13.021 – Marine Hydrodynamics
Lecture 7

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Chapter 3 – Ideal Fluid Flow

Ideal fluid: inviscid ($\nu = 0$) and incompressible ($\frac{D\rho}{Dt} = 0$) fluid.

$$R_e = \frac{\text{inertia}}{\text{viscous}} = \frac{UL}{\nu}$$

For 'typical' problems we are interested in: (L $\geq$ 1m, U $\geq$ 1m/s) $\nu_{water} = 10^{-6} m^2/s$

$$\frac{\nu}{UL} = R_e^{-1} << 1; (\leq 10^{-6}),$$

i.e. viscous effect << inertial effects, so an ideal fluid is a good approximation.

Governing Equations

- Continuity:

$$\nabla \cdot \vec{v} = 0$$

- Momentum (Navier Stokes $\Rightarrow$ Euler equation):

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p - g\hat{j}$$

By neglecting the viscous stress term ($\nu \nabla^2 \vec{v}$) in the Navier-Stokes equation, it reduces to the Euler equation. N-S is a second order p.d.e. (2^{nd} order in ∇^2), but Euler eq. is a first order p.d.e.

Boundary Conditions for Euler equations (Ideal Flow):

- KBC:

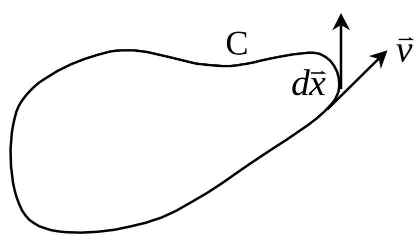
$$\vec{v} \cdot \hat{n} = \underbrace{\vec{u} \cdot \hat{n} = U_n}_{\text{given}} \quad \leftarrow \text{“No Flux” - free slip}$$

Note: ”No slip” condition $\vec{v} \cdot \hat{t} = \vec{U} \cdot \hat{t}$ does not apply since $\nu = 0$.

- DBC: $p = \ldots$ Given (pressure) (Cannot specify tangential τ_{ij} since $\nu = 0$)

Cirulation – Kelvin’s Theorem

Γ :Circulation (around closed contour C)



$$\Gamma = \int_C \underbrace{\vec{v} \cdot d\vec{x}}_{\substack{\text{tangential} \\ \text{velocity}}}$$

where C is an arbitrary contour. Γ is instantaneous, Eulerian idea, a ”snapshot”.

Kelvin’s Theorem (KT) :

For ideal fluid under conservative body forces,

$$\frac{d\Gamma}{dt} = 0 \text{ for any material contour } C,$$

i.e., Γ remains constant. Proof: cf JNN pp 103 (Mathematical Proof). This is a statement of conservation of angular momentum.

Kinematics of a small deformable body:

1. Uniform translation $\rightarrow$ Linear momentum
2. Rigid body rotation $\rightarrow$ Angular Momentum
3. Pure strain$\rightarrow$ no linear or angular momentum involved.(No change in volume)
4. Volume dilation

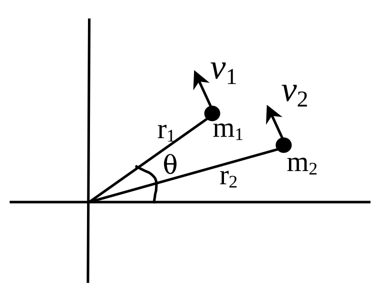
For ideal fluid under conservative body forces:

1. Can change
2. **By K.T., cannot change.**
3. Can change
4. not allowed (incompressible fluid).

Angular momentum is conserved: Ideal flow.

1. Angular Momentum $\times$ angular velocity $\vec{\omega}$.

   e.g.



Angular momentum:

$$\vec{L} = \vec{r} \times \vec{v} = mvr = mr^2\dot{\theta}$$

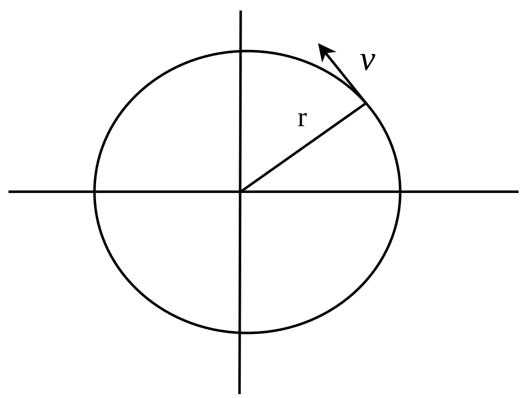
Conservation of angular momentum:

$$m_1 v_1 r_1 = m_2 v_2 r_2,$$

but $m_1 = m_2 \Rightarrow v_1 r_1 = v_2 r_2$.

Note: conservation of angular momentum does not imply constant angular velocity.

2. A circular material volume V_m.



$$\int_0^{2\pi} d\theta r_1 v_1 = \int_0^{2\pi} d\theta r_2 v_2$$

3. For arbitrary material volume V_m, C_m

$$\Gamma_1 = \int_{C_1} \vec{v}_1 \cdot d\vec{x} = \int_{C_2} \vec{v}_2 \cdot d\vec{x} = \Gamma_2$$

Vorticity:

$$\vec{\omega} = \nabla \times \vec{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial x}{\partial z} \right) \hat{i} + \left(\frac{\partial x}{\partial z} - \cdots \right) \hat{j} + \cdots$$

Relationship of vorticity to circulation - Apply Stokes' Theorem:

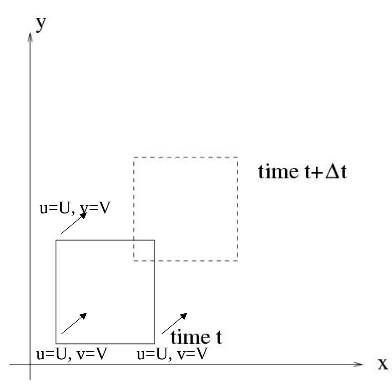
$$\Gamma = \oint_C \vec{v} \cdot d\vec{x} = \iint_S \left(\nabla \times \vec{v} \right) \cdot \hat{n} dS \text{ where } \oiint_C \vec{v} \cdot d\vec{x} = \oiint_S \vec{\omega} \cdot \hat{n} dS = \text{ Flux of vorticity out of } S$$

What is vorticity?

For example, special case: 2D flow - $w = 0$; $\frac{\partial}{\partial z} = 0$; $\omega_y = \omega_x = 0$ and

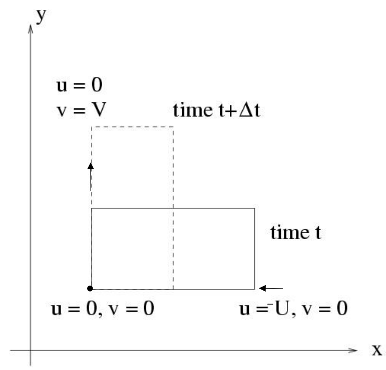
$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

1. Translation: $u =$ constant, $v =$ constant



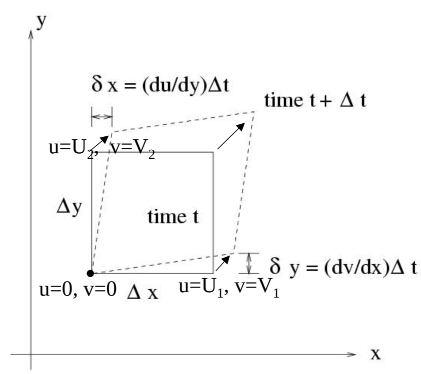
$$\frac{\partial v}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \Rightarrow \omega_z = 0 \rightarrow \text{no vorticity}$$

2. Pure Strain: (no change in volume)



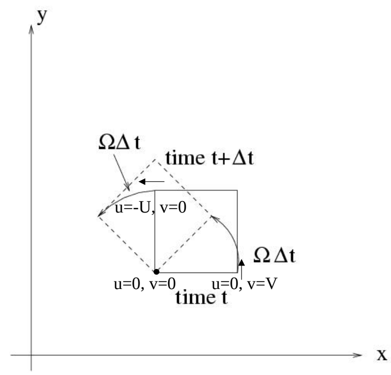
$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}; \quad \text{u = -v}; \quad \frac{\partial u}{\partial y} = 0; \quad \frac{\partial v}{\partial x} = 0 \Rightarrow \omega_z = 0$$

3. Angular deformation



$$\vec{\omega} = 0 \text{ only if } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \rightarrow \delta x = \delta y (\text{for } \Delta x = \Delta y)$$

4. Pure Rotation



Pure rotation with angular velocity Ω

$$\frac{\partial v}{\partial x} = \Omega; \quad \frac{\partial u}{\partial y} = -\Omega; \quad \omega_z = 2\Omega$$

i.e. vorticity $\propto$ 2(angular velocity).

Irrotational Flow:

$$\vec{\omega} \equiv 0 \text{ everywhere } \Leftrightarrow \Gamma \equiv 0 \text{ for any } C$$

Suppose at $t = t_o$, flow *is* irrotational, i.e. $\Gamma \equiv 0$ for all C. Then for ideal fluid under conservative body forces, Kelvin's theorem states that $\Gamma \equiv 0$ for all C for all time t. i.e., once irrotational, always irrotational (Special case of Kelvin's theorem).