6.434J/16.391J	Mar 2
Statistics for Engineers and Scientists	Handout #6
MIT, Spring 2006	Due: Mar 9

Problem Set 3

Problem 1: Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform p.d.f. $f(x|\theta) = 1/\theta$, for $0 < x < \theta$ and for some unknown parameter $\theta > 0$.

(a) Find a maximum likelihood estimator of θ, say T_n.

(b) Find a bias of T_n.

(c) Based on (b), derive an unbiased estimator of θ, say W_n.

(d) [*Extra Credit*] Compare variances of T_n and W_n.

(e) [*Extra Credit*] Show that T_n is a consistence sequence of estimators.

Problem 2: Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables, each $\mathcal{N}(\mu, \sigma^2)$ with both μ and σ^2 unknown:

$$(\mu, \sigma^2) \in \Theta \triangleq \{(x, y) \,|\, -\infty < x < +\infty, y > 0\}.$$

(a) Find a maximum likelihood estimator of (μ, σ^2).

(b) Suppose that $n = 131$ and that $X_1, X_2, \ldots, X_{131}$ are taken to be the average temperature observed at Boston, MA, since the 1872 that are given in Table 1. Use MATLAB to find the numerical values of the estimator of (μ, σ^2) for this data set, which is available as `hw3dataset.m` at:

`http://web.mit.edu/fmkashif/www/`
`6.434J-16.391J.htm`

Problem 3: Let $X_1, X_2, \ldots, X_n$ be independent random variables, each $\mathcal{N}(\mu, 1)$. Find an unbiased estimator of μ^2 that is a function of $\overline{X}_n$.
[*Hint*: Consider a bias of $(\overline{X}_n)^2$.]

Problem 4: A one-bit analog-to-digital (A/D) converter is defined by a single threshold α and has two outputs, 0 and 1. Suppose a random variable X with probability density function $f_X(.)$ is input to this A/D and the output is defined as Y.

(a) Find the MMSE estimator of x based on the observation y. (Question does not ask for estimator to be linear.)

(b) The output of the A/D is input to a binary symmetric channel characterized by a single parameter $0 \leq p \leq 1$. Let Z be the output of the channel, with the following conditional pdf:

$$f_{Z|Y}(z|y) = \begin{cases} p, & z \neq y \\ 1-p, & z = y \end{cases}$$

Find the optimum (MMSE) estimator of x based on the observation z.

(c) For the special cases of $p = 0$, $p = 1$ and $p = 0.5$, interpret the results of part (b).

Problem 5: Suppose X and Y are jointly Gaussian, and $z = Fx + g$, where F and g are known. Show that MMSE estimator of z given y is given by $\hat{z} = F\hat{x} + g$, where $\hat{x}$ is the MMSE estimator of x given y. Find an expression for the mean square estimation error of z given y.

Table 1: Average temperatures (in F) at Boston, MA. (*source*: National Weather Service Eastern Region)

Year	Average	Year	Average	Year	Average	Year	Average
1872	53.0	1899	50.1	1926	49.0	1953	53.6
1873	48.2	1900	50.9	1927	51.8	1954	51.4
1874	48.6	1901	49.0	1928	51.3	1955	51.4
1875	46.5	1902	49.7	1929	51.4	1956	50.6
1876	47.9	1903	49.5	1930	52.3	1957	52.5
1877	50.1	1904	47.1	1931	53.0	1958	50.0
1878	50.2	1905	49.1	1932	52.4	1959	51.8
1879	48.4	1906	50.0	1933	50.9	1960	51.4
1880	50.2	1907	48.7	1934	48.8	1961	51.0
1881	49.6	1908	51.1	1935	48.9	1962	49.8
1882	48.9	1909	50.5	1936	49.8	1963	51.0
1883	47.9	1910	50.8	1937	51.2	1964	50.1
1884	49.0	1911	50.9	1938	51.1	1965	49.6
1885	47.6	1912	50.6	1939	49.8	1966	51.4
1886	48.3	1913	52.3	1940	48.5	1967	49.5
1887	48.5	1914	49.7	1941	51.2	1968	50.4
1888	47.3	1915	51.2	1942	50.7	1969	51.2
1889	50.6	1916	49.7	1943	49.9	1970	50.9
1890	49.1	1917	47.9	1944	50.9	1971	51.2
1891	50.4	1918	49.8	1945	51.0	1972	50.4
1892	49.4	1919	51.1	1946	51.7	1973	53.0
1893	47.9	1920	50.0	1947	51.2	1974	50.9
1894	50.3	1921	52.4	1948	50.7	1975	52.8
1895	49.8	1922	51.3	1949	53.6	1976	52.2
1896	49.2	1923	50.3	1950	51.2	1977	52.5
1897	49.9	1924	50.4	1951	52.2	1978	50.3
1898	50.8	1925	51.7	1952	52.5	1979	52.1

Table 1: (Continued)

Year	Average	Year	Average	Year	Average	Year	Average
1980	50.6	1986	50.8	1992	50.2	1998	53.0
1981	51.5	1987	50.4	1993	51.6	1999	52.7
1982	51.0	1988	51.3	1994	52.2	2000	50.6
1983	53.2	1989	50.3	1995	51.4	2001	52.5
1984	51.6	1990	53.2	1996	50.9	2002	56.2
1985	51.0	1991	53.4	1997	50.9		