Problem Set 6

Problem 1: Consider the following hypothesis testing problem. There are k i.i.d. observations, $X_1, X_2, \ldots, X_k$. The hypotheses are

$$\begin{aligned} \mathsf{H}_1 : \quad & X_i \sim \mathcal{N}(0, \sigma_1^2), \text{ for } i = 1, 2, \ldots, k \\ \mathsf{H}_0 : \quad & X_i \sim \mathcal{N}(0, \sigma_0^2), \text{ for } i = 1, 2, \ldots, k, \end{aligned}$$

where $\sigma_1 > \sigma_0 > 0$ are constants.

(a) Find the likelihood ratio $\Lambda(\mathbf{x})$, for $\mathbf{x} \triangleq (x_1, x_2, \ldots, x_k)$.

(b) Assume the threshold is η. Thus, the likelihood ratio test is of the form

$$\Lambda(x) \underset{H_0}{\overset{H_1}{\gtrless}} \eta\,.$$

Show that a sufficient statistic for a true hypothesis is $l(X_1, X_2, \ldots, X_k) = \sum_{i=1}^{k} X_i^2$.

(c) Compute the threshold γ for the test

$$l(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma\,.$$

in terms of η, σ_0, and σ_1.

(d) Find expressions for probability of false alarm, P_{F}, and probability of miss, P_{M}.

(e) What is the threshold for the minimax criterion when the costs are $C_{01} = C_{10} = 1$ and $C_{00} = C_{11} = 0$?

Problem 2: Let $Y = \sum_{i=1}^{N} X_i$ denote an observation, which is a sum of N random variables. Here, X_i's are statistically independent random variables, each with a Gaussian density $\mathcal{N}(0, \sigma^2)$, for some constant σ. The number of variables in the sum is itself a random variable, independent of random variables X_i's, with a Poisson distribution

$$\mathbb{P}\{N = n\} = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, 2, \ldots .$$

Again, λ is a known constant.

Find the likelihood ratio test for deciding the two hypotheses,

$$\begin{aligned} \mathsf{H}_1 : & \qquad N \leq 1 \\ \mathsf{H}_0 : & \qquad N > 1. \end{aligned}$$

Problem 3: Let $N_i \sim \mathcal{N}(0, \sigma^2)$ be i.i.d. Gaussian random variables, for $i = 1, \ldots, n$. Consider a hypothesis testing problem, with two hypotheses,

$$\begin{aligned} \mathsf{H}_0 : & \qquad X_i = N_i \\ \mathsf{H}_1 : & \qquad X_i = a + N_i, \end{aligned}$$

where $a > 0$ is a constant.

(a) Design the likelihood ratio test with threshold η.

(b) Given that $\frac{a^2}{\sigma^2} = 0.5$, find the minimum number of samples n to have

$$\begin{aligned} P_{\mathrm{F}} & \leq 0.05 \\ P_{\mathrm{D}} & \geq 0.9. \end{aligned}$$

Problem 4: A binary hypothesis testing is made on the basis of the following observed statistics

$$\begin{aligned} \mathsf{H}_0 : & \qquad f_X(x \,|\, \mathsf{H}_0) = \frac{x}{\sigma^2} \exp\left\{ -\frac{x^2}{2\sigma^2} \right\} U(x) \\ \mathsf{H}_1 : & \qquad f_X(x \,|\, \mathsf{H}_1) = \frac{x}{\sigma^2} \exp\left\{ -\frac{x^2 + \alpha^2}{2\sigma^2} \right\} I_0\left(\frac{\alpha x}{\sigma^2} \right) U(x). \end{aligned}$$

where α and σ are known, $U(\cdot)$ is a unit step function, and $I_0(\cdot)$ is a Bessel function of imaginary order.

(a) Specify the likelihood ratio test if $\mathbb{P}\{H = \mathsf{H}_0\} = \mathbb{P}\{H = \mathsf{H}_1\} = \frac{1}{2}$.

(b) Write an expression for the probability of false alarm, P_{F}.

(c) Write an expression for the probability of detection, P_{D}.

(d) Write an expression for the probability of missed detection, P_{M}.

(e) Write an expression for the probability of error, P_{E}.

Problem 5: The probability density of $X_1, X_2, \ldots, X_n$ on the two hypotheses is

$$f_{X_i \mid H}(x_i \mid \mathsf{H}_k) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{-\frac{(x_i - m_k)^2}{2\sigma_k^2}\right\},$$

for $k \in \{0, 1\}$. Assume that the observations are independent.

(a) Find the likelihood ratio, $\Lambda(x_1, \ldots, x_n)$. Express the test in terms of the following quantities

$$l_\alpha = \sum_{i=1}^{n} x_i$$
$$l_\beta = \sum_{i=1}^{n} x_i^2.$$

(b) Consider a special case when

$$m_0 = 0$$
$$m_1 = 0$$
$$\sigma_1^2 = \sigma_s^2 + \sigma_n^2$$
$$\sigma_0 = \sigma_n,$$

where σ_s, σ_n are known. Find the decision regions.

Problem 6: The observation X is a Gaussian random variable with one of these five densities:

$$f_{X\,|\,H}\,(x\,|\,\mathsf{H}_k) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-m_k)^2}{2\sigma^2}\right\}.$$

Here, $-\infty < x < \infty$, $1 \leq k \leq 5$, and

$$\begin{aligned} m_1 &= -2m \\ m_2 &= -m \\ m_3 &= 0 \\ m_4 &= m \\ m_5 &= 2m. \end{aligned}$$

The hypotheses are equally likely to occur, and the criterion is the minimum error probability P_{E}.

(a) Draw the decision regions on the real axis.

(b) Compute the error probability.