Hydrostatic Pressure in Liquids

$$\gamma = \rho g = \text{specific weight} = \left[N/m^3 \right]$$

$$\sigma = 5.67\cdot 10^{-8} \left[\frac{W}{m^2K^4}\right]$$

Power= Force * Velocity = VI

$$p_2 - p_1 = -\gamma\left(z_2 - z_1\right) \quad F = \int p dA$$

$$p_{down} = p^{up} + \gamma\left|\Delta z\right|$$

Any two points at the same elevation in a continuous mass of same static fluid will be at the same pressure.

Accelerating liquids

$$\theta = \tan^{-1}\frac{a_x}{g + a_z} = \text{angle between horiz and surface}$$

Buoyancy

$$F_B = \gamma \cdot \text{Volume Displaced}$$

$$\bar{x}_{BB'} = \bar{x}_{B'} - \bar{x}_B = \frac{1}{V_{immersed}} \int_{V_{immersed}} x dV$$

$$MB = \frac{\left|\bar{x}_{BB'}\right|}{\tan\theta} = \frac{I_O}{V_{submurged}} = MG + GB \quad \text{stable if } MG > 0$$

$$I_O = \int_{waterline} x^2 dA \approx \int_{\text{original waterline}} x^2 dA$$

Reynolds transport theorem

$$\vec{v}_{rel} = \text{velocity},\ M = \text{mass},\ \vec{n} = \text{outward normal}$$

$$\frac{dB_{CM}}{dt} = \frac{d}{dt}\int_{CV} \rho b dV + \int_{CS} \rho b\left(\vec{v}_{rel}\cdot\vec{n}\right)dA \quad b = B/M$$

Conservation of mass

$$\frac{d}{dt}\int_{CV}\rho dV = -\int_{CS}\rho\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

$$\frac{dM_{CV}}{dt} = \sum_{in}\dot{m}_{in} - \sum_{out}\dot{m}_{out}$$

$$\int_{cv}\frac{\partial\rho}{\partial t} + \int_{cs}\rho\left(\mathbf{V}\cdot\mathbf{n}\right)dA = \left(\frac{dm}{dt}\right)_{system}$$

Linear Momentum

$$\vec{n}_u = \text{unit vector}, \vec{n} = \text{normal vector, outward positive}$$

$$\frac{d}{dt}\int_{CV}\rho\vec{v}dV = \int_{CS}\left(-P\vec{n}_u\right)dA + \int_{CS}\left(\vec{\tau}\right)dA + \int_{CV}\rho\vec{g}dV - \int_{CS}\rho\vec{v}\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

$$\frac{d}{dt}\int_{CV}\rho\vec{v}_{rel}dV = \int_{CS}\left(-P\vec{n}_u\right)dA + \int_{CS}\left(\vec{\tau}\right)dA + \int_{CV}\rho\vec{g}dV + \sum_{in}\left(\dot{m}\vec{v}_{rel}\right)_{in} - \sum_{out}\left(\dot{m}\vec{v}_{rel}\right)_{out}$$

$$\sum F = \frac{d}{dt}\int_{CV}\rho\vec{v}_{rel}dV + \int_{CS}\rho\vec{v}\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

Bernoulli Equation

Important to approximate the unsteady term.

$$P_{absolute} = P_{gauge} + P_{atm}$$

$$\int_1^2\frac{\partial\vec{v}}{\partial t}ds + \int_1^2\frac{1}{\rho}dp + \frac{1}{2}\left(\vec{v}_2^{\,2} - \vec{v}_1^{\,2}\right) + g\left(z_2 - z_1\right) = 0$$

$$\frac{p_2 - p_1}{\rho} + \frac{\left(\vec{v}_2^{\,2} - \vec{v}_1^{\,2}\right)}{2} + g\left(z_2 - z_1\right) = 0$$

Angular Momentum

$$\frac{d}{dt}\left[\int_{CV}\left(\vec{r}\times\vec{v}_{rel}\right)\rho dV\right] = \sum\left(\vec{r}\times\vec{F}_{surface}\right) + \int_{CV}\left(\vec{r}\times\vec{g}\right)\rho dV + \vec{T}_{shaft} - \int_{CS}\left(\vec{r}\times\vec{v}_{rel}\right)\rho\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

$$\frac{d}{dt}\left[\int_{CV}\left(\vec{r}\times\vec{v}_{rel}\right)\rho dV\right] = \sum\left(\vec{r}\times\vec{F}_{surface}\right) + \int_{CV}\left(\vec{r}\times\vec{g}\right)\rho dV + \vec{T}_{shaft} + \sum_{in}\dot{m}_{in}\left(\vec{r}\times\vec{v}_{rel}\right)_{in} - \sum_{out}\dot{m}_{out}\left(\vec{r}\times\vec{v}_{rel}\right)_{out}$$

$$\sum\mathbf{M}_o = \frac{d}{dt}\left(\int_{cv}\left(\mathbf{r}\times\vec{v}_{rel}\right)\rho dV\right) + \int_{cs}\left(\mathbf{r}\times\vec{v}_{rel}\right)\rho\left(\vec{v}_{rel}\cdot\mathbf{n}\right)dA$$

Pipe Flows

$$\mathrm{R_e} = \frac{\rho\vec{v}d}{\mu} = \frac{\vec{v}d}{\nu} \qquad d = \text{pipe diameter} \quad \nu = \frac{\mu}{\rho}$$

$$tran = 2300$$

$$\frac{p_1}{\rho} + \frac{1}{2}\alpha_1\vec{v}_1^{\,2} + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}\alpha_2\vec{v}_2^{\,2} + gz_2 + gh_f$$

$$h_f = f\frac{L}{d}\frac{\vec{v}^2}{2g} \quad f = \frac{8\tau_w}{\rho\vec{v}^2} \quad h_f = f\frac{L}{d}\frac{\vec{v}^2}{2g} = \Delta z + \frac{\Delta p}{\rho g}$$

$$h_{f,lam} = \frac{64\mu}{\rho\vec{v}d}\frac{L}{d}\frac{\vec{v}^2}{2g} = \frac{128\mu LQ}{\pi\rho g d^4}$$

$$\frac{1}{\sqrt{f}} \approx -1.8\log\left[\frac{6.9}{\mathrm{Re}_d} + \left(\frac{\varepsilon/d}{3.7}\right)^{1.11}\right]$$

$$x = fcn\left(\mathrm{Re}_d\right) = \frac{gd^3h_f}{Ln^2} = \frac{f\,\mathrm{Re}_d{}^2}{2}$$

$$\mathrm{Re}_d = -\sqrt{8x}\,\log\left(\frac{e/d}{3.7}+\frac{1.775}{\sqrt{x}}\right)$$

$$Q = \int u\,dA$$

First Law

$$\left(E_2-E_1\right)_{iso} = 0 \quad \left(E_2-E_1\right) = Q_{1-2}-W_{1-2}$$

$$E_{\text{uncoupled sys (coupled)}} = E_{kin}+E_{ela}+E_{grav}+U_{ther}+\left(U_{coupled}\right)$$

$$-dW = F\cdot dr \qquad -W_{1-2} = \int_{r_1}^{r_2} F\cdot dr$$

$$\dot{Q}_{1-2}-\dot{W}_{1-2} = \frac{dU}{dt}$$

Constitutive Relations

$$E_{kin} = \frac{1}{2}mv^2 \;\; F = ma = m\frac{dv}{dt} \;\; -W_{1-2} = \int_{v_1}^{v_2} mv\,dv$$

$$E_{ela} = \frac{1}{2}kx^2 \quad F = kx$$

$$E_{gra} = mgh \quad F = mg$$

$$U_{the} = CT_1 = mcT_1 \qquad S_2-S_1 = mc\ln\left(\frac{T_2}{T_1}\right)$$

$$PV = mRT \qquad dW_{rev} = P_1dV \qquad W_{1-2} = \int_{V_1}^{V_2} P_1dV$$

$$U_{gas} = mc_vT \quad c_p = c_v+R \quad g = \frac{c_p}{c_v} \quad PV^g = \text{constant}$$

$$Ma = \frac{\vec{v}}{\sqrt{gRT}}$$

$$S_2-S_1 = mc_p\ln\left(\frac{T_2}{T_1}\right)-mR\ln\left(\frac{P_2}{P_1}\right)$$

$$S_2-S_1 = mc_v\ln\left(\frac{T_2}{T_1}\right)+mR\ln\left(\frac{V_2}{V_1}\right)$$

$$S_2-S_1 = mc_v\ln\left(\frac{P_2}{P_1}\right)+mc_p\ln\left(\frac{V_2}{V_1}\right)$$

Reversible adiabatic

$$-W_{1-2} = mc_vT_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{g-1}{g}}-1\right] = mc_vT_1\left[\left(\frac{V_2}{V_1}\right)^{1-g}-1\right]$$

Second Law

$$\left(S_2-S_1\right)_{iso} \geq 0 = S_{gen} \qquad S_{trans}{}^{AB} = \int_{1_A}^{2_A}\frac{dQ}{T_A}$$

$$\left(S_2-S_1\right) = \int_{1_A}^{2_A}\frac{dQ}{T_A}+S_{gen}$$

Flow Field

$$r\,g_x-\frac{\partial p}{\partial x}+m\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right) = r\,\frac{du}{dt}$$

$$r\,g_y-\frac{\partial p}{\partial y}+m\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}+\frac{\partial^2v}{\partial z^2}\right) = r\,\frac{dv}{dt}$$

$$r\,g_z-\frac{\partial p}{\partial z}+m\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}+\frac{\partial^2w}{\partial z^2}\right) = r\,\frac{dw}{dt}$$

$$t_{xy} = m\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \quad t_{xz} = m\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

$$t_{zy} = m\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$F_{shear} = b\int_0^x t_w\left(x\right)dx$$

Heat Transfer

Conduction

$$\dot{Q} = -kA_c\left(\frac{dT}{dx}\right) \quad k = \left[\frac{W}{mK}\right] \quad \dot{Q} = \frac{kA_c}{L}\left(T_1-T_2\right)$$

Convection

$$\dot{Q} = h_cA_s\left(T_s-T_\infty\right)$$

Radiation

$$\dot{Q} = h_rA_s\left(T_1-T_2\right) \quad h_r = 4e_1s\,T_m{}^3 \quad T_m = \left(\frac{T_1+T_2}{2}\right)$$

Thermal Resistance

$$\dot{Q} = \left(\frac{T_1-T_2}{R_{total}}\right)$$ convection in parallel with radiation

$$R_{cond} = \frac{L}{kA} \qquad R_{conv} = \frac{1}{h_cA} \qquad R_{rad} = \frac{1}{h_rA}$$

$$R_{cond,radial} = \frac{\ln\left(R_o/R_i\right)}{2p\,kL}$$

Heat Diffusion Equation alpha = thermal diffusivity

Cartesian

$q\&_{gen}$ = rate of energy dissipated per unit mass by all energy storage methods other than thermal. $q\&$= rate of energy generated ie

$$q\&_{gen} = \frac{q\&}{r}$$

$$\frac{\partial T}{\partial t} = \frac{k}{rc}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{q\&}{rc} \quad a=\frac{k}{rc}$$

Cylindrical

$$rc_p\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial f}\left(k\frac{\partial T}{\partial f}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+q\&$$

Spherical

$$rc_p\frac{\partial T}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2 q}\frac{\partial}{\partial f}\left(k\frac{\partial T}{\partial f}\right)+\frac{1}{r^2\sin q}\frac{\partial}{\partial q}\left(k\sin q\frac{\partial T}{\partial q}\right)+q\&$$

Entropy Generation Equation

$$s\&_{gen} = \frac{k}{rT^2}\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2+\left(\frac{\partial T}{\partial z}\right)^2\right]+\frac{q\&_{gen}}{T}$$

$S\&_{gen} = r s\&_{gen}dxdydz$ s=per unit mass

Fin Temperature Distributions

$$q\equiv T-T_\infty \quad q_b = q(0)=T_b-T_\infty \quad m^2\equiv\frac{hP}{kA_c}$$

$$M\equiv\sqrt{hPkA_c}\,q_b$$

Convection $hq(L) = -k\left.\frac{dq}{dx}\right|_{x=L}$

$$\frac{q}{q_b}=\frac{\cosh m(L-x)+\left(h/mk\right)\sinh m(L-x)}{\cosh mL+\left(h/mk\right)\sinh mL}$$

$$q_f = M\frac{\sinh mL+\left(h/mk\right)\cosh mL}{\cosh mL+\left(h/mk\right)\sinh mL}$$

Adiabatic

$$\left.\frac{dq}{dx}\right|_{x=L}=0 \quad \frac{q}{q_b}=\frac{\cosh m(L-x)}{\cosh mL} \quad q_f = M\tanh mL$$

Prescribed Temperature

$$q(L)=q_L \quad \frac{q}{q_b}=\frac{\left(q_L/q_b\right)\sinh mx+\sinh m(L-x)}{\sinh mL}$$

$$q_f = M\frac{\cosh mL - q_L/q_b}{\sinh mL}$$

Infinite fin

$$q(L)=0 \quad \frac{q}{q_b}=e^{-mx} \quad q_f = M$$

Fin Efficiency

$$h_f = \frac{q_f}{q_{\max}}=\frac{q_f}{hA_f q_b}$$

Total array efficiency

$$A_{total}=NA_{fin}+A_{\text{exposed surface}} \quad Q\&_{\max}=h_cA_{total}q_b$$

$$h_{surface}=1-\frac{NA_{fin}}{A_{total}}\left(1-h_{fin}\right) \quad Q\&_{total}=h_{surface}Q\&_{\max}$$

Biot Number

$$L_c = V/A_s$$

$$Bi=\frac{\text{internal conduction resistance}}{\text{external convection resistance}}=\frac{L/k_{solid}A_c}{1/h_cA_s}\approx\frac{h_cL_c}{k_{solid}}$$

$$Q\&_{1-2}-W\&_{1-2}=\frac{dU}{dt}$$

$Bi = 1$ Lumped thermal capacitance model

Reversible in terms of the solid. Principal resistance to heat transfer lies within the fluid. Temperature of solid can be modeled as uniform at all times even though it is changing in time.

$$a=\frac{k}{rc}\quad t=\frac{rcV}{h_cA_s}=\left(\frac{1}{h_cA_s}\right)(rcV)=R_{th}C_{th}$$

$$\frac{q}{q_i}=e^{-\frac{h_cA_s}{rcV}t}=e^{-Bi\cdot Fo}\quad Fo=\frac{a\,t}{L_c^2}=\frac{t}{t_c}$$

tc is time for disturbance to diffuse characteristic length

$$\frac{h_cA_st}{rcV}=\frac{h_ct}{rcL_c}=\left(\frac{h_cL_c}{k_{solid}}\right)\left(\frac{k_{solid}}{rc}\right)\left(\frac{t}{L_c^2}\right)=\left(\frac{h_cL_c}{k_{solid}}\right)\left(\frac{at}{L_c^2}\right)=Bi\cdot Fo$$

$$rcV\frac{dT}{dt}=-h_cA_s\left(T-T_\infty\right)\quad \left(S_{gen}\right)_{solid}=0$$

$$\left(S_{gen}\right)_{fluid}=\frac{(rcV)_{solid}}{2}\left(\frac{\Delta T}{T_\infty}\right)^2=(rcV)_{solid}\left[\frac{\Delta T}{T_\infty}-\ln\left(1+\frac{\Delta T}{T_\infty}\right)\right]$$

$$\left(S_{trans}\right)_{solid}=\int_0^{t_{final}}\frac{Q\&}{T}dt=\int_0^{t_{final}}\frac{h_cA_s\left(T-T_\infty\right)}{T}dt=rcV\ln\left(\frac{T_f}{T_i}\right)$$

$Bi\to\infty$ Fluid behaves as heat reservoir. Principal resistance to heat transfer lies within the solid. Temperature of fluid can be modeled as uniform at all times even though it is changing in time.

$$\frac{q}{q_b}=\sum_{n=0}^{\infty}\frac{2(-1)^n}{\left(n+\frac{1}{2}\right)p}e^{-\left(n+\frac{1}{2}\right)^2p^2Fo}\cos\left(n+\frac{1}{2}\right)p\frac{x}{L}$$

$$\frac{\dot{Q}}{A_c} = \frac{2k_{solid}(T_i - T_\infty)}{L} e^{-\left(\frac{\pi}{2}\right)^2 Fo} \quad Fo \geq .2$$

$$\frac{\dot{Q}}{A_c} = \frac{k_{solid}(T_i - T_\infty)}{\sqrt{\pi\alpha t}} \quad Fo \leq .05$$

Semi Infinite Model

$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ with boundary conditions

$$\theta = \frac{T-T_i}{T_s-T_i} \quad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$erf\eta = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du \quad erfc\eta = 1 - erf\eta$$

Case I Constant Surface temperature

$$\theta = \frac{T-T_i}{T_s-T_i} = erfc\eta \quad \frac{\dot{Q}}{A_c} = \frac{k_{solid}(T_s-T_i)}{\sqrt{\pi\alpha t}}$$

Case II Constant Heat Flux at Surface

$$T-T_i = \frac{\dot{Q}}{Ak}\left[\sqrt{\frac{4\alpha t}{\pi}}\, e^{-\frac{x^2}{4\alpha t}} - x\,erfc\left(\frac{x}{\sqrt{4\alpha t}}\right)\right]$$

Case III Convective Heat Transfer at the Surface

$$\frac{T-T_i}{T_s-T_i} = erfc\eta - e^{\left[\frac{h_c x}{k} + \left(\frac{h_c}{k}\right)^2 \alpha t\right]} erfc\left[\eta + \frac{h_c}{k}\sqrt{\alpha t}\right]$$

Case IV Surface Energy Pulse

$$T-T_i = \frac{E}{\rho c\sqrt{\pi\alpha t}} e^{-\frac{x^2}{4\alpha t}}$$

Case V Periodic Variation of Surface Temperature

$$\frac{T-T_i}{T_s-T_i} = e^{-x\sqrt{\omega/2\alpha}} \sin\left[\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right]$$

Two Semi-Infinite Solids in Simple Thermal Communication

$$T_s = \frac{T_{A,i}\sqrt{(k\rho c)_A} + T_{B,i}\sqrt{(k\rho c)_B}}{\sqrt{(k\rho c)_A}+\sqrt{(k\rho c)_B}}$$

Reversible Cycles

Carnot Cycle

$$\oint dS_{transfer} = \oint\frac{dQ}{T} = \frac{Q_H}{T_H}+\frac{Q_L}{T_L}=0$$

$$\oint dW = (T_H-T_L)\frac{Q_H}{T_H} = Q_H\left(1-\frac{T_L}{T_H}\right)$$

$$\eta = \frac{\oint dW}{Q_H} \qquad \eta_{rev} = \frac{Q_H\left(1-\frac{T_L}{T_H}\right)}{Q_H} = 1-\frac{T_L}{T_H}$$

$$COP = \frac{Q_L}{-\oint dW} = \frac{1}{-\frac{Q_H}{Q_L}-1} \qquad COP_{rev} = \frac{1}{\frac{T_H}{T_L}-1}$$

$$COP_{HP} = \frac{Q_H}{\oint dW} = \frac{Q_H}{Q_H+Q_L} \qquad (COP_{HP})_{rev} = \frac{1}{1-\frac{T_L}{T_H}}$$

The energy transferred in the form of a heat transfer from a higher temperature source has a higher “quality” (greater value for energy conversion purposes) than the energy transferred as a heat transfer from a lower temperature source because it carries less entropy.

$$\oint dS_{transfer} + \oint dS_{gen} = \frac{Q_H}{T_H}+\frac{Q_L}{T_L}+\oint dS_{gen} = 0$$

$$\oint(dW)_{rev} - \oint(dW)_{irrev} = T_L\oint dS_{gen}$$

$$\eta_{irrev} = \eta_{rev} - \frac{T_L}{Q_H}\oint dS_{gen}$$

$$COP_{irrev} = \frac{1}{\frac{T_H}{T_L}-1+\frac{T_H}{Q_L}\oint dS_{gen}}$$

Temperature Distributions

Plane Wall (x=0 at middle of wall, x= +L or –L at ends of wall, -L=T1

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1-\frac{x^2}{L^2}\right)+\frac{T_{s,2}-T_{s,1}}{2}\frac{x}{L}+\frac{T_{s,1}+T_{s,2}}{2}$$

$$q''(x) = \dot{q}x - \frac{k}{2L}(T_{s,2}-T_{s,1})$$

$$q(x) = \left[\dot{q}x - \frac{k}{2L}(T_{s,2}-T_{s,1})\right]A_x$$

Cylindrical Wall

$$T(r) = T_{s,2}+\frac{\dot{q}r_2^2}{4k}\left(1-\frac{r^2}{r_2^2}\right) - \left[\frac{\dot{q}r_2^2}{4k}\left(1-\frac{r_1^2}{r_2^2}\right)+(T_{s,2}-T_{s,1})\right]\frac{\ln(r_2/r)}{\ln(r_2/r_1)}$$

$$q''(r) = \frac{\dot{q}r}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1-\frac{r_1^2}{r_2^2}\right)+(T_{s,2}-T_{s,1})\right]}{r\ln(r_2/r_1)}$$

$$R_{cond} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

Spherical Wall

$$T(r) = T_{s,2}+\frac{\dot{q}r_2^2}{6k}\left(1-\frac{r^2}{r_2^2}\right) - \left[6\frac{\dot{q}r_2^2}{4k}\left(1-\frac{r_1^2}{r_2^2}\right)+(T_{s,2}-T_{s,1})\right]\frac{1/r-1/r_2}{1/r_1-1/r_2}$$

$$q''(r) = \frac{\dot{q}r}{3} - \frac{k\left[\frac{\dot{q}r_2^{\,2}}{6k}\left(1-\frac{r_1^{\,2}}{r_2^{\,2}}\right)+\left(T_{s,2}-T_{s,1}\right)\right]}{r^2\left[1/r_1 - 1/r_2\right]}$$

$$R_{cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Summary of Governing Relations for a Control Volume

u = thermal energy $v = 1/\rho$ =specific volume

$$h = u + Pv\ \left[\frac{J}{kg}\right]\quad H = U + PV = mh$$

Reynolds Transport Theorem

$$\frac{dB_{CM}}{dt} = \frac{d}{dt}\int_{CV}\rho b\,dV + \int_{CS}\rho b\left(\vec{J}\cdot\vec{n}\right)dA\quad b = B/M$$

Conservation of Mass (Continuity Equation)

$$\frac{d}{dt}\int_{CV}\rho\,dV = -\int_{CS}\rho\left(\vec{J}\cdot\vec{n}\right)dA\quad \frac{dM_{CV}}{dt} = \sum_{in}\dot{m}_{in} - \sum_{out}\dot{m}_{out}$$

First Law of Thermodynamics

$$\frac{d}{dt}\int_{CV}\rho\left(h+\frac{v^2}{2}+gz\right)dV = \dot{Q}-\dot{W}_{shaft} - \int_{CS}\rho\left(h+\frac{v^2}{2}+gz\right)\left(\vec{v}_r\cdot\vec{n}\right)dA$$

$$\frac{dE_{CV}}{dt} = \dot{Q}-\dot{W}_{shaft} + \sum_{in}\dot{m}_{in}\left(h+\frac{v^2}{2}+gz\right)_{in} - \sum_{out}\dot{m}_{out}\left(h+\frac{v^2}{2}+gz\right)_{out}$$

Second Law of Thermodynamics

$$\frac{d}{dt}\int_{CV}\rho s\,dV = \sum_i\left(\frac{\dot{Q}}{T}\right)_i - \int_{CS}\rho s\left(\vec{J}\cdot\vec{n}\right)dA + \dot{S}_{gen}$$

$$\frac{dS_{CV}}{dt} = \sum_i\left(\frac{\dot{Q}}{T}\right)_i + \sum_{in}\left(\dot{m}s\right)_{in} - \sum_{out}\left(\dot{m}s\right)_{out} + \dot{S}_{gen}$$

Linear Momentum Equation

$$\frac{d}{dt}\int_{CV}\rho\vec{J}\,dV = \int_{CS}\rho\left(-P\vec{n}\right)dA + \int_{CS}\left(\vec{t}\right)dA + \int_{CV}\rho\vec{g}\,dV + \int_{CS}\rho\vec{J}\left(\vec{J}_r\cdot\vec{n}\right)dA$$

$$\frac{d}{dt}\int_{CV}\rho\vec{J}\,dV = \int_{CS}\rho\left(-P\vec{n}\right)dA + \int_{CS}\left(\vec{t}\right)dA + \int_{CV}\rho\vec{g}\,dV + \sum_{in}\left(\dot{m}\vec{J}\right)_{in} - \sum_{out}\left(\dot{m}\vec{J}\right)_{out}$$

Angular Momentum Equation

$$\frac{d}{dt}\left[\int_{CV}\left(\vec{r}\times\vec{J}\right)\rho\,dV\right] = \sum\left(\vec{r}\times\vec{F}_{surface}\right) + \int_{CV}\left(\vec{r}\times\vec{g}\right)\rho\,dV + \vec{T}_{shaft} - \int_{CS}\left(\vec{r}\times\vec{J}\right)\rho\left(\vec{J}_r\cdot\vec{n}\right)dA$$

$$\frac{d}{dt}\left[\int_{CV}\left(\vec{r}\times\vec{J}\right)\rho\,dV\right] = \sum\left(\vec{r}\times\vec{F}_{surface}\right) + \int_{CV}\left(\vec{r}\times\vec{g}\right)\rho\,dV + \vec{T}_{shaft} + \sum_{in}\dot{m}_{in}\left(\vec{r}\times\vec{J}\right)_{in} - \sum_{out}\dot{m}_{out}\left(\vec{r}\times\vec{J}\right)_{out}$$

general coordinate information

$$dA = r\,dr\,d\theta$$

quasi static= passing through a series of equilibrium states.

$\text{Acceleration} := \frac{m}{s^2}$ $\text{Area} := m^2$

$\text{Density} := \frac{kg}{m^3}$ $\text{Energy} := J$

$J = 1 \cdot kg \cdot m^2 \cdot s^{-2}$ $\text{Force} := N$

$N = 1 \cdot kg \cdot m \cdot s^{-2}$ $\text{HeatTransferRate} := W$

$W = 1 \cdot kg \cdot m^2 \cdot s^{-3}$ $\text{Heatflux} := \frac{W}{m^2}$

$\text{HeatGenRate} := \frac{W}{m^3}$ $\text{HeatTransferCoeff} := \frac{W}{m^2 \cdot K}$

$\text{KViscosity} := \frac{m^2}{s}$ $\text{LatentHeat} := \frac{J}{kg}$

$\frac{J}{kg} = 1 \cdot m^2 \cdot s^{-2}$ $\text{Length} := m$

$\text{Mass} := kg$ $\text{MassDensity} := \frac{kg}{m^3}$

$\text{MassFlowRate} := \frac{kg}{s}$ $\text{MassTransferCoef} := \frac{1}{m}$

$\text{Power} := W$ $W = 1 \cdot kg \cdot m^2 \cdot s^{-3}$

$\text{PressureStress} := \frac{N}{m^2}$ $\text{SpecificHeat} := \frac{J}{kg \cdot K}$

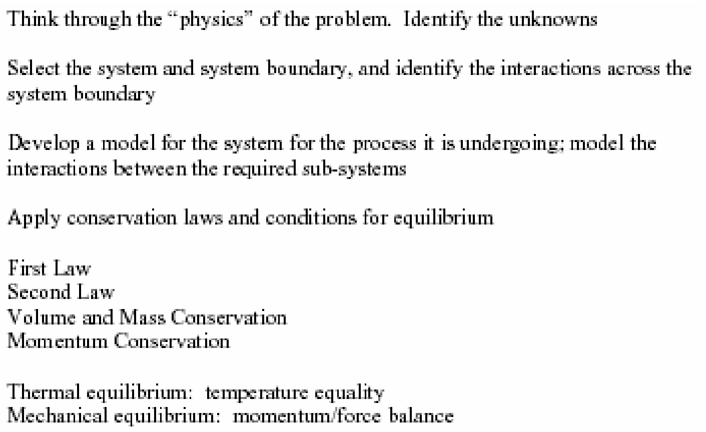
$\text{Temperature} := K$ $\text{TempDiff} := K$

$\text{ThermalCond} := \frac{W}{m \cdot K}$ $\frac{W}{m \cdot K} = 1 \cdot kg \cdot m \cdot s^{-3} \cdot K^{-1}$

$\text{ThermalResist} := \frac{K}{W}$ $\frac{K}{W} = 1 \cdot kg^{-1} \cdot m^{-2} \cdot s^{3} \cdot K$

$\text{DViscosity} := \frac{N \cdot s}{m^2}$ $\text{Volume} := m^3$

$\text{VolumeFlowRate} := \frac{m^3}{s}$



Think through the “physics” of the problem. Identify the unknowns

Select the system and system boundary, and identify the interactions across the system boundary

Develop a model for the system for the process it is undergoing; model the interactions between the required sub-systems

Apply conservation laws and conditions for equilibrium

First Law
Second Law
Volume and Mass Conservation
Momentum Conservation

Thermal equilibrium: temperature equality
Mechanical equilibrium: momentum/force balance

Entropy Transfer

The 2nd law of Thermodynamics is written as follows.

$$S_2 - S_1 = \int_1^2 \frac{\delta Q}{T_{boundary}} + S_{gen}, \text{ where } S_{gen} \geq 0.$$

In the RHS, we have two terms. The first one is called *entropy transfer*, which is due to heat transfer across the system boundary. The other is the *entropy generation* term within the system. The sum of these two is the total *change of entropy* of the system, which is in the LHS. The entropy generation is always non-negative. This is one method to state the 2nd law of Thermodynamics.

Usually, direct evaluation of entropy generation within a system is almost impossible. So, we usually try to compute entropy transfer and the total change of entropy to evaluate associated entropy generation. However, the problem is that evaluation of entropy transfer is not a simple matter either.

There are three important cases, where you can evaluate the entropy transfer easily.

The first case is when the system boundary is considered as *adiabatic*. There is no heat transfer, so there will be no entropy transfer. This case is trivial but still important.

The second case is when the system is modeled as a *heat reservoir*. Because a heat reservoir has constant temperature at any instance, and is *usually* considered to have uniform temperature distribution including its boundary, the entropy transfer is given as follows.[1]

$$\int_1^2 \frac{\delta Q}{T_{boundary}} = \int_1^2 \frac{\delta Q}{T_{HR}} = \frac{Q_{HR,1-2}}{T_{HR}}.$$

The third case is when the heat transfer process is assumed to be *quasi-static*. In this case, the temperature distribution in the system is uniform, so the boundary temperature is same as the temperature of the system. Also, we can use differential form of the 1st law, because the system experiences a series of infintesimal state changes between equilibrium states. Therefore,

$$\int_1^2 \frac{\delta Q}{T_{boundary}} = \int_1^2 \frac{\delta Q}{T_{system}}, \text{ where } \delta Q - \delta W = dE_{system}.$$

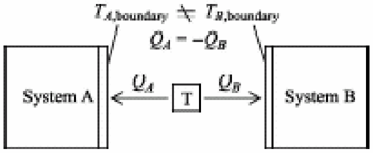
Using the 1st law, we can usually express δQ as a function of T_{system}. Then, we can put it into the expression, and evaluate the integral.

Note that the evaluation of entropy transfer is sensitive to how you model the system. If you model something as a heat reservoir, it will give a different result from the one you get with a system modeled as a pure thermal system. You need to think physically to justify your modeling in general.

Although the principles mentioned so far are simple, it seems that some students are confused by the different ways of evaluating entropy transfer[2], so I'll try to clarify some points here.

Major Point Misunderstood:

Entropy transfer from system A into system B always has same size as, but only has opposite sign from entropy transfer from system B into system A.



In general, entropy transfer cannot be dealt in the same way you deal heat or work transfers. Assuming a quasi-static process, we have

$$\delta Q_A = -\delta Q_B.$$

However, the boundary temperature of system A and system B can be different. So,

$$-\int_1^2 \frac{\delta Q_A}{T_{A,boundary}} \neq \int_1^2 \frac{\delta Q_B}{T_{B,boundary}}.$$

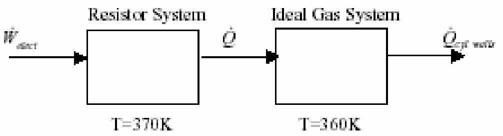
You may argue that the boundary temperatures of two systems should be same in principle. If you considered heat transfer processes only by conduction and convection, it might be right.[3] However, one important heat transfer process violates it. *Radiation* process is an interaction at a distance, so the boundary temperatures of two systems need not be same to each other.

Then, what happened in (c) of Problem 2 in Quiz 2? The solution[4] said: *the entropy transferred in from the resistor is equal to 0.2703 W/K*, which has the same size as the entropy transfer out from the resistor!

Now, we can examine the case with two different methods.

Method 1 (My naive method only based on thermodynamic models)

In the model given in the solution, we had



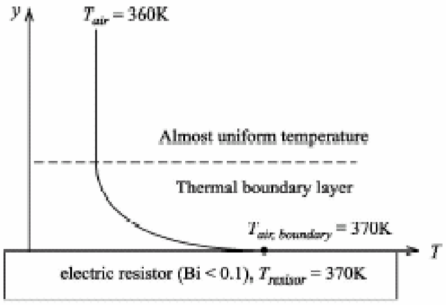
If we think that the ideal gas system has a uniform temperature distribution, the answer is

$$\dot{S}_{transfer\ in,gas} = \frac{\dot{Q}}{T_{gas}} = \frac{100\,\text{W}}{360\,\text{K}} = 0.2778\,\text{W/K},$$

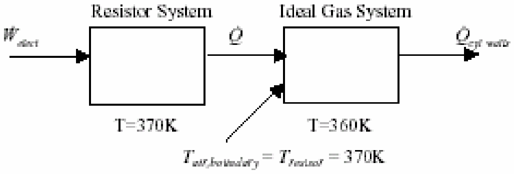
which has different size from the entropy transfer out from the resistor. In this case, entropy generation in the ideal gas system is 0, and all entropy generation is due to the electric resistor and the *boundary between the resistor and the gas*, where heat transfer occurs across a finite temperature difference.

Method 2 (More physical way based on knowledge of heat transfer)

We have learned a little bit of heat transfer, so let's think about the reality here. The electric resistor itself would have thin coil type geometry, and so, a small Biot number is expected. So, we may assume that it has a uniform temperature. On the other hand, within the air, we will have a thermal boundary layer as shown in the following figure.



Note that the heat transfer process would be mostly due to convection, so the continuity of temperature would hold. Hence, although we didn't explicitly mention it in the model, the model implies that the boundary temperature would be same as the temperature of the resistor. If we use this additional physical argument, the correct model would look like the following figure.



With this physical reinterpretation of the model, the conclusion made in the solution of (c) of Problem 2 in Quiz 2 is valid. As we can see, in this case, the temperature gradient physically lies within the gas, so *the entropy generation due to heat transfer process across finite temperature difference is in the gas.*

As you can see, the conclusion is dependent on how you model and interpret the system. Method 1 and Method 2 give different results at least apparently. However, they are only different ways of seeing the same phenomenon. For example, they will give *same answer for the entropy generation for the total system including the gas and the resistor*. The difference only comes from how you model the thermal boundary layer. If it is considered as a boundary between two subsystems, you are following Method 1. If it is part of the gas, you are following Method 2.[5] In either case, *you should explain your model carefully to make it understood.*

Final IMPORTANT Remark:

You should use Method 2, because now you have enough knowledge of heat transfer process, which can be used to decide the boundary temperature. Method 2 has been also used in the official Quiz solution, as you know. Hence, I believe that it is officially accepted in MIT.

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[1] I put 'usually' because now we have a slightly improved version of heat reservoir model, which can have its boundary temperature different from its inside temperature, T_{HR}. Read Footnote 5 carefully.

[2] Specifically, the difference between my way of explaining Problem 2 in Quiz 2 and the *more physical* way seems to make some confusion.

[3] Actually, the assumption of temperature continuity is not always a valid model even for conduction processes. For more information, see 'contact resistance' section in Incropera & DeWitt.

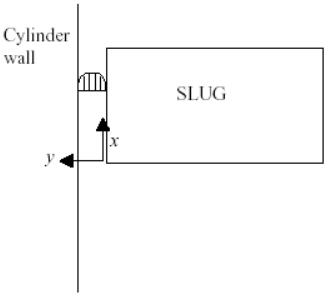
[4] During preparation of this material, I found that the solution had a wrong unit. In the solution, the entropy transfer rate was given with the unit J/K, but it should be W/K.

[5] The same analogy applies to Problem 2 in Problem Set 9. Now, heat reservoirs can have its boundary temperature different from its inside temperature, and you may figure out what the boundary temperature would be by analysis of heat transfer process. Then, entropy transfer can be calculated based on the temperature at the boundary and the amount of heat transfer experienced. The entropy change still can be calculated based on the temperature inside and the amount of heat transfer experienced. Then, the entropy transfer for a heat reservoir is now given by $S_{HR,transfer} = \int \frac{\delta Q}{T_{HR,boundary}}$, which is in general different from the entropy change of the heat reservoir $\Delta S_{HR} = \int \frac{\delta Q}{T_{HR}}$.

Problem 1 [1]

(a) To find out the shear force acting on the slug, we need to compute the shear stress on the wall. To do that, we model the flow through the clearance between the slug and the cylinder as a highly viscous flow, which is fully developed and steady. Then, the differential equation for the velocity field is

$$\frac{d^2u}{dy^2} = \mu \frac{dP}{dx}.$$



The boundary conditions are

$u(0) = 0,\ u(\Delta t) = 0.$

Solving this equation gives the flow field:

$$u(y) = -\frac{1}{2\mu}\frac{dP}{dx}(\Delta t\, y - y^2)$$

Also, from mass conservation, we have

$$Q = \pi D \int_{y=0}^{y=\Delta t} u(y)\, dy = -\frac{\pi D \Delta t^3}{12\mu}\frac{dP}{dx}.$$

Therefore,

$$\frac{dP}{dx} = -\frac{12\mu Q}{\pi D \Delta t^3},\quad u(y) = \frac{6Q}{\pi D \Delta t}\left(\frac{\Delta t\, y - y^2}{\Delta t^2}\right)$$

The wall shear force is given by

$$F_{shear} = \tau_w \pi D l = \mu \pi D l \left.\frac{du}{dy}\right|_{y=0} = \frac{6\mu Q}{\Delta t}\cdot\frac{l}{\Delta t}.$$

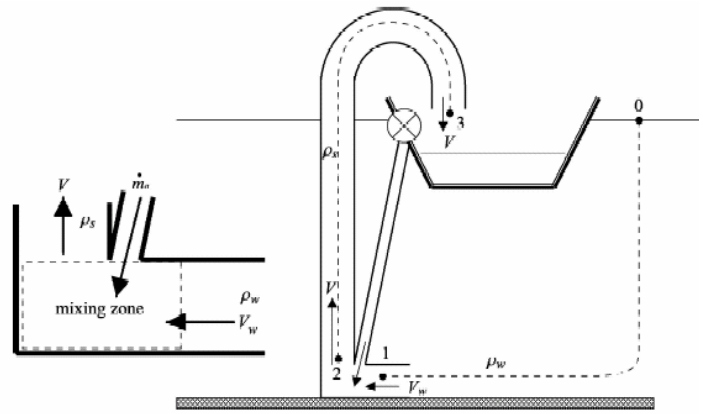
(b) The displacement, Δh, can be found by the force balance for the slug. That is,

$$k\Delta h = F_{shear} + \frac{\pi D^2}{4}\left[P_{bottom} - P_{top}\right] = \frac{6\mu Q}{\Delta t}\cdot\frac{l}{\Delta t} - \frac{\pi D^2}{4}\left[-\frac{12\mu Q}{\pi D \Delta t^2}\cdot\frac{l}{\Delta t}\right]$$

$$= \frac{6\mu Q}{\Delta t}\cdot\frac{l}{\Delta t} + \frac{3\mu Q}{\Delta t}\cdot\frac{l}{\Delta t}\frac{D}{\Delta t} = \frac{3\mu Q}{\Delta t}\cdot\frac{l}{\Delta t}\left[2 + \frac{D}{\Delta t}\right] \approx \frac{3\mu Q}{\Delta t}\cdot\frac{l}{\Delta t}\left[\frac{D}{\Delta t}\right]. \qquad \because D/\Delta t >> 1$$

$$\therefore \Delta h = \frac{3\mu Q}{k\Delta t}\cdot\frac{l}{\Delta t}\left[\frac{D}{\Delta t}\right].$$

Note that the shear force is much smaller than the force due to pressure drop. Also, we have neglected non-uniform pressure effect at each side, which will turn out to be small anyway.



(a) Because of mixing of air and water, which produces a lower density fluid at point 2, buoyancy drives the mixture up through the pipe.

(b) As described in (a), we use Bernoulli equation from 0 to 1 and from 2 to 3. That is,

$$P_2 + \frac{1}{2}\rho_s V^2 = P_3 + \rho_s gH + \frac{1}{2}\rho_s V^2 = P_{atm} + \rho_s gH + \frac{1}{2}\rho_s V^2.$$

$$\therefore P_2 = P_{atm} + \rho_s gH.$$

$$P_1 + \frac{1}{2}\rho_w V_w^2 = P_0 + \rho_w gH = P_{atm} + \rho_w gH.$$

$$\therefore P_1 = P_{atm} + \rho_w gH - \frac{1}{2}\rho_w V_w^2. \quad \text{(Eq.1)}$$

$P_1 \equiv P_2$ (mixing zone)

Therefore,

$$P_{atm} + \rho_s gH = P_{atm} + \rho_w gH - \frac{1}{2}\rho_w V_w^2.$$

$$V_w = \sqrt{2gH\left(1 - \frac{\rho_s}{\rho_w}\right)}.$$

(c) From Eq.1 in (b), we have

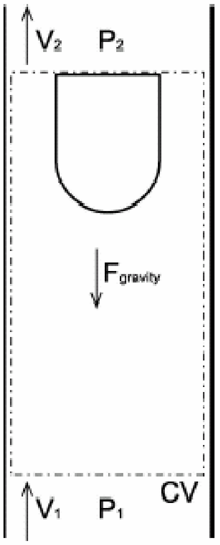
$P_{entry} = P_1 = P_{atm} + \rho_s gH.$ [1]

(d) From mass conservation for mixing zone, we have

$\dot{m}_a + \rho_w V_w A = \rho_s V A.$

Therefore,

$$V = \frac{\rho_w}{\rho_s}V_w + \frac{\dot{m}_a}{\rho_s A} = \frac{\rho_w}{\rho_s}\sqrt{2gH\left(1 - \frac{\rho_s}{\rho_w}\right)} + \frac{\dot{m}_a}{\rho_s A}.$$



(a) First, we take our CV as shown in the figure. We use mass conservation principle for the CV and Bernoulli eq. from section 1 to section 2.

$$\frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS}\rho(\vec{v}_{rel}\cdot\hat{n})\,dA = 0.$$

Here,

$$\frac{d}{dt}\int_{CV}\rho\, dV = 0.$$

$$\int_{CS}\rho(\vec{v}_{rel}\cdot\hat{n})\,dA = -\rho V_1 \cdot \pi R_0^2 + \rho V_2 \cdot \pi(R_0^2 - R^2).$$

$$V_1 = \frac{\dot{\forall}}{\pi R_0^2}.$$

$$\therefore V_2 = \frac{\dot{\forall}}{\pi(R_0^2 - R^2)}.$$

Bernoulli eq. becomes

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \rho gH + \frac{1}{2}\rho V_2^2.$$

$$\therefore P_2 = P_1 + \frac{1}{2}\rho\left(\frac{\dot{\forall}}{\pi R_0^2}\right)^2 - \left[\rho gH + \frac{1}{2}\rho\left(\frac{\dot{\forall}}{\pi(R_0^2 - R^2)}\right)^2\right]$$

$$= P_1 - \rho gH + \frac{\rho\dot{\forall}^2}{2\pi^2 R_0^4}\left[\frac{R^2(R^2 - 2R_0^2)}{(R_0^2 - R^2)^2}\right].$$

(b) We use linear momentum principle in vertical direction.

$$\frac{d}{dt}\int_{CV}\rho\, v_y\, dV + \int_{CS}\rho\, v_y(\vec{v}_{rel}\cdot\hat{n})\,dA_1 = \sum F_y.$$

For the CV, it becomes

$$\frac{d}{dt}\int_{CV}\rho\, v_y\, dV = 0. \text{ (steady state by requirement)}$$

$$\int_{CS}\rho\, v_y(\vec{v}_{rel}\cdot\hat{n})\,dA = -\rho\dot{\forall}V_1 + \rho\dot{\forall}V_2 = \frac{\rho\dot{\forall}^2}{\pi}\frac{R^2}{R_0^2(R_0^2 - R^2)}.$$

$$\sum F_y = F_{P1} - F_{P2} - F_{gravity}$$

$$F_{P1} - F_{P2} = \pi R_0^2 P_1 - \pi R_0^2 P_2 = \pi R_0^2\left\{\frac{\rho\dot{\forall}^2}{2\pi^2 R_0^4}\left[\frac{R^2(2R_0^2 - R^2)}{(R_0^2 - R^2)^2}\right]\right\} + \pi\rho\, g\, R_0^2 H$$

$$F_{gravity} = -\rho g\left[\pi R_0^2 H - \frac{8}{3}\pi R^3\right] - \rho_s g\cdot\frac{8}{3}\pi R^3 = -\pi\rho\, g\, R_0^2 H - \frac{8}{3}\pi(\rho_s - \rho)gR^3.$$

$$\therefore \sum F_y = \left\{\frac{\rho\dot{\forall}^2}{2\pi R_0^2}\left[\frac{R^2(2R_0^2 - R^2)}{(R_0^2 - R^2)^2}\right]\right\}.$$

Therefore,

$$\frac{\rho\dot{\forall}^2}{\pi R_0^2}\frac{R^2}{(R_0^2 - R^2)} = \left\{\frac{\rho\dot{\forall}^2}{2\pi R_0^2}\left[\frac{R^2(2R_0^2 - R^2)}{(R_0^2 - R^2)^2}\right]\right\} - \frac{8}{3}\pi(\rho_s - \rho)gR^3.$$

Solving for $\dot{\forall}$ yields

$$\dot{\forall}^2 = \frac{16}{3}\pi^2\left(\frac{\rho_s}{\rho} - 1\right)gR^3 R_0^2\left[\frac{(R_0^2 - R^2)}{R^2}\right]^2.$$

$$\therefore \dot{\forall} = 4\pi R R_0\left\{\left(\frac{R_0}{R}\right)^2 - 1\right\}\sqrt{\frac{1}{3}\left(\frac{\rho_s}{\rho} - 1\right)gR}.$$

Definitions and such

When to use Bernoulli.

1. Steady flow. (otherwise, use unsteady version and estimate unsteady term.
2. Incompressible flow- Mach number less than .3.
3. Frictionless flow

4. Flow along a single streamline
5. No shaft work-no pumps or turbines.
6. No heat transfer, ie adiabatic.

Heat Transfer

The energy transfer interaction that occurs between pure thermal systems

Temperature

The property of a system which indicates the potential for heat transfer with other systems. Two systems are unequal in temperature when they experience heat transfer.

For a heat transfer in the absence of work transfer, the heat transfer is positive for the low temperature system which increases in energy and negative for the high temperature systems which decreases in energy

A quasi -static process is a model for a dynamic process in which the state of the system is changing at a rate which is slow compared to the rate at which the system approaches the equilibrium state by means of energy and entropy transfer processes internal to the system boundary. Internally, the system appears to be in equilibrium at all times throughout the process even though its state is changing with time. The equilibrium properties thus provide a complete description of the state of the system at all times throughout the process.

A reversible process generates no entropy, is a sequence of equilibrium states, and proceeds in the reverse direction just as readily as in the forward direction and takes an infinitely long time to be carried out.

Head loss is the change in the sum of the pressure and gravity head, the change in height of the hydraulic grade line or height change of the energy grade line

System:

A system is defined as an quantity of matter or region of space to which attention is directed for purpose of analysis.

Boundary:

The quantity of matter or region of space which forms the system is delineated by a boundary, a surface either real or imaginary.

State and Property:

State is used to signify the condition of a system at a specific instant. The state of a system is characterized by a collection of observable, macroscopic quantities, called properties. A property is one of those observable macroscopic quantities which are definable at a particular instant without reference to the system's history.