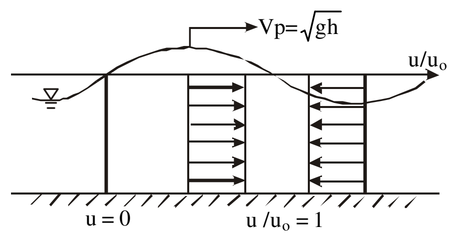
Rule of thumb $\dfrac{u}{u_o} = \dfrac{v}{v_o} \approx 4\%$ at $y = -\dfrac{\lambda}{2}$

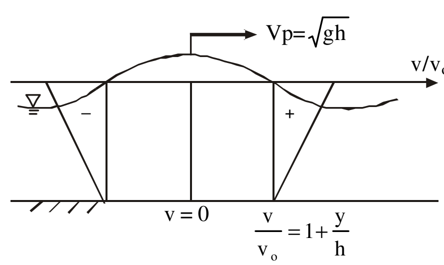
$(\cosh \ kh \sim 1, \sinh \ kh \sim kh)$

Shallow water / Long waves: kh << 1

$$u = \frac{A\omega}{kh}\cos(kx - \omega t) = \eta\sqrt{\frac{g}{h}}$$

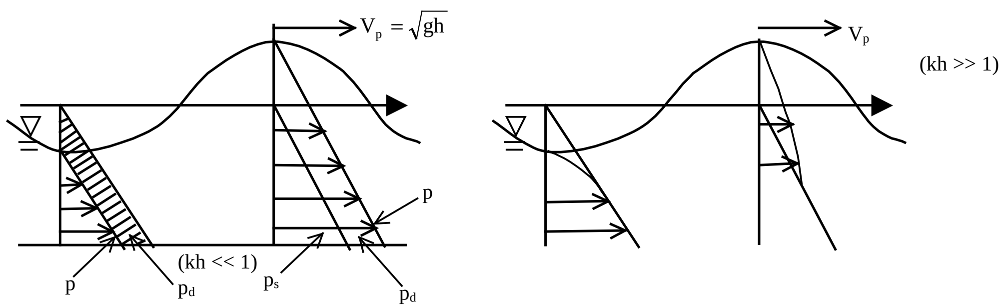
$$v = A\omega\left(1 + \frac{y}{h}\right)\sin(kx - \omega t)$$





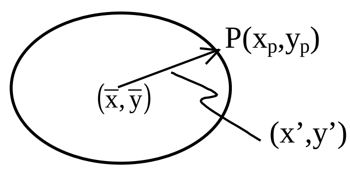
<u>Pressure Field:</u> dynamic pressure $p_d = -\rho \frac{\partial \phi}{\partial t}; p = p_d - \rho g y$

Finite depth	Deep water
$p_d = \rho g A \frac{\cosh k(y+h)}{\cosh kh} \cos(kx - \omega t)$ $= \rho g \frac{\cosh k(y+h)}{\cosh kh} \eta$ $\frac{p_d}{p_{d_o}}$ same picture as $\frac{u}{u_o}$; $\frac{p_d(-h)}{p_{d_o}} = \frac{1}{\cosh kh}$ shallow water: $p_d = \rho g \eta$ (no decay) $p = \rho g(\eta - y) \leftarrow$ "hydrostatic" approximation	$p_d = \rho g e^{ky} \eta$ $p = \rho g \left[\eta e^{ky} - y\right]$ $\frac{p_d(-h)}{p_{d_o}} = e^{-ky}$



Particle Orbit/ Velocity (Lagrangian).

Let $x_p(t), y_p(t)$ be the position of the particle P, then $x_p(t) = \bar{x} + x'(t); y_p(t) = \bar{y} + y'(t)$ where $(\bar{x}; \bar{y})$ is the mean position of P.



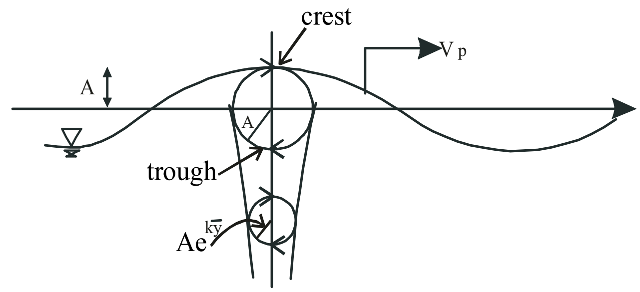
$$
\begin{aligned}
v_p &\approx v\left(\bar{x},\bar{y},t\right)\left(\bar{x},\bar{y}\right) \\
u_p &= \frac{dx_p}{dt} = u\left(\bar{x},\bar{y},t\right) + \underbrace{\frac{\partial u}{\partial x}\left(\bar{x},\bar{y},t\right)x' + \frac{\partial u}{\partial y}\left(\bar{x},\bar{y},t\right)y' + \ldots}_{\textbf{ignore - linear theory}}
\end{aligned}
$$

$$
\begin{aligned}
x_p &= \bar{x} + \int dt\ \ u\left(\bar{x},\bar{y},t\right) = \bar{x} + \int dt\ \ \omega A\frac{\cosh k\left(\bar{y}+h\right)}{\sinh kh}\cos\left(k\bar{x}-\omega t\right) \\
x' &= -A\frac{\cosh k\left(\bar{y}+h\right)}{\sinh kh}\sin\left(k\bar{x}-\omega t\right) \\
y' &= \int dt\ \ v\left(\bar{x},\bar{y},t\right) = \int dt\ \ \omega A\frac{\sinh k\left(\bar{y}+h\right)}{\sinh kh}\sin\left(k\bar{x}-\omega t\right) \\
&= A\frac{\sinh k\left(\bar{y}+h\right)}{\sinh kh}\cos\left(k\bar{x}-\omega t\right)\ \ \textbf{on}\ \bar{y}=0, y' = A\cos\left(k\bar{x}-\omega t\right) = \eta
\end{aligned}
$$

$$
\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \ \textbf{or}\ \frac{\left(x_p-\bar{x}\right)}{a^2} + \frac{\left(y_p-\bar{y}\right)}{b^2} = 1
$$

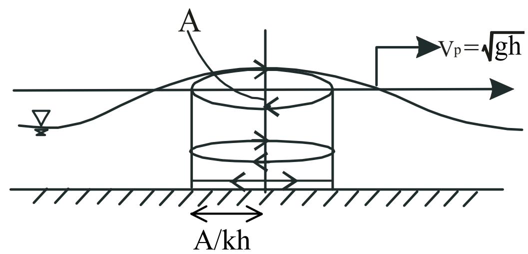
where $a = A\frac{\cosh k(\bar{y}+h)}{\sinh kh}; b = A\frac{\sinh k(\bar{y}+h)}{\sinh kh}$, i.e. the particle orbits form closed ellipses with horizontal and vertical axes a and b.

(a) deep water $kh >> 1$: $a = b = Ae^{k\bar{y}}$
circular orbits with radii $Ae^{k\bar{y}}$ decreasing exponentially with depth

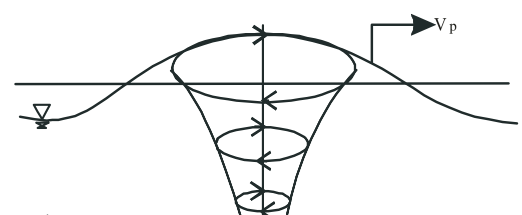


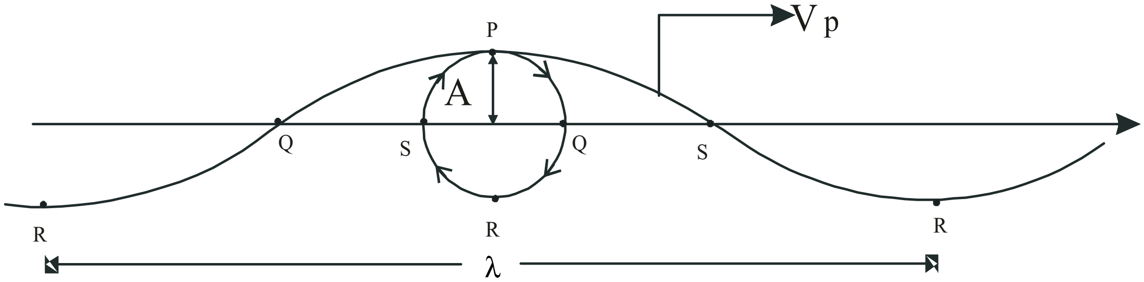
(b) shallow water $kh << 1$:

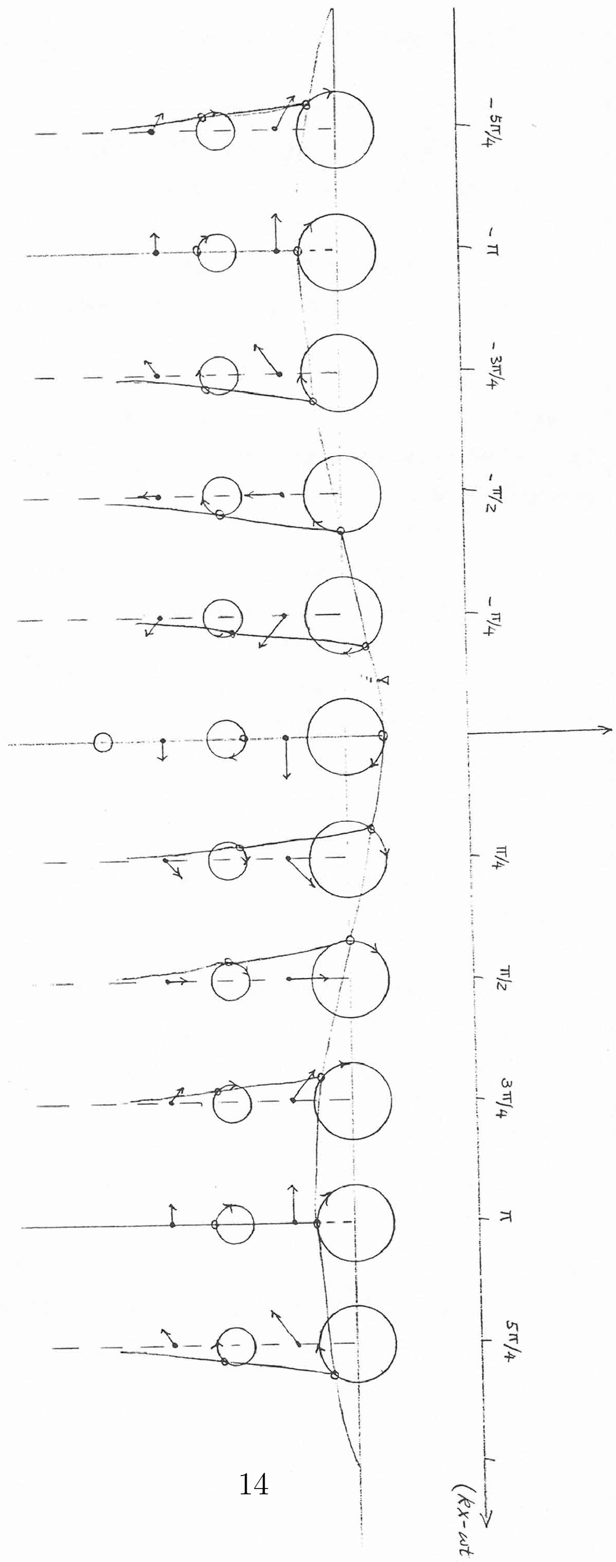
$a = \frac{A}{kh} =$ const. ; $b = A(1+\frac{y}{h})$
decreases linearly with depth



(c) Intermediate depth






$-5\pi/4$
$-\pi$
$-3\pi/4$
$-\pi/2$
$-\pi/4$
$\pi/4$
$\pi/2$
$3\pi/4$
π
$5\pi/4$
$(kx-\omega t)$

Summary of Plane Progressive Wave Characteristics

$f(y)$	Deep water/ short waves $kh > \pi$ (say)	Shallow water/ long waves $kh << 1$
$\frac{\cosh k(y+h)}{\cosh kh} = f_1(y) \sim$ e.g.p_d	e^{ky}	1
$\frac{\cosh k(y+h)}{\sinh kh} = f_2(y) \sim$ e.g.u, a	e^{ky}	$\frac{1}{kh}$
$\frac{\sinh k(y+h)}{\sinh kh} = f_3(y) \sim$ e.g. v, b	e^{ky}	$1 + \frac{y}{h}$

$C(x) = \cos(kx - \omega t)$	$S(x) = \sin(kx - \omega t)$
$\frac{\eta}{A} = C(x)$	
$\frac{u}{A\omega} = C(x)\, f_2(y)$	$\frac{v}{A\omega} = S(x)\, f_3(y)$
$\frac{p_d}{\rho g A} = C(x)\, f_1(y)$	
$\frac{y'}{A} = C(x)\, f_3(y)$	$\frac{x'}{A} = -S(x)\, f_2(y)$
$\frac{a}{A} = f_2(y)$	$\frac{b}{A} = f_3(y)$