S1 Omitted Proofs

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S1  Omitted Proofs

S1.1  Section 4: Equilibrium Analysis

Theorem 1: Transversality Conditions and Admissibility of the Candidate Equilibrium Strategy. Recall that the candidate value function is of the form

\[ V(\theta, M) = v_0 + v_1 \theta + v_2 M + v_3 \theta^2 + v_4 \theta M. \]

Let \( X_t := (\theta_t, M_t), t \geq 0 \). While the initial condition of \( X := (X_t)_{t \geq 0} \) in the game is random, verifying the optimality of consumer’s strategy requires evaluating payoffs at all histories of \( X_t, t \geq 0 \), i.e., at all possible realizations \( X_t = x \), where \( x \in \mathbb{R}^2 \) is deterministic. Thus, let \( (X_t^{Q,x})_{t \geq 0} \) denote the dynamic of \( X \) under an admissible strategy \( Q := (Q_t)_{t \geq 0} \) when the initial condition is \( x = (\vartheta, m) \in \mathbb{R}^2 \). In this proof, the expectation operator \( \mathbb{E}_0[\cdot] \)
conditions on this realized value, and the corresponding variance and covariance operators are also indexed by 0.

By Theorem 3.5.3 in Pham (2009), the transversality conditions to verify are:

1. For every \( x \in \mathbb{R}^2 \), and any admissible strategy \( Q \), \( \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[V(X_t^{Q,x})] \geq 0 \).

2. For every \( x \in \mathbb{R}^2 \), \( \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[V(X_t^{Q,x})] \leq 0 \) where \( \dot{Q}_t = \delta \mu + \alpha \theta_t^0 + \beta M_t^{Q,x}, \ t \geq 0 \).

We proceed in two lemmas.

**Lemma 1.** For any admissible strategy \( Q \),

\[
\lim_{t \to \infty} e^{-rt} \mathbb{E}_0[M_t^{Q,x}] = \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[\theta_t^0] = \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[(\theta_t^0)^2] = 0.
\]

Also, \( \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[(M_t^{Q,x})^2] < 0 \).

**Proof:** Let \( x = (\theta, m) \in \mathbb{R}^2 \). That \( \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[\theta_t^0] = \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[(\theta_t^0)^2] = 0 \) follows directly from \( (\theta_t^0)_{t \geq 0} \) being mean-reverting, as this implies that both the mean and variance of \( \theta_t \) are bounded. To see that \( \lim_{t \to \infty} e^{-rt} \mathbb{E}_0[M_t^{Q,x}] = 0 \), observe first that

\[
M_t^{Q,x} = me^{-\phi t} + [\mu - \phi \lambda Y][1 - e^{-\phi t}] + \int_0^t e^{-\phi(t-s)} \lambda Q_s ds + \int_0^t e^{-\phi(t-s)} \sigma_d Z_t, \ t \geq 0.
\]

Thus, it suffices to show that the transversality condition holds for \( J_t := \int_0^t e^{-\phi(t-s)} \lambda Q_s ds \), as the rest of the terms trivially vanish in the limit. By Cauchy-Schwarz, however,

\[
e^{-rt} \mathbb{E}_0[J_t] \leq \left( e^{-rt} \int_0^t e^{-2\phi(t-s)} \lambda^2 ds \right)^{1/2} \left( e^{-rt} \mathbb{E}_0[Q_s^2] ds \right)^{1/2} \to 0 \text{ as } t \to \infty.
\]

We claim that \( C(Q) < \infty \) for any admissible strategy. Let \( V^Q \) the corresponding payoff. By (iii) in the notion of admissibility, we can separate this payoff into

\[
V^Q = \mathbb{E}_0 \left[ \int_0^\infty e^{-rt}[\theta_t Q_t - Q_t^2/2]dt \right] - \mathbb{E}_0 \left[ \int_0^\infty e^{-rt} P_t Q_t dt \right],
\]

which are both finite. Also, since the first term is integrable, Fubini’s Theorem applies, and thus

\[
\mathbb{E}_0 \left[ \int_0^\infty e^{-rt}[\theta_t Q_t - Q_t^2/2]dt \right] = \int_0^\infty e^{-rt} \mathbb{E}_0[\theta_t Q_t] - \mathbb{E}_0[Q_t^2]/2 dt.
\]
Moreover, by Tonelli, $C(Q) = \int_0^\infty e^{-rt}E_0[Q_t^2]dt$. But since $E_0[\int_0^T Q_t^2 dt] < +\infty$ for all $T > 0$ (part (ii) in the definition of an admissible strategy), $C(Q) = +\infty$ implies that $E_0[Q_t^2] = O(e^{rt})$ for $\rho \geq r$ for large $t$. Using that $E_0[\theta_tQ_t] < (E_0[\theta_t^2])^{1/2}(E_0[Q_t^2])^{1/2}$, and that $E_0[\theta_t^2]$ is bounded due to mean reversion, we deduce that the tail $\int_T^\infty e^{-rt}[E_0[\theta_tQ_t] - E_0[Q_t^2]/2]dt$ cannot converge, and so the consumer’s payoff is $-\infty$, a contradiction. It follows that $\lim_{t \to \infty} e^{-rt}E_0[|J_t|] = 0$, and hence, that $\lim_{t \to \infty} e^{-rt}E_0[M_t^{Q,x}] = 0$.

To conclude, observe that in $(M_t^{Q,x})^2$ the only non-trivial terms are

$$K_t := \left( \int_0^t e^{-\phi(t-s)}Q_sds \right)^2 \quad \text{and} \quad L_t := \int_0^t e^{-\phi(t-s)}Q_sds \int_0^t e^{-\phi(t-s)}dZ_t^\xi.$$  

However, 

$$e^{-rt}E_0[K_t] \leq e^{-rt} \int_0^t e^{-2\phi(t-s)}dsE_0\left[\int_0^t Q_s^2ds\right] \leq \frac{1 - e^{-2\phi t}}{2\phi}C(Q) < \frac{C(Q)}{2\phi}.$$  

Also, by the same logic

$$e^{-rt}[L_t] \leq (e^{-rt}E_0[K_t])^{1/2} \left( e^{-rt}E_0\left[\left( \int_0^t e^{-\phi(t-s)}dZ_t^\xi \right)^2 \right] \right)^{1/2} \leq \left( \frac{C(Q)}{2\phi} \right)^{1/2} \left( \frac{1 - e^{-2\phi t}}{2\phi} \right)^{1/2}.$$  

Thus, $\lim_{t \to \infty} e^{-rt}E_0[L_t] = 0$, from where $\limsup_{t \to \infty} e^{-rt}E_0[(M_t^{Q,x})^2] < \infty$.

**Lemma 2.** (a) Under any admissible strategy $Q$, $\limsup_{t \to \infty} E_0[V(X_t^{Q,x})] \geq 0$. (b) Under the candidate equilibrium strategy $\hat{Q}$, $\lim_{t \to \infty} e^{-rt}E_0[V(X_t^{\hat{Q},x})] = 0$. (c) $(\hat{Q}_t)_{t \geq 0}$ is admissible.

**Proof.** To prove (a), we first show that $\lim_{t \to \infty} e^{-rt}E_0[\theta_t^2 M_t^{Q,x}] = 0$ for any admissible $Q$. Observe first that, by Cauchy-Schwarz,

$$|e^{-rt}E_0[\theta_t^2 M_t^{Q,x}]| \leq \sqrt{\left( e^{-rt}E_0[\theta_t^2]^2 \right)^{1/2} \left( e^{-rt}E_0[\theta_t^2 M_t^{Q,x}]^2 \right)^{1/2}}.$$  

Since $f, g > 0$, we have that $0 \leq \limsup_{t \to \infty} fg \leq \limsup_{t \to \infty} f \limsup_{t \to \infty} g$. But $\limsup_{t \to \infty} g < \infty$ and $\limsup_{t \to \infty} f = \lim_{t \to \infty} f = 0$, and so $\limsup_{t \to \infty} |e^{-rt}E_0[\theta_t^2 M_t^{Q,x}]| = 0$. However, it is easy to see that $\lim_{t \to \infty} e^{-rt}E_0[\theta_t^2 M_t^{Q,x}] = 0$ if and only if $\limsup_{t \to \infty} |e^{-rt}E_0[\theta_t^2 M_t^{Q,x}]| = 0$.

With this in hand, and using the previous lemma,

$$\lim_{t \to \infty} e^{-rt}E_0[v_0 + v_1\theta_t^2 + v_2 M_t^{Q,x} + v_4(\theta_t^2)^2 + v_5\theta_t^2 M_t^{Q,x}] = 0.$$  

Thus, $\limsup_{t \to \infty} e^{-rt}E_0[V(X_t^t)] = \limsup_{t \to \infty} e^{-rt}E_0[v_3(M_t^{Q,x})^2]$. However, the last term is non-negative due to $v_3 = (\alpha + 2\beta)/2\lambda > 0$. 

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To show (b), observe that under the candidate equilibrium strategy,
\[ dM_t = [-\phi M_t + \rho + \lambda(\alpha \theta_t + \beta M_t)]dt + \lambda \sigma_t dZ_t^\xi, \quad t \geq 0, \]
where \( \rho = \phi[\mu - \lambda Y] + \lambda \delta \mu \). Thus, the dynamics of \((\theta_t, M_t)_{t \geq 0}\) admit a solution given by
\[
\begin{align*}
\theta_t^\theta &= e^{-\kappa t} \theta + \mu[1 - e^{-\kappa t}] + \sigma_t^\theta \int_0^t e^{-\kappa(t-s)} dZ_s, \quad \text{and} \\
M_t^{Q,x} &= e^{-\phi \beta \lambda t} m + \rho \frac{1 - e^{\phi \beta \lambda t}}{\phi - \beta \lambda} + \lambda \alpha \int_0^t e^{-(\phi \beta \lambda)(t-s)} \theta_s^\theta ds + \sigma_t \int_0^t e^{-(\phi \beta \lambda)(t-s)} dZ_t^\xi.
\end{align*}
\]
Since \( E_0[\theta_t] \) is bounded over \( t \in \mathbb{R}_+ \), so is \( E_0[M_t^{Q,x}] \). Also, \( \text{Var}_0[\theta_t] = \sigma_t^2[1 - e^{-2\kappa t}] / 2 \kappa \). We conclude that \( \lim_{t \to \infty} e^{-\tau t} E_0[\Psi_t] = 0 \) for \( \Psi_t \in \{\theta_t, M_t^{Q,x}, \theta_t^2\} \). Furthermore,
\[
\text{Cov}_0[\theta_t^\theta, M_t^{Q,x}] = \lambda \alpha \int_0^t e^{-(\phi \beta \lambda)(t-s)} \text{Cov}_0[\theta_s^\theta, \theta_t^\theta] ds
\]
where \( \text{Cov}_0[\theta_t^\theta, \theta_s^\theta] = \sigma_t^2 e^{-\kappa(s-t)} - e^{-\kappa(t-s)} \), \( t \geq s \). Thus, \( \lim_{t \to \infty} e^{-\tau t} E_0[\theta_t^\theta M_t^{Q,x}] = 0 \).

Now, \( \text{Var}_0 \left[ \int_0^t e^{-(\phi \beta \lambda)(t-s)} dZ_t^\xi \right] = [1 - e^{-2(\phi \beta \lambda) t}] / [2(\phi - \beta \lambda)] \), which is bounded. Also,
\[
\begin{align*}
\text{Var}_0 \left[ \int_0^t e^{-(\phi \beta \lambda)(t-s)} \theta_s^\theta ds \right] &= \text{Cov}_0 \left[ \int_0^t e^{-(\phi \beta \lambda)(t-s)} \theta_s^\theta ds, \int_0^t e^{-(\phi \beta \lambda)(t-s)} \theta_s^\theta ds \right] \\
&= e^{-2(\phi \beta \lambda) t} \int_0^t \int_0^t e^{(\phi \beta \lambda)(u+v)} \text{Cov}_0[\theta_u^\theta, \theta_v^\theta] dudv,
\end{align*}
\]
where the last equality follows from integrability and Fubini’s Theorem. Since
\[
\text{Cov}_0[\theta_t^\theta, \theta_s^\theta] = \frac{\sigma_t^2}{2 \kappa} \left[ e^{-\kappa \max\{u,v\}} - e^{-\kappa(u+v)} \right],
\]
it is easy to verify that \( \text{Var}_0 \left[ \int_0^t e^{-(\phi \beta \lambda)(t-s)} \theta_s^\theta ds \right] \) is also bounded, and so \( \text{Var}_0[M_t^{Q,x}] \) is bounded. We conclude that \( \lim_{t \to \infty} e^{-\tau t} \text{Var}_0[M_t^{Q,x}] = \lim_{t \to \infty} e^{-\tau t} E_0[(M_t^{Q,x})^2] = 0 \).

Finally, to show (c), observe that \( (\theta_t^\theta - \hat{P}_t \hat{Q}_t - (\hat{Q}_t)^2/2, \quad \text{where} \quad \hat{P}_t := \delta \mu + (\alpha + \beta)M_t^{Q,x}, \quad t \geq 0, \quad \text{is quadratic in } (\theta_t^\theta, M_t^{Q,x}). \] Thus, there is \( C > 0 \) large enough such that
\[
||\theta_t^\theta \hat{Q}_t - (\hat{Q}_t)^2/2| + |\hat{P}_t \hat{Q}_t| \leq C[1 + (\theta_t^\theta)^2 + (M_t^{Q,x})^2].
\]
But from the previous arguments, the second moments \( E_0[(\theta_t^\theta)^2] \) and \( E_0[(M_t^{Q,x})^2] \) are bounded over \( \mathbb{R}_+ \), and hence, (iii) in the admissibility requirement holds. It is easy to see that (ii) holds via an identical argument, and observe that we already showed that the controlled dynamics admit a solution under the candidate equilibrium strategy. This concludes the proof. \( \square \)
S1.2 Section 5: Information Revelation

Proof of Proposition 4. Suppose that $\xi^t := (\xi_s : 0 \leq s < t)$ is observed by firm $t$, and let $M_t^* := \mathbb{E}[\theta_t | \mathcal{F}_t^\xi]$, $t \geq 0$, where $(\mathcal{F}_t^\xi)_{t \geq 0}$ denotes the filtration generated by $(\xi_t)_{t \geq 0}$. When the quantity demanded follows $Q_t = \delta \mu + \alpha \theta_t + \beta M_t^*$, recorded purchases obey

$$d\xi_t = (\delta \mu + \alpha \theta_t + \beta M_t^*)dt + \sigma_\xi dZ_t^\xi,$$

where $(M_t^*)_{t \geq 0}$ satisfies the filtering equation

$$dM_t^* = -\kappa(M_t^* - \mu)dt + \frac{\alpha \gamma(\alpha)}{\sigma_\xi^2} [d\xi_t - (\delta \mu + [\alpha + \beta] M_t^* dt)].$$

In this SDE, $\gamma(\alpha)$ is the unique positive solution to $x \mapsto -2\kappa x + \sigma_\xi^2 - (\alpha x/\sigma_\xi)^2 = 0$ (Theorem 12.1 in Liptser and Shiryaev, 1977). As a function of $(\theta, \xi)$, $Q_t$ is stationary Gaussian, as $\phi - \beta \lambda^* = \nu(\alpha, \beta) - \beta \alpha \gamma(\alpha)/\sigma_\xi^2 = \kappa + \alpha^2 \gamma(\alpha)/\sigma_\xi^2 > 0$. Denote the previous process by $(Y_t)_{t \geq 0}$, and note that

$$Y_t = e^{-\nu(\alpha, \beta)t} Y_0 + \int_0^t e^{-\nu(\alpha, \beta)(t-s)} d\xi_s, \quad t \geq 0,$$

with $Y_0$ normally distributed.
as a function of the public history, where $\nu(\alpha, \beta)$ is strictly positive by assumption.

Defining $X_t = \rho^* + \lambda^* Y_t^{\nu(\alpha, \beta)}$, it is easy to verify that
\[
dX_t = \left[ \lambda^*(\delta \mu + \alpha \theta_t + \beta X_t) - \nu(\alpha, \beta)[X_t - \rho^*] \right] + \lambda^* \sigma_x dZ_t^x
= \left[ -(\kappa + \lambda^* \alpha)X_t + \kappa \mu + \lambda^* \alpha \theta_t \right] dt + \lambda^* \sigma_x dZ_t^x,
\]
where in the last equality we used that $\nu(\alpha, \beta) = \kappa + \lambda^*(\alpha + \beta)$ and that $\lambda^* \delta \mu + \nu(\alpha, \beta) = \mu \kappa$. We conclude that $M_t^\ast - X_t$ satisfies $d[M_t^\ast - X_t] = -(\kappa + \lambda^* \alpha)\right[M_t^\ast - X_t\left] dt$, and therefore that $M_t^\ast - X_t = [M_0 - X_0] e^{-(\kappa + \lambda^*) t}$ for all $t \geq 0$.

Notice, however, that since $(X_t)_{t \geq 0}$ is stationary, stationarity of $(M_t^\ast)_{t \geq 0}$ implies that $M_0^\ast - X_0 \equiv 0$ a.s. To see this, notice first that $M_0^\ast - X_0$ cannot be random: otherwise, the constraint that $\text{Var}[M_t]$ must be independent of time becomes
\[
\text{Var}[X_t] + e^{-2(\alpha + \lambda^*) t} \text{Var}[M_0 - X_0] + 2e^{-(\kappa + \lambda^*) t} \text{Cov}[X_t, M_0 - X_0] = \text{constant, independent of } t
\]
which cannot hold for all $t \geq 0$. Thus, $M_0 - X_0 = C \in \mathbb{R}$. From here, however, $C = 0$, as the requirement that $\mathbb{E}[M_t^\ast]$ is independent of time would be violated otherwise. Consequently, if beliefs are stationary,
\[
M_t^\ast = X_t = \rho^* + \lambda^* Y_t^{\nu(\alpha, \beta)} = \left[ \frac{1}{\nu(\alpha, \beta)} \left( \kappa \mu - \frac{\alpha \gamma(\alpha) \delta \mu}{\sigma_x^2} \right) \right] + \frac{\alpha \gamma(\alpha)}{\sigma_x^2} Y_t^{\nu(\alpha, \beta)}, \text{ for all } t \geq 0.
\]

To prove the converse, consider $(\theta_t, Y_t)_{t \geq 0}$ with $\phi = \nu(\alpha, \beta)$ as in Proposition 1 (i.e., stationary Gaussian, driven by $Q_t = \delta \mu + \alpha \theta_t + \beta M_t$, and where $M_t = \mathbb{E}[\theta_t | Y_t], t \geq 0$). We aim to show that $(M_t)_{t \geq 0}$ coincides with $(M_t^\ast)_{t \geq 0}$ path-by-path of $(Y_t)_{t \geq 0}$. In fact, because $M_t = \mu + \lambda [Y_t - \bar{Y}]$, with $\lambda$ and $\bar{Y}$ as in Proposition 1, the task reduces to showing that,
\[
\lambda = \lambda^* \text{ and } \mu - \lambda \bar{Y} = \rho^*
\]
when $\phi = \nu(\alpha, \beta)$.

Recall that if $\alpha > 0$ and $\beta < 0$, stationarity implies that $\lambda > 0$ (this can be seen from (7) in Proposition 1 in the paper). Thus, $\lambda = \Lambda(\phi, \alpha, \beta)$, where the right-hand side is defined in (A.12) in Appendix A; therefore, the first equality follows directly from (ii) in Lemma 6 (the proof of which can be found in Section 1.5 in this Appendix).

To show the second equality, recall that
\[
\bar{Y} = \frac{\mu [\alpha + \beta + \delta]}{\phi}.
\]
Thus, when $\phi = \nu(\alpha, \beta) = \kappa + \lambda^*(\alpha + \beta)$ and $\lambda = \lambda^*$,
\[
\mu - \lambda \bar{Y} = \frac{\mu \nu(\alpha, \beta) - \lambda \mu [\alpha + \beta + \delta]}{\nu(\alpha, \beta)} = \frac{\mu \kappa - \lambda^* \delta \mu}{\nu(\alpha, \beta)} = \rho^*.
\]
This concludes the proof. \qed
S1.3 Section 6: Welfare Analysis

Expression for Consumer Surplus. In this section, expectation (and hence, variance and covariance) operators are with respect to the prior distribution of \((\theta_0, Y_0)\).

Recall that, in equilibrium,
\[
\mathbb{E}[\theta_t] = \mu, \quad \text{Var}[\theta_t] = \frac{\sigma^2}{2\kappa}, \quad M_t = \mu + \lambda [Y_t - \bar{Y}], \quad \lambda := \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]}, \quad \text{and} \quad G(\phi) = \frac{\text{Cov}[\theta_t, Y_t]^2}{\text{Var}[Y_t]\text{Var}[\theta_t]}.
\]

Thus,
\[
\mathbb{E}[\theta_t M_t] = \mathbb{E}[M_t^2] = \text{Var}[M_t] + \mu^2 = \frac{\text{Cov}[\theta_t, Y_t]^2}{\text{Var}[Y_t]} + \mu^2 = \text{Var}[\theta_t] G(\phi) + \mu^2.
\]

Recall that consumer surplus is defined as the consumer’s ex ante equilibrium payoff normalized by the discount rate, i.e., \(CS(\phi) := \mathbb{E}[Q_t(\theta_t - P_t - Q_t/2)]\), where \(P_t = \delta(\phi)\mu + [\alpha(\phi) + \beta(\phi)]M_t\) and \(Q_t = \delta(\phi)\mu + \alpha(\phi)\theta_t + \beta(\phi)M_t\). Omitting the dependence on \(\phi\) of all equilibrium coefficients, therefore,
\[
CS(\phi) = \delta\mu \left[ \mathbb{E}[\theta_t] \left( 1 - \frac{\alpha}{2} \right) - \left( \alpha + \frac{3\beta}{2} \right) \mathbb{E}[M_t] - \frac{3\delta\mu}{2} \right] + \alpha \left[ \mathbb{E}[\theta_t^2] \left( 1 - \frac{\alpha}{2} \right) - \left( \alpha + \frac{3\beta}{2} \right) \mathbb{E}[M_t \theta_t] - \frac{3\delta\mu}{2} \mathbb{E}[\theta_t] \right] + \beta \left[ \mathbb{E}[\theta_t M_t] \left( 1 - \frac{\alpha}{2} \right) - \left( \alpha + \frac{3\beta}{2} \right) \mathbb{E}[M_t^2] - \frac{3\delta\mu}{2} \mathbb{E}[M_t] \right].
\]

Using the above expressions for the first two moments of \((\theta_t, M_t)\),
\[
CS(\phi) = \frac{\sigma^2}{2\kappa} G(\phi) \left[ -\alpha \left( \alpha + \frac{3\beta}{2} \right) + \beta \left( 1 - \frac{\alpha}{2} \right) - \beta \left( \alpha + \frac{3\beta}{2} \right) \right] + \frac{\sigma^2}{2\kappa} \left[ \alpha \left( 1 - \frac{\alpha}{2} \right) \right]
\]
\[

\begin{aligned}
= & -3(\alpha + \beta)^2/2 + \alpha^2/2 + \beta =: L(\phi) \\
+ & \mu^2 \left[ -\alpha \left( \alpha + \frac{3\beta}{2} \right) + \beta \left( 1 - \frac{\alpha}{2} \right) - \beta \left( \alpha + \frac{3\beta}{2} \right) + \alpha \left( 1 - \frac{\alpha}{2} \right) \right] \\
= & -3(\alpha + \beta)^2/2 + \alpha + \beta \\
+ & \delta\mu^2 \left[ \left( 1 - \frac{3(\alpha + \beta)}{2} - \frac{3\alpha}{2} - \frac{3\beta}{2} \right) - \frac{3\delta}{2} \right].
\end{aligned}
\]

Collecting terms in the last two lines yields
\[
\mu^2 \left[ -3(\alpha + \beta)^2/2 + \alpha + \beta + \delta - 3\delta(\alpha + \beta) - \frac{3\delta^2}{2} \right]
\]
\[
= \mu^2 \left[ \alpha + \beta + \delta - \frac{3}{2} \left( (\alpha + \beta)^2 + 2\delta(\alpha + \beta) + \delta^2 \right) \right]
\]
\[
= \mu^2(\alpha + \beta + \delta) \left[ 1 - \frac{3}{2} (\alpha + \beta + \delta) \right]
\]
\[
= \mathbb{E}[P_t] \left( \mu - \frac{3}{2} \mathbb{E}[P_t] \right),
\]
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from where we obtain the desired expression for consumer surplus, i.e., (24) in the paper.

Finally, notice that we can write the term that multiplies $\sigma_\phi^2 G(\phi)/2\kappa$ as

$$L(\phi) := -\frac{3(\alpha + \beta)^2}{2} + \frac{\alpha^2}{2} + \beta = -\alpha[2\beta] + \beta[1 - \alpha] - \frac{3\beta^2}{2} < 0.$$ 

On the other hand, since $-1/2 < \beta < 0$ and $\alpha > 0$, and $0 < \alpha + \beta < 1$,

$$L(\phi) = \frac{\alpha(\phi)^2}{2} + \beta(\phi) - \frac{3}{2}(\alpha(\phi) + \beta(\phi))^2 > 0 - \frac{1}{2} - \frac{3}{2} = -2.$$ 

This concludes the proof. □

**Proof of Proposition 8.** The final step of the proof requires proving (A.26), i.e.,

$$\lim_{\phi \to 0, +\infty} \frac{[\alpha(\phi) - 1]^2}{R(\phi) - 1/8} = 0, \quad \text{and} \quad \lim_{\phi \to 0, +\infty} \frac{G(\phi)}{R(\phi) - 1/8} > 0,$$

where $R(\phi) := [\alpha(\phi) + \beta(\phi) + \delta(\phi)](1 - \frac{3}{2}[\alpha(\phi) + \beta(\phi) + \delta(\phi)])$.

To this end, we can use the expressions (A.11) and (A.15) in Appendix A for $\beta$ and $\delta$, respectively, to obtain,

$$\frac{[\alpha(\phi) - 1]^2}{R(\phi) - 1/8} = -\frac{8(\alpha - 1)}{(\kappa + r + \phi)(-\alpha \kappa + \alpha \phi + \kappa + r + \phi) \times \alpha^2(-2r + \kappa + 2\phi + \alpha r + 3\phi^2) - \alpha(2\kappa + r)(\kappa + r + \phi) + (\kappa + r + \phi)^2]{\alpha^2(-\kappa^2 + 4r^2 - \kappa r + 13\phi + 9\phi^2) + \alpha(2\kappa + r)(\kappa + r + \phi) - (\kappa + r + \phi)^2}. $$

In addition, we can use expression (A.25) for $G$ to obtain,

$$\frac{G(\phi)}{R(\phi) - 1/8} = \frac{8[\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi]}{\alpha(r + 2\phi)(\kappa - \alpha r + r + \phi)(-\alpha \kappa + \alpha \phi + \kappa + r + \phi) \times \alpha^2(-2r + \kappa + 2\phi + \alpha r + 3\phi^2) - \alpha(2\kappa + r)(\kappa + r + \phi) + (\kappa + r + \phi)^2]{\alpha^2(-\kappa^2 + 4r^2 - \kappa r + 13\phi + 9\phi^2) + \alpha(2\kappa + r)(\kappa + r + \phi) - (\kappa + r + \phi)^2},$$

where we have omitted the dependence of $\alpha$ on $\phi$. Consider now the first expression. To examine the case $\phi \to +\infty$, write $\alpha - 1 = \lambda \alpha \beta /[r + \kappa + \phi]$ in the first ratio. Recalling that $(\alpha, \beta, \lambda) \to (1, -1/2, \sigma_\phi^2/\kappa \sigma_\phi^2)$ (proof of Proposition 3 in the paper), we have that the numerator is of $O(\phi^4)$ for $\phi$ large. In contrast, it is easy to see that the denominator is of $O(\phi^5)$. Thus, the first expression converges to zero as $\phi \to +\infty$. Regarding the case $\phi \to 0$, it is easy to see that since $\alpha \to 1$, the denominator converges to $4(\kappa + r)r^3 > 0$, while the numerator converges to zero. Thus, the first expression attains the same limiting value as $\phi \to 0$.

Turning to the second expression, using that $\kappa, r > 0$ and $\alpha \to 1$ as $\phi \to +\infty$, the numerator is $O(\phi^5)$ for $\phi$ large, where the associated constant is $8 \times 4 \times 16$. Similarly, the denominator is also $O(\phi^5)$ for $\phi$ large, with constant $2 \times 1 \times 2 \times 8$. Thus, the limit is 16. Finally, when $\phi \to 0$, the denominator converges to $4r^4\kappa$. Instead, the numerator converges to $64r^5$. Thus, the limit is $16r/\kappa > 0$. This concludes the proof. □
S1.4 Section 7: Hidden Scores

Proof of Proposition 10. The proof parallels the steps followed for proving Theorem 1. If \((\theta_t, Y_t)_{t\geq 0}\) is as in Proposition 1 with \(Q_t = \delta^h \mu/2 + \alpha^h Q_t + \beta^h M_t\) and \(\beta^h = -\alpha^h/2\), we have that

\[
\lambda^h = \frac{\alpha^h \sigma^2_\phi (\phi - \beta^h \lambda^h)}{(\alpha^h)^2 \sigma^2_\phi + \kappa \sigma^2_\xi (\phi + \kappa - \beta^h \lambda^h)} \quad \text{and} \quad \bar{Y}^h = \frac{\mu [\delta^h/2 + \alpha^h + \beta^h]}{\phi}.
\]

In addition, (ii) in the same Proposition becomes \(\phi - \beta^h \lambda^h = \phi + \lambda^h \alpha^h/2 > 0\).

Observe that \(\alpha^h \neq 0\) in equilibrium as well: otherwise, using that \(\phi - \beta^h \lambda^h > 0\) in the equation for \(\lambda^h\) implies that \(\lambda^h = 0\), and so price is constant—but this leads to a demand with unit weight on the type. Using that \(\alpha^h \neq 0\) and \(\phi - \beta^h \lambda^h > 0\), the same equation implies that \(\lambda^h \neq 0\).

Recalling that \(P_t = -\mathbb{E}[Q_t|Y_t]/\zeta^h\), and using that \(M_t = \mu + \lambda^h [Y_t - \bar{Y}^h]\),

\[
dP_t = -\frac{\alpha^h \lambda^h}{2 \zeta^h} dY_t = -\frac{\alpha^h \lambda^h}{2 \zeta^h} [(Q_t - \phi Y_t) dt + \sigma_\xi dZ_t] = \left[ -\frac{\alpha^h \lambda^h}{2 \zeta^h} Q_t - \phi \left( P_t + \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2 \zeta^h} \right) \right] dt - \frac{\alpha^h \lambda^h}{2 \zeta^h} \sigma_\xi dZ_t, \quad (S.1)
\]

where \(\rho^h := 1 - \lambda^h [\delta^h/2 + \alpha^h + \beta^h]/\phi\). Thus, the consumer’s problem is to maximize her utility as in Section 3 subject to (S.1) and the law of motion of her type.

We guess a value function \(V = v_0 + v_1 \theta + v_2 P + v_3 P^2 + v_4 \theta^2 + v_5 \theta P\), which gives the first-order condition

\[
q = \theta - P - \frac{\alpha^h \lambda^h}{2 \zeta^h} \left[ v_2 + 2 v_3 P + v_5 \theta \right] = -\frac{\alpha^h \lambda^h}{2 \zeta^h} v_2 + \left[ 1 - \frac{\alpha^h \lambda^h}{2 \zeta^h} v_5 \right] \theta + \left[ -1 - \frac{\alpha^h \lambda^h}{\zeta^h} v_3 \right] P.
\]

As a result, we obtain the matching-coefficients conditions

\[
\delta^h \mu = -\frac{\alpha^h \lambda^h}{2 \zeta^h} v_2, \quad \alpha^h = 1 - \frac{\alpha^h \lambda^h}{2 \zeta^h} - v_5 \quad \text{and} \quad \zeta^h = -1 - \frac{\alpha^h \lambda^h}{\zeta^h} v_3. \quad (S.2)
\]

Moreover, by the Envelope Theorem,

\[
(r + \phi) [v_2 + 2 v_3 P + v_5 \theta] = q \left[ -1 - v_3 \frac{\alpha^h \lambda^h}{\zeta^h} \right] - 2 v_3 \phi \left[ P + \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2 \zeta^h} \right] - \kappa v_5 (\theta - \mu),
\]

which leads to the system

\[
(r + \phi) v_2 = \delta^h \mu \left[ -1 - v_3 \frac{\alpha^h \lambda^h}{\zeta^h} \right] - 2 v_3 \phi \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2 \zeta^h} + \kappa \mu v_5
\]

\[
2 (r + \phi) v_3 = \zeta^h \left[ -1 - v_3 \frac{\alpha^h \lambda^h}{\zeta^h} \right] - 2 v_3 \phi
\]

\[
(r + \phi) v_5 = \alpha^h \left[ -1 - v_3 \frac{\alpha^h \lambda^h}{\zeta^h} \right] - \kappa v_5.
\]
Using that $v_2, v_3$ and $v_5$ can be written as a function of $\delta^h \mu, \alpha^h$ and $\zeta^h$, respectively, and dividing by $\zeta^h$ in each equation, we obtain the following system

\[
\begin{align*}
-(r + \phi) \frac{2\delta^h \mu - \alpha^h}{2\alpha^h} &= \delta^h \mu + 2\phi \frac{c^h + 1 - \delta^h \mu + \alpha^h \mu}{2\alpha^h} + \kappa \mu \frac{2(1 - \alpha^h)}{\alpha^h} \\
-2(r + 2\phi) \frac{c^h + 1 - \delta^h \mu + \alpha^h \mu}{2\alpha^h} &= \zeta^h \\
(r + \phi + \kappa) \frac{2(1 - \alpha^h)}{\alpha^h} &= \alpha^h.
\end{align*}
\]

(S.3)

Observe that the last equation is independent of the other two. Also, the second equation is linear in $\zeta^h$ given $\alpha^h$, while the first equation is linear in $\delta^h$ given $\zeta^h$ and $\alpha^h$. Thus, we can solve for $\alpha^h, \zeta^h$, and $\delta^h$ sequentially. We proceed by finding $\alpha^h$ first.

It is immediate that $\alpha^h \in (0, 1)$: (i) if $\alpha^h < 0$, the last equation in system (S.3) reads \[ \phi - \beta^h \lambda^h = \phi + \frac{\alpha^h}{2} \lambda^h = (r + \kappa) \left( \frac{1}{\alpha^h} - 1 \right) + \frac{\alpha^h}{\phi} < 0, \] which contradicts stationarity; (ii) if $\alpha^h = 1$, the equation for $\lambda^h$ implies that $\lambda^h > 0$, and so the last equation in (S.3) yields that $\alpha^h = 0$, a contradiction; (iii) and if $\alpha^h > 1$, it follows that $\lambda^h < 0$ from the same equation, but the equation for $\lambda^h$ yields $\lambda^h > 0$ in a stationary linear Markov equilibrium. Since we already know that $\alpha^h \neq 0$, it follows that $\alpha^h \in (0, 1)$, from where we conclude that $\lambda^h > 0$ using again the equation for $\lambda^h$.

Since $\alpha^h > 0$ and $\beta^h = -\alpha^h/2 < 0$, the unique possible value for $\lambda^h > 0$ is given by $\Lambda(\phi, \alpha^h, -\alpha^h/2)$ as defined in (A.12) in the paper. The last equation in (S.3) then reads $A^h(\phi, \alpha^h) = 0$, where

\[ A^h(\phi, x) := (r + \kappa + \phi)(x - 1) - xA(\phi, x, -x/2) \left[ -\frac{x}{2} \right], \quad (\phi, x) \in (0, \infty) \times [0, 1]. \]

The final steps of the proof are as follows:

1. Existence and uniqueness of solution to $A^h(\phi, x) = 0$, $x \in [0, 1]$. Lemma 4 in Appendix A of the paper (i.e., the existence and uniqueness of $\alpha \in (0, 1)$ s.t. $A(\phi, \alpha) = 0$ in the observable-score case), carries over to this setting. To see this, recall that $A(\phi, x) = (r + \kappa + \phi)(x - 1) - xA(\phi, x, B(\phi, x))B(\phi, x)$, where $B(\phi, x) \in (-x/2, 0)$. It is then easy to verify that the steps that showed that $x \in [0, 1] \mapsto H(\phi, x) := -A(\phi, x, B(\phi, x))B(\phi, x)$ is strictly increasing also imply that $x \in [0, 1] \mapsto H^h(\phi, x) := -A(\phi, x, -x/2)[-x/2]$ is strictly increasing. This is because, when $B(\phi, x)$ is replaced by $-x/2$ in $H$: $B_\alpha(\phi, \alpha) + B(\phi, \alpha) < 0$ becomes $-\alpha < 0$; $\ell_\alpha(\phi, \alpha) > 0$ becomes $\sigma_0^2 \alpha > 0$; and $-\alpha B_\alpha(\phi, \alpha) + B(\phi, \alpha) \geq 0$ becomes $0 \geq 0$, which were the critical steps to prove that $H$ was increasing. That $\phi \mapsto \alpha^h(\phi)$ is of class $C^2$ follows from identical arguments too.
2. Determination of the rest of the coefficients. Returning to \( \zeta^h \), it is easy to see from the second equation in \((S.3)\) that

\[
\zeta^h = -\frac{2(r + 2\phi)}{\lambda^h \alpha^h + 2(r + 2\phi)} \in (-1, 0),
\]

where the bounds follow from \( \alpha^h \lambda^h > 0 \).

Regarding \( \delta^h \), observe that the first equation in system \((S.3)\) is trivially satisfied if \( \mu = 0 \); in this case, the constant term in the demand function is simply zero. When \( \mu \neq 0 \), we can eliminate \( \mu \) on both sides to obtain

\[
-(r + \phi)\frac{2\delta^h}{\alpha^h \lambda^h} = \delta^h + \phi \frac{\zeta^h + 1}{\zeta^h \lambda^h \alpha^h} [\delta^h + \alpha^h \rho^h] + \kappa \mu \frac{2(1 - \alpha^h)}{\lambda^h \alpha^h}
\]

where \( \rho^h = 1 - \lambda^h[\delta^h/2 + \alpha^h + \beta^h]/\phi \). Also, from the second equation in \((S.3)\), \((\zeta^h + 1)/(\zeta^h \lambda^h \alpha^h) = -1/[2(r + 2\phi)]\). Thus, the coefficient that multiplies \( \delta^h \) in the previous equation is given by

\[
\frac{1}{\alpha^h \lambda^h} \left[ -2(r + \phi) - \lambda^h \alpha^h + \frac{\phi}{2(r + 2\phi)} \alpha^h \lambda^h - \frac{\phi}{4(r + 2\phi)} (\alpha^h \lambda^h)^2 \right].
\]

But observe that \( \phi \alpha^h \lambda^h/[2(r + 2\phi)] \in (0, \alpha^h \lambda^h/4) \), and so the second term dominates the third. We conclude that the previous expression is strictly negative, which implies that the equation for \( \delta^h \) admits a solution for all parameters.

The rest of the unknowns are determined as follows. First, \( v_2, v_3 \) and \( v_5 \) are determined from the matching-coefficient conditions \((S.2)\) using \( \delta^h, \alpha^h \) and \( \zeta^h \); it is easy to see that all these equations admit a solution. The coefficients \( v_1 \) and \( v_4 \) can in turn be obtained via the Envelope Theorem. Specifically,

\[
(r + \kappa)[v_1 + 2v_4 \theta + v_5 P] = (\delta^h \mu + \alpha^h \theta + \zeta^h P) \left[ 1 - v_5 \frac{\alpha^h \lambda^h}{2\zeta^h} \right] - v_5 \phi \left[ P + \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2\zeta^h} \right] - 2v_4 \kappa (\theta - \mu)
\]

yields the additional equations

\[
2(r + \kappa)v_4 = \alpha^h \left[ 1 - v_5 \frac{\alpha^h \lambda^h}{2\zeta^h} \right] - 2v_4 \kappa \Rightarrow v_4 = \frac{(\alpha^h)^2}{2(r + 2\kappa)} \quad \text{and}
\]

\[
(r + \kappa)v_1 = \delta^h \alpha^h \mu - v_5 \phi \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2\zeta^h} \Rightarrow v_1 = \frac{\delta^h \alpha^h \mu}{r + \kappa} - v_5 \phi (\delta^h \mu + \alpha^h \rho^h \mu) \frac{2\zeta^h}{r + \kappa}.
\]

The coefficient \( v_0 \) in turn corresponds to

\[
v_0 = \frac{1}{r} \left[ -(\delta^h \mu)^2 + v_2 \left( \delta^h \mu \frac{\alpha^h \lambda^h}{2\zeta^h} - \phi \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2\zeta^h} \right) + v_1 \kappa \mu + \sigma^2 v_3 + \left( \frac{\alpha^h \lambda^h \sigma^2}{2\zeta^h} \right)^2 v_4 \right],
\]

which is obtained by equating the constant terms in the HJB equation.
3. Transversality conditions and admissibility of the candidate equilibrium strategy.

Recall that under any admissible strategy,
\[
dP_t = \left[ -\frac{\alpha^h\lambda^h}{2\zeta^h} Q_t - \phi \left( P_t + \frac{\delta^h \mu + \alpha^h \rho^h \mu}{2\zeta^h} \right) \right] dt - \frac{\alpha^h\lambda^h}{2\zeta^h} \sigma \xi dZ^\xi_t,
\]
whereas under the candidate equilibrium strategy, \( Q_t = \delta^h \mu + \alpha^h \theta_t + \zeta^h P_t, \)
\[
dP_t = \left[ - \left( \phi + \frac{\alpha^h \lambda^h}{2} \right) P_t - \left( \frac{\alpha^h}{2\zeta^h} \right) \theta_t - \mu \left( \frac{\delta^h \alpha^h \lambda^h}{2\zeta^h} + \phi \frac{\delta^h + \alpha^h \rho^h}{2\zeta^h} \right) \right] dt - \frac{\alpha^h\lambda^h}{2\zeta^h} \xi dZ^\xi_t.
\]
Since \( \phi \) and \( \phi + \alpha^h \lambda^h/2 \) are strictly positive, both dynamics have the exact same structure as the corresponding ones for \((M_t)_{t \geq 0}\) in the observable-scores case. Moreover, from (S.2), the coefficient on \( P^2 \) in the value function is \( v_3 = - (\zeta^h + 1) \zeta^h /[\alpha^h \lambda^h] > 0 \), where the last inequality follows from \( \zeta^h \in (-1, 0) \). It can be easily seen that these facts imply that all the arguments used to prove Lemmas 1 and 2 in Section 1.1 of this online Appendix also apply to the hidden-scores case.

4. Upper bound for \( \zeta^h \). To conclude, we derive the upper bound for \( \zeta^h \). Using the last equation in the system (S.3),
\[
\lambda^h \alpha^h = \frac{(r + \kappa + \phi)2(1 - \alpha^h)}{\alpha^h}.
\]
Consequently,
\[
-\zeta^h = \frac{(r + 2\phi)\alpha^h}{(r + 2\phi)\alpha^h + (r + \kappa + \phi)(1 - \alpha^h)} = \frac{1}{1 + (r + \kappa + \phi)/(1 - \alpha^h)}.
\]
However, it is easy to see from (A.17)–(A.18) (proof of (ii) in Proposition 3 in Appendix A) that the lower bound \( \alpha^h \geq [r + \kappa + \phi]/[r + \kappa + 2\phi] \) also holds in the hidden case. Thus,
\[
1 - \alpha^h \leq \frac{\phi}{r + \kappa + \phi} \Rightarrow -\zeta^h \geq \frac{1}{1 + \phi/[r + 2\phi]} = \frac{r + 2\phi}{r + 3\phi}.
\]
\[\square\]

**Proof of Proposition 11.** We begin by proving (iii). A simple rearrangement of terms in \( A(\phi, \alpha^o) = 0 \) and \( A^h(\phi, \alpha^h) = 0 \) shows that \( \alpha^o \) and \( \alpha^h \) are defined solutions to
\[
-2(r + \kappa + \phi) + \alpha (2r + \kappa + \phi)
+ \alpha \left\{ \left[ \frac{2\sigma^2 \alpha[B(\phi, \alpha)]}{\kappa \sigma^2} + \phi + \kappa \right]^2 - \frac{4\sigma^2 \alpha B(\phi, \alpha)}{\kappa \sigma^2} \right\}^{1/2} - \frac{\sigma^2 \alpha[B(\phi, \alpha)]}{\kappa \sigma^2} = 0,
\]
and
\[
-2(r + \phi + \kappa) + \alpha (2r + \phi + \kappa)
+ \alpha \left\{ \left[ \frac{\alpha^2 \sigma^2}{2\kappa \sigma^2} + \phi + \kappa \right]^2 + \frac{2\alpha^2 \sigma^2 \phi}{\kappa \sigma^2} \right\}^{1/2} - \frac{\alpha^2 \sigma^2}{2\kappa \sigma^2} = 0,
\]
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respectively, where \( B(\phi, \alpha) \in (-\alpha/2, 0) \). With this in hand, given \( \alpha \in (0, 1) \), define

\[
y \mapsto h(y; \alpha) := \left( \frac{\sigma_0^2 \alpha[y + \phi]}{\kappa \sigma_x^2} + \phi + \kappa \right)^2 - \frac{4 \sigma_0^2 \alpha y \phi}{\kappa \sigma_x^2} - \frac{\sigma_0^2 \alpha[y + \phi]}{\kappa \sigma_x^2},
\]

and notice that the first equation can be written as \( \tilde{A}(\phi, \alpha) := -2(r + \kappa + \phi) + \alpha (2r + \kappa + \phi) + \alpha h(B(\phi, \alpha); \alpha) = 0 \), whereas the second can be written as \( A^h(\phi, \alpha) := -2(r + \kappa + \phi) + \alpha (2r + \kappa + \phi) + \alpha h(-\alpha/2; \alpha) = 0 \).

We now show that, given \( \alpha \in [0, 1], y \mapsto h(y; \alpha) \) is strictly decreasing over \((-\alpha/2, 0)\).

In fact, observe that \( h'(y; \alpha) < 0 \) if and only if

\[
(\frac{\sigma_0^2 \alpha[y + \phi]}{\kappa \sigma_x^2} + \phi + \kappa) \frac{\sigma_0^2 \alpha}{\kappa \sigma_x^2} - \frac{2 \sigma_0^2 \alpha \phi}{\kappa \sigma_x^2} < \frac{\sigma_0^2 \alpha}{\kappa \sigma_x^2} \left( \left( \frac{\sigma_0^2 \alpha[y + \phi]}{\kappa \sigma_x^2} + \phi + \kappa \right)^2 - \frac{4 \sigma_0^2 \alpha y \phi}{\kappa \sigma_x^2} \right)^{1/2}.
\]

If the left-hand side is negative, the result follows immediately. Suppose to the contrary that \((\frac{\sigma_0^2 \alpha[y + \phi]}{\kappa \sigma_x^2} + \phi + \kappa) - 2\phi > 0\). Squaring both sides of the inequality under study yields

\[
-4 \left( \frac{\sigma_0^2 \alpha[y + \phi]}{\kappa \sigma_x^2} + \phi + \kappa \right) \phi + 4\phi^2 < -\frac{4 \sigma_0^2 \alpha y \phi}{\kappa \sigma_x^2} \Leftrightarrow 0 < \frac{\sigma_0^2 \alpha^2}{\kappa \sigma_x^2} + \kappa,
\]

which is always true. We conclude that \( \tilde{A}(\phi, \alpha) < \tilde{A}^h(\phi, \alpha) \) for all \( \alpha \in [0, 1] \). But since \( \tilde{A}(\phi, \alpha) = 2A(\phi, \alpha) \) and \( \tilde{A}^h(\phi, \alpha) = 2A^h(\phi, \alpha), \) and both \( \alpha \mapsto A(\phi, \alpha) \) and \( \alpha \mapsto A^h(\phi, \alpha) \) are increasing (proofs of Theorem 1 and Proposition 10), it follows that \( \alpha^o(\phi) > \alpha^h(\phi) \).

As for (i) and (ii), recall that in the observable case \( Q_t = \delta \mu + \alpha \theta_t + \beta M_t \) and \( P_t = \delta \mu + (\alpha + \beta) M_t \). Thus, omitting the dependence of all equilibrium coefficients on \( \phi \),

\[
\alpha^o = \alpha, \quad \zeta^o = \frac{\beta}{\alpha + \beta} \quad \text{and} \quad \pi_1^o = (\alpha + \beta) \lambda,
\]

where \( \lambda = \Lambda(\phi, \alpha, B(\phi, \alpha)) \).

To show (i), notice that from the second equation for the system (A.10) that defines \((\delta, \alpha, \beta)\) in the observable case,

\[
(r + 2\phi) \frac{\alpha + 2\beta}{\lambda} = \beta^2 \Rightarrow \zeta^o = \frac{\beta}{\beta + \alpha} = \frac{-2(r + 2\phi)}{2(r + 2\phi) - 2\lambda \beta}.
\]

On the other hand, in the hidden-scores case,

\[
\zeta^h = \frac{2(r + 2\phi)}{2(r + 2\phi) + \alpha^h \lambda^h}.
\]
Thus, we must compare $-2\lambda \beta$ with $\alpha^h \lambda^h$. However, from the last equation in (A.10) in Appendix A, and the last equation in (S.3) in this Appendix,

$$(r + \kappa + \phi) \frac{2(1 - \alpha)}{\alpha} = -2\lambda \beta,$$

and

$$(r + \kappa + \phi) \frac{2(1 - \alpha^h)}{\alpha^h} = \alpha^h \lambda^h,$$

But since $1 > \alpha^o > \alpha^h > 0$, we have $0 < -2\lambda \beta < \alpha^h \lambda^h$, from where $-1 < \zeta^o < \zeta^h < 0$.

Regarding (ii), we use again the second equation in (A.10) to obtain

$$\pi^o_h = (\alpha + \beta) \lambda = \frac{[\lambda \beta]^2}{r + 2 \phi} - \beta \lambda = \frac{4[\lambda \beta]^2 - 4 \beta \lambda (r + 2 \phi)}{4(r + 2 \phi)} \frac{[-2\lambda \beta + (r + 2 \phi)]^2 - (r + 2 \phi)^2}{4(r + 2 \phi)}.$$

However, using the expression for $\zeta^h$ in the hidden-scores case,

$$\pi^h := -\frac{\alpha^h \lambda^h}{2 \zeta^h} = \frac{[\alpha^h \lambda^h]^2 + 2(r + 2 \phi) \alpha^h \lambda^h}{4(r + 2 \phi)} = \frac{[\alpha^h \lambda^h + (r + 2 \phi)]^2 - (r + 2 \phi)^2}{4(r + 2 \phi)}.$$

Because $0 < -2\lambda \beta < \alpha^h \lambda^h$, it follows that $0 < \pi^o_h < \pi^h$.

We conclude by proving (iv). To find the expected price and quantities in the hidden case, observe that the equation for $\delta^h$ is given by the first equation in (S.3) in this Appendix:

$$-(r + \phi) \frac{2\delta^h \mu}{\alpha^h \lambda^h} = \delta^h \mu + \phi \frac{\zeta^h + 1}{\alpha^h \lambda^h \zeta^h} [\delta^h \mu + \alpha^h \rho^h \mu] + \kappa \mu \frac{2(1 - \alpha^h)}{\alpha^h \lambda^h},$$

where $\rho^h := 1 - \lambda^h [\delta^h + \alpha^h] / [2\phi]$. Also, from the second and third equations in (S.3) again,

$$\frac{\zeta^h + 1}{\alpha^h \lambda^h \zeta^h} = -\frac{1}{2(r + 2 \phi)} \text{ and } \frac{2(1 - \alpha^h)}{\alpha^h \lambda^h} = \alpha^h - (r + \phi) \frac{2(1 - \alpha^h)}{\alpha^h \lambda^h}.$$

Consequently, we can write

$$\frac{2(r + \phi)}{\alpha^h \lambda^h} [\mu - (\delta^h \mu + \alpha^h \mu)] = \delta^h \mu + \alpha^h \mu - \frac{\phi}{2(r + 2 \phi)} [\delta^h \mu + \alpha^h \rho^h \mu]. \quad (S.4)$$

On the other hand, on-path prices and quantities in the hidden case take the form $P^h_t = -[\delta^h \mu + \alpha^h M_t] / 2 \zeta^h$ and $Q^h_t = \delta^h \mu + \alpha^h \theta_t + \zeta^h P^h_t$. Hence,

$$P^h(\phi) := \mathbb{E}[P^h_t] = -\mu \frac{\delta^h + \alpha^h}{2 \zeta^h} \text{ and } Q^h(\phi) := \mathbb{E}[Q^h_t] = \mu \frac{\delta^h + \alpha^h}{2} = -\zeta^h P^h(\phi).$$

From here, $P^h(\phi) = Q^h(\phi)$ if and only if $\mu = 0$. In this case, expected prices and quantities in the observable and hidden case all coincide, and their common value is zero. We assume $\mu > 0$ in what follows; in particular, $P^h(\phi) > Q^h(\phi)$ due to $\zeta^h \in (-1, 0)$. 

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Let $\bar{P}^h(\phi) = P^h(\phi)/\mu$ and observe that $-\zeta^h \bar{P}^h(\phi) = (\alpha^h + \delta^h)/2$. In addition,

$$\rho^h := 1 - \lambda^h \frac{\delta^h + \alpha^h}{2\phi} \Rightarrow \delta^h \mu + \alpha^h \rho^h \mu = \mu(\alpha^h + \delta^h) - \lambda^h \alpha^h \frac{\delta^h + \alpha^h}{2\phi} = -\mu \zeta^h \bar{P}^h(\phi) \left[ 2 - \frac{\lambda^h \alpha^h}{\phi} \right].$$

Plugging these expressions into (S.4), and multiplying the resulting equation by $\mu/2\zeta^h$ yields

$$\frac{2(r + \phi)}{\alpha^h \lambda^h} \left[ \frac{1}{2\zeta^h} + \bar{P}^h(\phi) \right] = -\bar{P}^h(\phi) + \bar{P}^h(\phi) \frac{\phi}{2(r + 2\phi)} \left[ 1 - \frac{\alpha^h \lambda^h}{2\phi} \right]$$

$$\Rightarrow \bar{P}^h(\phi) = \frac{1}{2(r + \phi)} \left[ \frac{r + \phi}{\alpha^h \lambda^h} \right] - \frac{\alpha^h \lambda^h}{4(r + 2\phi)} = \frac{2(r + \phi)}{8(r + \phi)(r + 2\phi) + (\alpha^h \lambda^h)^2 - 2\phi \alpha^h \lambda^h + 4\alpha^h \lambda^h(r + 2\phi)} (S.5)$$

where in the last equality we used that $\zeta^h = -2(r + 2\phi)/[2(r + 2\phi) + \alpha^h \lambda^h]$. The expression is well-defined because $8(r + \phi)(r + 2\phi) + (\alpha^h \lambda^h)^2 - 2\phi \alpha^h \lambda^h + 4\alpha^h \lambda^h(r + 2\phi) = 4(r + \phi)[\alpha^h \lambda^h + 2(r + 2\phi)] + (\alpha^h \lambda^h)^2 + 2\phi \alpha^h \lambda^h > 0$.

We first prove $\bar{Q}^\phi(\phi) > \bar{Q}^h(\phi)$. Since $\bar{Q}^h = -\zeta^h \bar{P}^h$, it follows that

$$\bar{Q}^h = \frac{1}{2 + \frac{(\alpha^h \lambda^h)^2 - 2\phi \alpha^h \lambda^h + 4\alpha^h \lambda^h(r + 2\phi)}{4(r + \phi)(r + 2\phi)}} = \frac{1}{2 + \frac{\alpha^h \lambda^h}{2(r + \phi)} + \frac{\alpha^h \lambda^h + 4\alpha^h \lambda^h(r + 2\phi)}{2(r + 2\phi)}}.$$

Also, from (A.19) in the proof of Proposition 3 (Appendix A in the paper),

$$\bar{P}^\phi(\phi) := \frac{P^\phi(\phi)}{\mu} = \frac{r + \phi}{2(r + \phi) + \lambda(\alpha + \beta)}. \quad \text{(S.6)}$$

Since $\bar{Q}^\phi(\phi) = \bar{P}^\phi(\phi)$, the desired inequality holds if and only if

$$\frac{\alpha^h \lambda^h}{2(r + \phi)} < \frac{\alpha^h \lambda^h}{2(r + 2\phi)} \iff \frac{\alpha^h \lambda^h}{2} \frac{r + \phi}{2(r + 2\phi)} = \frac{(r + \phi)(\alpha - 1)}{\alpha \beta} \geq \lambda(\alpha + \beta). \quad \text{(*)}$$

Now, using the equations that define $\alpha$ and $\alpha^h$ we obtain

$$\lambda = \frac{(r + \kappa + \phi)(\alpha - 1)}{\alpha \beta} \quad \text{and} \quad \frac{\alpha^h \lambda^h}{2} = \frac{(r + \kappa + \phi)(1 - \alpha^h)}{\alpha^h}.$$

Also, using that $\beta = B(\phi, \alpha) = -\alpha^2(r + 2\phi)/[2(r + 2\phi) \alpha - (r + \kappa + \phi)(\alpha - 1)]$, it is easy to see that

$$\frac{\alpha + \beta}{\alpha \beta} = \frac{\alpha(r + 2\phi) - (r + \kappa + \phi)(\alpha - 1)}{-\alpha^2(r + 2\phi)}.$$

The inequality (*) then holds if and only if

$$\frac{(1 - \alpha^h)[2(r + \kappa + \phi)(1 - \alpha^h) + (4r + 6\phi)\alpha^h]}{2(\alpha^h)^2(r + 2\phi)} > \frac{(1 - \alpha)(r + \kappa + \phi)(1 - \alpha) + \alpha(r + 2\phi)}{\alpha^2(r + 2\phi)}.$$

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Since \( \alpha > \alpha^h \), we have that \( 2(1 - \alpha^h)^2\alpha^2[r + \kappa + \phi] > 2(1 - \alpha)^2(\alpha^h)^2[r + \kappa + \phi] \). It therefore suffices to show that

\[
(1 - \alpha^h)\alpha^2(4r + 6\phi)\alpha^h > (1 - \alpha)(\alpha^h)^2\alpha(2r + 4\phi)
\]

\[
\Leftrightarrow [1 - \alpha^h]\alpha^2(2r + 4\phi)\alpha^h + 2[1 - \alpha^h]\alpha^2(r + \phi)\alpha^h > (1 - \alpha)(\alpha^h)^2\alpha(2r + 4\phi)
\]

\[
\Leftrightarrow \alpha\alpha^h[\alpha - \alpha^h] + 2[1 - \alpha^h]\alpha^2(r + \phi)\alpha^h > 0,
\]

which is always true.

To establish the ranking of prices, we introduce the following:

**Lemma 3.** A sufficient condition for \( \bar{P}^h(\phi) < \bar{P}^a(\phi) \) is

\[
\alpha(\alpha - \alpha^h)(r + 2\phi) - (1 - \alpha)^2\alpha^h(r + \kappa + \phi) < 0, \forall \phi > 0. \tag{S.7}
\]

**Proof.** From (S.5) in this Appendix, and using that \( 8(r + \phi)(r + 2\phi) + (\alpha^h\lambda^h)^2 - 2\phi\alpha^h\lambda^h + 4\alpha^h\lambda^h(r + 2\phi) = 4(r + \phi)[\alpha^h\lambda^h + 2(r + 2\phi)] + (\alpha^h\lambda^h)^2 + 2\phi\alpha^h\lambda^h > 0 \), we can write

\[
\bar{P}^h(\phi) = \frac{1}{2 + \frac{\alpha^h\lambda^h[\alpha^h\lambda^h + 2\phi]}{2(r + \phi)[2(r + 2\phi) + \alpha^h\lambda^h]}},
\]

As a result, using (S.6),

\[
\bar{P}^a(\phi) < \bar{P}^h(\phi) \Leftrightarrow \frac{\alpha^h\lambda^h + 2\phi}{2} - \frac{\alpha^h\lambda^h}{2(r + 2\phi) + \alpha^h\lambda^h} < \lambda(\alpha + \beta).
\]

Using again that \( \alpha^h\lambda^h = 2(r + \kappa + \phi)(1 - \alpha^h)/\alpha^h \), \( \lambda = (r + \kappa + \phi)(\alpha - 1)/[\alpha\beta] \), and

\[
\frac{\alpha + \beta}{\alpha\beta} = \frac{\alpha(r + 2\phi) - (r + \kappa + \phi)(\alpha - 1)}{-\alpha^2(r + 2\phi)},
\]

The inequality of interest becomes

\[
\alpha^2(r + 2\phi)(1 - \alpha^h)[\alpha^h\lambda^h + 2\phi] < (1 - \alpha)\alpha^h[\alpha^h\lambda^h + 2(r + 2\phi)]
\]

\[
\times [\alpha(r + 2\phi) - (r + \kappa + \phi)(\alpha - 1)]
\]

\[
\Leftrightarrow \alpha^2(r + 2\phi)[\alpha^h\lambda^h + 2\phi] < (1 - \alpha)^2\alpha^h[\alpha^h\lambda^h + 2(r + 2\phi)](r + \kappa + \phi)
\]

\[
-2\alpha^2\alpha^h(r + \phi)(r + 2\phi)
\]

\[
+\alpha^h\alpha(r + 2\phi)[\alpha^h\lambda^h + 2(r + 2\phi)]
\]

\[
\Leftrightarrow \alpha^2(r + 2\phi)[\alpha^h\lambda^h + 2\phi + 2\alpha^h(r + \phi)] < (1 - \alpha)^2\alpha^h[\alpha^h\lambda^h + 2(r + 2\phi)](r + \kappa + \phi)
\]

\[
+\alpha^h\alpha(r + 2\phi)[\alpha^h\lambda^h + 2(r + 2\phi)].
\]

But since \( \alpha^h < 1 \), the left-hand side is less than \( \alpha^2(r + 2\phi)[\alpha^h\lambda^h + 2(r + 2\phi)] \). Inserting the latter expression on the left-hand side and dividing by \( \alpha^h\lambda^h + 2(r + 2\phi) > 0 \), we conclude
that the desired inequality is implied by \( \alpha^2(r + 2\phi) < (1 - \alpha)^2 \alpha^h(r + \kappa + \phi) + \alpha^h \alpha(r + 2\phi) \).

This concludes the proof of the lemma.

Using the previous lemma, we can rewrite (S.7) as

\[
\alpha^h > \frac{\alpha^2(r + 2\phi)}{\alpha(r + 2\phi) + (1 - \alpha)^2(r + \kappa + \phi)} := A^P(\phi, \alpha).
\]

Let \( s = \sigma^2 / \sigma^2_h \). We first write the equilibrium condition for \( \alpha \) as in (A.22) in the paper for the observable case, and in terms of \( \alpha^h \) and \( \phi \) only for the hidden case. Specifically,

\[
0 = \frac{(\alpha - 1)(\kappa + r + \phi)(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}{\alpha^3(r + 2\phi)} + \frac{\alpha(\kappa - \alpha(\kappa + r) + r + \phi)}{\alpha^3 + \kappa s(\kappa - \alpha r + r + \phi)}
\]

\[
0 = \frac{2(\alpha^h - 1)(\kappa + r + \phi)}{(\alpha^h)^2} + \frac{\alpha^h(\kappa - \alpha^h(\kappa + r) + r + \phi)}{(\alpha^h)^3 + \kappa s(\kappa - \alpha^h r + r + \phi)}.
\]

Solving both equations for \( s \), we obtain the following expressions

\[
s = \frac{\alpha^3(2(-\kappa(r + 2r + 3\phi^2) + \alpha(2r^2 + \kappa + r + \phi) - (r + 3\phi) + r + \phi))}{(1 - \alpha)\kappa(r + 2(1 - \alpha) + \phi)(\kappa + r(1 - \alpha) + \phi)(\kappa(1 - \alpha) + \alpha(r + 3\phi) + r + \phi)},
\]

\[
s = \frac{(\alpha^h)^3 \left( \kappa - \alpha^h(\kappa + r + 2\phi) + r + \phi \right)}{2\kappa(\alpha^h - 1)(\kappa + r + \phi)(\kappa - \alpha^h r + r + \phi)}.
\]

In particular, observe that since \( \alpha^h > [r + \kappa + \phi] / [r + \kappa + 2\phi] \), \( S_h(\alpha^h) \) is increasing.

Now fix \( \alpha = \alpha(\phi) \) and consider the difference \( S^o(\alpha) - S^h(\alpha^P(\alpha)) \). After simplifications, we obtain

\[
S^o(\alpha) - S^h(\alpha^P(\alpha)) = [(\kappa + r(1 - \alpha) + \phi)(\kappa(1 - \alpha) + \alpha(r + 3\phi) + r + \phi)] \frac{S^o(\alpha)}{2} \times
\]

\[
2 \frac{\left( \kappa - \alpha r + r + \phi \right)(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}{\left( \kappa + \alpha^2(r + \kappa + \phi) - \alpha(2r + \kappa) + r + \phi \right)^2 \alpha^3(r + 2\phi)^3} - \frac{\left( -\alpha \kappa + \alpha \phi + \kappa + r + \phi \right)(\alpha^2((\kappa + \phi)^2 + 2\kappa r) - \alpha(2r + \kappa)(\kappa + r + \phi) + (\kappa + r + \phi)^2)}{\left( \kappa + \alpha \phi + \kappa + r + \phi \right)(\alpha^2((\kappa + \phi)^2 + 2\kappa r) - \alpha(2r + \kappa)(\kappa + r + \phi) + (\kappa + r + \phi)^2)}.
\]

Finally, it can be verified that the term in parentheses is strictly positive for all \( \alpha \in (0, 1) \) and for all \( (\kappa, \phi, r) \in \mathbb{R}_+^3 \)—see scores.nb in our websites. Because in equilibrium we must have \( S^o(\alpha) = S^h(\alpha^h) = s \), it follows that \( \alpha^h(\phi) > A^P(\alpha(\phi)) \) for all \( \phi \). This concludes the proof of the proposition.

\[ \square \]
S1.5 Appendix A: Proofs

Proof of Lemma 6. We start by showing (ii). To this end, recall that when $\alpha > 0$ and $\beta < 0$, the quadratic in $\lambda$
\begin{equation}
\lambda = \frac{\alpha \sigma_\theta^2(\phi - \beta \lambda)}{\alpha^2 \sigma_\theta^2 + \sigma_\xi^2 \kappa(\phi - \beta \lambda + \kappa)},
\end{equation}
has a unique strictly positive root, which we denoted by $\Lambda(\phi, \alpha, \beta)$. It then suffices to show that $\alpha \gamma(\alpha)/\sigma_\xi^2 > 0$ solves the previous equation.

To this end, we omit the dependence of $\nu$ on $(\alpha, \beta)$ and of $\gamma$ on $\alpha$ in what follows. Rewrite (S.8) at $\phi = \nu$ as
\begin{equation}
-\kappa \sigma_\xi^2 \beta \lambda^2 + \lambda [\alpha^2 \sigma_\theta^2 + \kappa \sigma_\xi^2 (\nu + \kappa) + \alpha \sigma_\theta^2 \beta] - \alpha \sigma_\theta^2 \nu = 0.
\end{equation}
However, using that $\lambda \kappa \sigma_\xi^2 (\nu + \kappa) = \lambda \kappa \sigma_\xi^2 \left(2 \kappa + \frac{\alpha \gamma}{\sigma_\xi^2} [\alpha + \beta]\right)$ and $\alpha \sigma_\theta^2 \nu = \alpha \sigma_\theta^2 \left(\kappa + \frac{\alpha \gamma}{\sigma_\xi^2} [\alpha + \beta]\right)$, we obtain
\begin{equation}
0 = \lambda \alpha^2 \sigma_\theta^2 + 2 \kappa^2 \sigma_\xi^2 \lambda + 2 \kappa \lambda \alpha^2 \gamma + \kappa \lambda \alpha \gamma / \beta - \kappa \lambda^2 \sigma_\xi^2 \beta - \alpha \sigma_\theta^2 \kappa - \frac{\alpha^2 \gamma \sigma_\theta^2}{\sigma_\xi^2} [\alpha + \beta] + \alpha \sigma_\theta^2 \beta \lambda
= \lambda \alpha^2 \sigma_\theta^2 + 2 \kappa^2 \sigma_\xi^2 \lambda + \kappa \lambda \alpha^2 \gamma - \alpha \sigma_\theta^2 \kappa - \frac{\alpha^2 \gamma \sigma_\theta^2}{\sigma_\xi^2} + \beta \left[\kappa \lambda \alpha \gamma - \kappa \lambda^2 \sigma_\xi^2 - \frac{\alpha^2 \gamma \sigma_\theta^2}{\sigma_\xi^2} + \alpha \sigma_\theta^2 \beta \lambda\right].
\end{equation}
Setting $\lambda = \alpha \gamma / \sigma_\xi^2$, the first and last term of the first line in the second equality cancel out, and the last bracket vanishes. Thus, we are left with
\begin{equation}
0 = 2 \kappa \alpha \left[2 \kappa \gamma + \frac{\alpha^2 \gamma^2}{\sigma_\xi^2} - \frac{\alpha^2 \gamma \sigma_\theta^2}{\sigma_\xi^2}\right],
\end{equation}
which is true by definition of $\gamma$.

We now prove that $\nu(\alpha, \beta)$ is an extreme point of $\phi \mapsto G(\phi, \alpha, \beta)$, and verify (i) in the process. For notational simplicity, we again omit any dependence on variables unless it is strictly necessary. Recall that
\begin{equation}
G = \frac{\alpha \Lambda}{\phi + \kappa - \beta \Lambda}.
\end{equation}
Thus, $G_\phi = 0$ if and only if $\Lambda_\phi(\phi + \kappa) = \Lambda$. We first check that this equality is satisfied at $(\nu(\alpha, \beta), \alpha, \beta)$.

From (ii), $\Lambda = \alpha \gamma / \sigma_\xi^2$ at the point of interest; hence, the claim reduces to showing that $\Lambda_\phi(\nu(\alpha, \beta), \alpha, \beta) = \alpha \gamma / \sigma_\xi^2 (\nu + \kappa)$. However, it is easy to check that
\begin{equation}
\Lambda_\phi = \frac{\alpha \sigma_\theta^2 [1 - \beta \Lambda_\phi] [\alpha^2 \sigma_\theta^2 + \kappa^2 \sigma_\xi^2]}{[\alpha^2 \sigma_\theta^2 + \kappa \sigma_\xi^2 (\phi + \kappa - \beta \Lambda)]^2}.
\end{equation}
Also, \( \nu + \kappa - \beta \Lambda(\nu(\alpha, \beta), \alpha, \beta) = 2\kappa + \alpha^2 \gamma / \sigma^2_\xi = \sigma^2_\phi / \gamma \), where the last equality comes from the definition of \( \gamma \). Thus,

\[
[a^2 \sigma^2_\phi + \kappa \sigma^2_\xi (\phi + \kappa - \beta \Lambda)]^2 \bigg|_{\phi=\nu} = \frac{\sigma^4_\phi[a^2 \gamma + \kappa \sigma^2_\xi]^2}{\gamma^2} = \frac{\sigma^4_\phi[a^2 \gamma + 2 \kappa \gamma \sigma^2_\xi + \kappa^2 \sigma^2_\xi]}{\gamma^2} = \frac{\sigma^4_\phi \sigma^2_\xi [a^2 \sigma^2_\phi + \kappa^2 \sigma^2_\xi]}{\gamma^2}.
\]

We conclude that at \((\nu(\alpha, \beta), \alpha, \beta)\),

\[
\Lambda_\phi = \frac{\gamma^2 \alpha}{\sigma^2_\phi \sigma^2_\xi} [1 - \beta \Lambda_\phi] \Rightarrow \Lambda_\phi \left[ \sigma^2_\phi \sigma^2_\xi + \gamma^2 \alpha \beta \right] = \gamma^2 \alpha \Rightarrow \Lambda_\phi = \frac{\gamma \alpha}{\sigma^2_\xi} \left. \right|_\Lambda(\nu(\alpha, \beta), \alpha, \beta) \times \frac{1}{2 \kappa + \frac{\alpha \gamma (\alpha + \beta)}{\sigma^2_\xi} \left/ \nu(\alpha, \beta) \right.}
\]

which shows that \( \nu \) is an extreme point of \( \phi \mapsto G(\phi, \alpha, \beta) \).

On the other hand, it is easy to verify that at an extreme point \( \phi \),

\[
G_{\phi \phi} = \frac{\alpha \sigma^2_\phi}{2 \kappa} \left[ \frac{\Lambda_\phi (\phi + \kappa)}{[\phi + \kappa - \beta \Lambda]^2} \right].
\]

Since \( \alpha > 0 \), the sign of \( G_{\phi \phi} \) is determined by \( \Lambda_{\phi \phi} \) at that point. We now show that \( \Lambda_{\phi \phi}(\phi, \alpha, \beta) < 0 \) for all \( \phi > 0 \), \( \alpha > 0 \) and \( \beta < 0 \), and hence, that any extreme point of \( \phi \mapsto G(\phi, \alpha, \beta) \) must be a strict local maximum. But this is enough to guarantee that \( \phi \mapsto G(\phi, \alpha, \beta) \) has a unique extreme point, and hence that \( \nu(\alpha, \beta) \) is a global maximum.

Recall that \( \Lambda(\phi, \alpha, \beta) = \left[ \sqrt{\ell^2(\phi, \alpha, \beta) - 4 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha \phi} - \ell(\phi, \alpha, \beta) \right] / [-2 \kappa \sigma^2_\xi \beta] \) where \( \ell(\phi, \alpha, \beta) := \alpha \sigma^2_\phi [\alpha + \beta] + \kappa \sigma^2_\xi (\phi + \kappa) \). Thus,

\[
\Lambda_\phi = \frac{1}{[-2 \kappa \sigma^2_\xi \beta]^1} \left[ \frac{\kappa \sigma^2_\xi (\phi, \alpha, \beta) - 2 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha - \kappa \sigma^2_\xi}{\sqrt{\ell^2(\phi, \alpha, \beta) - 4 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha \phi}} \right], \text{ and so}
\]

\[
\Lambda_{\phi \phi} = K_2(\phi) \left\{ \left( \frac{\kappa \sigma^2_\xi}{\ell(\phi, \alpha, \beta) - 4 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha} - (\kappa \sigma^2_\xi \ell(\phi, \alpha, \beta) - 2 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha)^2 \right) \right. \frac{J(\phi)}{J(\phi) := K_1 / [\ell^2(\phi, \alpha, \beta) - 4 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha]^{3/2}}.
\]

where \( K_2(\phi) := K_1 / [\ell^2(\phi, \alpha, \beta) - 4 \kappa (\sigma_\xi \sigma_\theta)^2 \beta \alpha]^{3/2} \). Since \( \beta < 0 \), we have that \( K_1 > 0 \), from where \( K_2 > 0 \). Moreover,

\[
J(\phi) = -4 \kappa^3 \sigma^2_\phi \sigma^2_\theta \beta \alpha \phi + 4 \kappa^2 \sigma^2_\xi \sigma^2_\phi \beta \alpha \ell(\phi, \alpha, \beta) - 4 \kappa^2 (\sigma_\xi \sigma_\theta)^4 \beta^2 \alpha^2
\]

\[
> 0, \text{ as } \beta < 0 \quad \text{def. of } \ell(\phi, \alpha, \beta)
\]

concluding the proof. \( \square \)
Proof of Lemma 7. Recall from Proposition 5 and its proof show that (i) \( \alpha \) is decreasing at any point satisfying \( \phi = \nu(\alpha(\phi), \beta(\phi)) \), and (ii) a point like that is shown to exist via a simple application of the Intermediate Value Theorem. Importantly, neither step (nor the derivation of the bounds in part (ii)) relies on knowledge of \([\alpha + \beta']\), which is used to establish the uniqueness part of the proposition only.

That \( \arg \min \alpha > \kappa \) follows from the proof of the same proposition. Also, \( \arg \min \alpha < +\infty \) follows directly from \( \alpha \in [1/2, 1], \lim_{\phi \to 0, +\infty} \alpha = 1 \), and \( \alpha \) being continuous.

To prove the last two parts, omit the dependence of \( \alpha \) and \( \beta \) on \( \phi \) and write

\[
\alpha + \beta = \alpha \left[ 1 - \frac{\alpha(r + 2\phi)}{2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)} \right] =: \alpha h(\phi, \alpha).
\]

Thus, \([\alpha + \beta]'(\phi) = \alpha'[h + \alpha h_\alpha] + \alpha h_\phi\), where

\[
h_\phi(\alpha, \phi) = \frac{\alpha(\alpha - 1)(r + 2\kappa)}{[2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)]^2} < 0.
\]

We will show that \( h + \alpha h_\alpha > 0 \) over \([\kappa, \infty)\), which implies that \( \alpha + \beta \) must be decreasing over \([\kappa, \arg \min \alpha]\), and hence, at any point satisfying \( \phi = \nu(\alpha(\phi), \beta(\phi)) \).

To this end, notice that

\[
h + \alpha h_\alpha > 0 \iff [2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)](r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)]
- \alpha(r + 2\phi)[2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)]
+ \alpha^2(r + 2\phi)[2(r + 2\phi) - (r + \kappa + \phi)] > 0
\]

\[
\iff 2(r + 2\phi)^2 \alpha^2 + (r + \kappa + \phi)^2(\alpha - 1)^2 - \alpha(3\alpha - 2)(r + 2\phi)(r + \kappa + \phi) > 0.
\]

If \( \phi \geq \kappa \), however,

\[
2(r + 2\phi)^2 \alpha^2 - \alpha(3\alpha - 2)(r + 2\phi)(r + \kappa + \phi) + \underbrace{(r + \kappa + \phi)^2(\alpha - 1)^2}_{> 0}
\]

\[
\geq \alpha(r + 2\phi)[2(r + 2\phi)\alpha - 3\alpha(r + \kappa + \phi) + 2(r + \kappa + \phi)]
\]

\[
\geq \alpha(r + 2\phi)(r + \kappa + \phi)[2 - \alpha] > 0,
\]

and the result follows. As a final step, observe that since \( \phi > 0 \) and \( \alpha < 1 \),

\[
2(r + 2\phi)^2 \alpha^2 - \alpha(3\alpha - 2)(r + 2\phi)(r + \kappa + \phi) + \underbrace{(r + \kappa + \phi)^2(\alpha - 1)^2}_{> 0}
\]

\[
\geq \alpha(r + 2\phi)[\alpha(\phi - r - 3\kappa) + 2(r + \kappa + \phi)]
\]

\[
\geq \alpha(r + 2\phi)[-r - 3\kappa + 2(r + \kappa)] = \alpha(r + 2\phi)[r - \kappa].
\]

which is non-negative when \( r \geq \kappa \). This concludes the proof. \( \square \)
Proof of Lemma 8. To establish the bounds for $\alpha$ and $\beta$, fix $\phi > \kappa$. Using the definition of $\beta$, and that $1/2 < \alpha < 1$,

$$|\beta| = |B(\phi, \alpha)| = \frac{\alpha^2}{2\alpha + \frac{(r + \kappa + \phi)}{r + 2\phi}(1 - \alpha)} > \frac{\alpha^2}{2} > \frac{1}{8},$$

so $-1/2 < -\alpha/2 < \beta < -1/8$.

To show the second part of the lemma, fix $(\alpha, \beta) \in I := [1/2, 1] \times [-1/2, -1/8]$ and consider the function $\phi \mapsto \Lambda(\phi, \alpha, \beta)/(r + \phi)$. Letting

$$\ell := \ell(\phi, \alpha, \beta) = \sigma_2^2\alpha[\alpha + \beta] + \kappa\sigma_2^2[\kappa + \phi] > 0$$

and $\Delta := -\kappa\sigma_2\sigma_\phi^2\alpha\beta > 0$, we have that

$$\Lambda(\phi, \alpha, \beta) = \frac{1}{r + \phi}\frac{\sqrt{\ell^2 + 4\Delta\phi - \ell}}{-2\kappa\sigma_2^2\beta},$$

so we will prove the property for $K(\phi)$. Since

$$K'(\phi) = \frac{1}{(r + \phi)^2}\left[(r + \phi)\left(\frac{\ell\phi + 2\Delta}{\sqrt{\ell^2 + 4\Delta\phi}} - \ell\phi\right) - (\sqrt{\ell^2 + 4\Delta\phi} - \ell)\right]
= \frac{1}{(r + \phi)^2[\sqrt{\ell^2 + 4\Delta\phi} - \ell]}\left[2\Delta(r + \phi) - [\sqrt{\ell^2 + 4\Delta\phi} - \ell][\sqrt{\ell^2 + 4\Delta\phi} + \ell\phi(r + \phi)]\right],$$

we are left with the task of showing that

$$2\Delta(r + \phi) < [\sqrt{\ell^2 + 4\Delta\phi} - \ell][\sqrt{\ell^2 + 4\Delta\phi} + \ell\phi(r + \phi)]
= \frac{4\Delta\phi}{\sqrt{\ell^2 + 4\Delta\phi} + \ell}$$

$$\Leftrightarrow [\sqrt{\ell^2 + 4\Delta\phi} + \ell](r + \phi) < 2\phi[\sqrt{\ell^2 + 4\Delta\phi} + \ell\phi(r + \phi)]
\Leftrightarrow \ell(r + \phi) < (\phi - r)\sqrt{\ell^2 + 4\Delta\phi} + 2\phi\ell\phi(r + \phi)
\Leftrightarrow (r + \phi)[\ell - 2\phi\ell\phi] < (\phi - r)\sqrt{\ell^2 + 4\Delta\phi}.$$

Observe now that $\ell\phi = \kappa\sigma_\phi^2$. Thus, left-hand side reads $(r + \phi)[\ell - 2\phi\ell\phi] = (r + \phi)[\sigma_2^2\alpha(\alpha + \beta) + \kappa\sigma_2^2[\kappa - \phi]]$. Since we are interested in the region $\phi \geq \kappa$, it suffices to prove the inequality of interest when its left-hand side is replaced with $(r + \phi)\sigma_2^2\alpha[\alpha + \beta] > 0$.

Whenever $r < \kappa \leq \phi$ we can square both sides and obtain

$$(r + \phi)^2(\sigma_2^2\alpha[\alpha + \beta])^2 < (\phi - r)^2[\ell^2 + 4\Delta\phi]
\Leftrightarrow (\sigma_2^2\alpha[\alpha + \beta])^2[\ell^2 + (\phi - \phi)^2] < (\phi - r)^2[2\sigma_2^2\alpha(\alpha + \beta)\kappa\sigma_2^2(\kappa + \phi) + [\kappa\sigma_2^2(\kappa + \phi)]^2]
= 4\phi
+ 4(\phi - r)^2\Delta\phi,$$
where in the last inequality we used the definition of $\ell$. Both terms on the right-hand side are positive. Thus, a sufficient condition is that for all $\phi \geq \kappa$,

$$(\sigma_0^2 \alpha [\alpha + \beta])^2 4r \phi < 4(\phi - r)^2 \Delta \phi \iff r(\sigma_0^2 \alpha [\alpha + \beta])^2 < (\phi - r)^2 \Delta.$$ 

Since $\sigma_0^2 \alpha [\alpha + \beta] < \sigma_0^2$ and $(\phi - r)^2 \Delta = (\phi - r)^2 \kappa (\sigma_0 \sigma_\xi)^2 \alpha |\beta| > (\kappa - r)^2 \kappa (\sigma_0 \sigma_\xi)^2 / 16$, the desired inequality will be guaranteed to hold for all $(\alpha, \beta) \in I$ if $r < r_\triangleright$, where $r_\triangleright \in (0, \kappa)$ is defined by the equality

$$r_\triangleright = \frac{\kappa (\kappa - r)^2 \sigma_\xi^2}{16}.$$ 

This concludes the proof. $\square$
S2 Discretized Model and Limit Demand Sensitivity

This appendix introduces a sequence of discrete-time counterparts of our continuous-time game that allows us to refine the concept of stationary linear Markov equilibrium by choosing a sensitivity of demand equal to -1. More specifically, we will show that, along such sequence, -1 is the limiting value of the sensitivity of demand arising from the consumer’s best-response problem as the period length shrinks to zero.

Fix $\Delta > 0$ and consider a consumer who interacts with a sequence of short-run firms in a stochastic game of period length $\Delta$. Specifically, at each $t \in T := \{0, \Delta, 2\Delta, 3\Delta, \ldots\}$ the consumer shops for a product that is supplied by a single firm ($\text{firm } t$). The timing of events over $[t, t + \Delta)$ is as in the baseline model: first, firm $t$ posts a price; second, having observed this price, the consumer chooses how much to buy; third, the purchase is recorded with noise, and subsequently incorporated into the score. The same sequence of events then repeats at $[t + \Delta, t + 2\Delta)$, but now with the next firm.

The discretized model consists of the dynamics

$$
\theta_{t+\Delta} = \theta_t - \kappa \Delta (\theta_t - \mu) + \sqrt{\Delta} \epsilon^\theta_{t+\Delta}
$$

$$
Y_{t+\Delta} = Y_t - \phi \Delta Y_t + Q_t \Delta + \sqrt{\Delta} \epsilon^\xi_{t+\Delta}
$$

where $\epsilon^\theta_t \sim N(0, \sigma^2_\theta)$ and $\epsilon^\xi_t \sim N(0, \sigma^2_\xi)$ are independent across time, and the sequences $(\epsilon^\theta_t)_{t \in T}$ and $(\epsilon^\xi_t)_{t \in T}$ independent from one another. Finally, the consumer’s utility over period $[t, t + \Delta)$ given $(\theta_t, P_t, Q_t) = (\theta, p, q)$ takes the form

$$
u^\Delta(\theta, p, q) = \left( (\theta - p)q - \frac{q^2}{2} \right) \Delta.
$$

It is easy to see that if the firms conjecture a strategy for the consumer that is linear in $(p, \theta, M)$ with weight $-\zeta \neq 0$ on the current price, then, from their perspective, realized prices, $(P_t)_{t \in T}$, and realized quantities, $(Q_t)_{t \in T}$, satisfy $P_t = E[Q_t|Y_t]/\zeta$, $t \in T$. Thus, firms set prices and conjecture past quantities according to

$$
P_t = \frac{\delta + (\alpha + \beta) M_t}{\zeta} \quad \text{and} \quad Q_t = \delta + \alpha \theta_t + \beta M_t, \quad t \in \{0, \Delta, 2\Delta, \ldots\},
$$

respectively, for some coefficients $\zeta, \alpha, \beta$ and $\delta$. We allow this conjectured coefficients to depends on $\Delta$. However, we make two assumptions. First, we restrict the analysis to the case $\zeta > 0$, $\alpha > 0$, $\beta < 0$ and $\alpha + \beta > 0$. Second, we assume that all the coefficients are bounded in a neighborhood of $\Delta = 0$, and also bounded away from zero. Observe that these are minimal properties that a meaningful dynamic extension of the outcome of a static interaction must have. In what follows, we omit the dependence of the coefficients on the period length.
We now proceed in three steps. First, we find an expression for the weight that the consumer’s best-response attaches to the current price when firms both set prices and form beliefs using (S.9). Call this weight \( -\hat{\zeta} \). Second, we show that at any history at which firm \( t \) sets a price different than the one prescribed by (S.9), the consumer optimally responds with the same linear strategy used along the path of (S.9); thus \( -\hat{\zeta} \) is effectively the sensitivity of demand. Third, we show that \( -\hat{\zeta} \) goes to -1 as \( \Delta \searrow 0 \).

Importantly, these steps hold under any linear conjecture by the firms (in particular, for \( \zeta \neq \hat{\zeta} \)), satisfying our requirements on bounds. Thus, \( \hat{\zeta} = 1 \) is a limiting property of the consumer’s best-response along the sequence of games.

**Step 1.** Since from each firm’s perspective the score carries past quantities that satisfy (S.9), \( M_t := \mathbb{E}[\theta_t|Y_t] = \rho + \lambda Y_t \) for some \( \rho \in \mathbb{R} \) and \( \lambda > 0 \) (potentially depending on \( \Delta \)). In this case,

\[
M_{t+\Delta} - M_t = \lambda[Y_{t+\Delta} - Y_t] = \lambda[-\phi \Delta(M_t - \rho)/\lambda + Q_t \Delta + \sqrt{\Delta} \epsilon _{t+\Delta}]
\]

\[
\Rightarrow M_{t+\Delta} = M_t - \phi \Delta(M_t - \rho) + \lambda Q_t \Delta + \lambda \sqrt{\Delta} \epsilon _{t+\Delta}, \quad t \in T.
\]

Let \( V \) denote the consumer’s value function when facing prices as stated in (S.9). Then, the following Bellman equation holds:

\[
V(\theta, M) = \max_{q \in \mathbb{R}} \left\{ \left[ \left( \theta - \frac{\delta + (\alpha + \beta)M}{\zeta} \right) q - q^2/2 \right] \Delta + e^{-r\Delta} \mathbb{E}[V(\theta', M')|(M, \theta)] \right\}
\]

s.t.

\[
\begin{align*}
\theta' &= \theta - \kappa \Delta(\theta - \mu) + \sqrt{\Delta} \epsilon ^\theta \\
M' &= M - \phi \Delta(M - \rho) + \lambda q \Delta + \lambda \sqrt{\Delta} \epsilon ^\xi.
\end{align*}
\]

We look for a quadratic value function, i.e., \( V(\theta, M) = v_0 + v_1 \theta + v_2 M + v_3 M^2 + v_4 \theta^2 + v_5 \theta M \), where we omit the dependence of the coefficients on \( \Delta \). Letting \( X := (\theta, M) \), we have that \( V(X') = V(X + DV(X)(X' - X) + \frac{1}{2}(X' - X)^{\top}D^2V(X' - X) \), and straightforward algebra shows that the Bellman equation further reduces to

\[
V(\theta, M) = \max_{q \in \mathbb{R}} \left\{ \left[ \left( \theta - \frac{\delta + (\alpha + \beta)M}{\zeta} \right) q - q^2/2 \right] \Delta + e^{-r\Delta} V(\theta, M) \right. \\
+ e^{-r\Delta} \left[ \kappa[\theta - \mu]V_\theta + [-\phi (M - \rho) + \lambda q]V_M + \frac{1}{2} V_{\theta \theta} [\Delta^2 \epsilon ^{\theta} + \sigma _{\theta}^2] \right] \\
+ e^{-r\Delta} V_{\theta M} [\kappa(\theta - \mu) [-\phi (M - \rho) + q \lambda \Delta]] \\
+ e^{-r\Delta} \frac{1}{2} V_{MM} [q^2 \Delta (M - \rho)^2 + \lambda^2 q^2 \Delta + \lambda^2 \sigma _{\theta}^2 - 2 \phi \lambda \Delta (M - \rho) q] \right\}.
\]

The first-order condition of this problem reads

\[
[1 - e^{-r\Delta} \lambda^2 \Delta V_{MM}] q = \theta - \frac{\delta + (\alpha + \beta)M}{\zeta} + e^{-r\Delta} (\lambda V_M + \Delta V_{\theta M} [-\kappa (\theta - \mu)] - V_{MM} \phi \Delta (M - \rho) \lambda),
\]

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from where the contemporaneous price has a weight equal to
\[ -\hat{\zeta} = -\frac{1}{1 - e^{-r\Delta^2\lambda^2\Delta V_{MM}}} = -\frac{1}{1 - 2e^{-r\Delta^2\lambda^2}\Delta v_3} \]
in the consumer’s linear best-response. As we show in step 3, \( \zeta \), which enters as a parameter in the consumer’s best-response problem, turns out to affect coefficient \( v_3 \). In particular, to show that \( \lim_{\Delta \to 0} -\hat{\zeta} = 1 \), it suffices that \( \Delta v_3 \downarrow 0 \) while the rest of the terms that accompany it remain bounded. We turn to the the sensitivity of demand first.

**Step 2.** Consider now a history at which firm \( t \) posts a price \( p \neq [\delta + (\alpha + \beta)M_t]/\zeta \). It is easy to see that at any such history the consumer’s problem is of the form

\[
\max_{q \in \mathbb{R}} \left\{ \left[ (\theta - p) q - q^2/2 \right] \Delta + e^{-r\Delta} E[V(\theta', M')|(M, \theta)] \right\} \\
\text{s.t.} \quad \theta' = \theta - \kappa \Delta (\theta - \mu) + \sqrt{\Delta} \epsilon \theta \\
M' = M - \phi \Delta (M - \rho) + \lambda q \Delta + \lambda \Delta \epsilon \xi.
\]

In fact, since the deviation is not observed by subsequent firms, the consumer's continuation payoff given any fixed continuation strategy is unaffected by the deviation. But this implies that her continuation value—i.e., her best continuation payoff among admissible strategies—must be given by \( V \) found by solving the Bellman equation of the previous step. As a result, the consumer’s optimal strategy is determined by the same first-order condition. In particular, \( -\hat{\zeta} \), the weight that the linear best-response attaches to the current price, is effectively the sensitivity of demand.

**Step 3.** It is straightforward to verify that \( v_3 \) can be found by setting the coefficient on \( M^2 \) in the Bellman equation equal to zero. Such equation is

\[ 4\xi^2\Delta \lambda^2 v_3^2 + 2\xi v_3 \{\xi[(1 - \Delta \phi)^2 - e^{\Delta r}] - 2\Delta(1 - \Delta \phi)\lambda(\alpha + \beta)\} + e^{\Delta r} \Delta(\alpha + \beta)^2 = 0. \]

Letting \( \Gamma := \xi[(1 - \Delta \phi)^2 - e^{\Delta r}] - 2\Delta(1 - \Delta \phi)\lambda(\alpha + \beta) \), the two solutions are given by

\[ v_3^\pm = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\xi^2\Delta \lambda^2 e^{\Delta r}(\alpha + \beta)^2}}{2\xi \Delta \lambda^2}. \]

The square root is well defined for small \( \Delta \) due to \( \Delta^2[\xi(r + 2\phi) + 2\lambda(\alpha + \beta)] \) and \( 4\Delta^2\lambda^2(\alpha + \beta)^2 \) being the terms that dominate for low \( \Delta \) in \( \Gamma^2 \) and \( 4\Delta^2\lambda^2 e^{\Delta r}(\alpha + \beta)^2 \) respectively.

We now show that \( \Delta v_3^\pm \downarrow 0 \) as \( \Delta \downarrow 0 \) (but as we show below, \( v_3^- \) is the root associated with the equilibrium examined in the paper). To this end, observe that \( \lambda \) also depends
on $\Delta$. A calculation presented at the end of this appendix shows that this value satisfies the equation $F(\Delta, \lambda) = 0$ where

$$F(\Delta, \lambda) := \lambda - \frac{\sigma^2 \alpha (1 - \kappa \Delta)[2(\phi - \beta \lambda) - (\phi - \beta \lambda)^2 \Delta]}{\sigma^2[2\kappa - \kappa^2 \Delta][(\phi - \beta \lambda)(1 - \kappa \Delta) + \kappa] + \sigma^2 \alpha^2[2 - \kappa \Delta - (\phi - \beta \lambda)(1 - \kappa \Delta) \Delta]}.$$  

It is easy to verify that, at $\Delta = 0$, the previous equation reduces to the quadratic function that determines the sensitivity of beliefs in the continuous-time game analyzed (equation (7) in the paper). Let $\lambda_0 = \Lambda(\phi, \alpha, \beta) > 0$, as in the paper. By definition of $\lambda_0$, $F(0, \lambda_0) = 0$. Moreover, since $\beta < 0$ and $\lambda_0 > 0$

$$\frac{\partial F}{\partial \lambda}(0, \lambda_0) = \frac{(\sigma^2 \alpha^2 + \sigma^2 \kappa[\phi + \kappa - \beta \lambda_0])^2 + \beta \sigma^2 \alpha \sigma^2 \alpha^2 + \kappa^2 \sigma^2 \xi)}{(\sigma^2 \alpha^2 + \sigma^2 \kappa[\phi + \kappa - \beta \lambda_0])^2} > \frac{\sigma^4 \alpha^3[\alpha + \beta] + \sigma^2 \kappa[2\alpha + \beta]}{(\sigma^2 \alpha^2 + \sigma^2 \kappa[\phi + \kappa - \beta \lambda_0])^2} > 0$$

where the last inequality follows from $\alpha + \beta > 0$. By the Implicit Function Theorem, therefore, the exists $\varepsilon > 0$ and a unique continuously differentiable function $\lambda(\Delta)$ such that $\lambda(0) = \lambda_0$, $F(\Delta, \lambda(\Delta)) = 0$, and $\lambda(\Delta) > 0$, for all $\Delta \in [0, \varepsilon]$.

Since $\lambda(\cdot)$ is bounded in that set, and both $\lambda(\cdot)$ and $\zeta$ being bounded away from zero, we conclude that

$$\Delta v_3^+ = -\frac{\Gamma \pm \sqrt{\Gamma^2 - 4\Delta^2 \lambda^2(\Delta) e^{\Delta \tau}(\alpha + \beta)^2}}{2\zeta \lambda^2(\Delta)} \rightarrow 0, \text{ as } \Delta \searrow 0,$$

due to the rest of the coefficients being bounded and $\Gamma := \zeta[(1 - \Delta \phi)^2 - e^{\Delta \tau}] - 2\Delta(1 - \Delta \phi)\lambda(\alpha + \beta)$ also vanishing in the limit. This proves step 3.

Before showing that $F(\Delta, \lambda) = 0$ is the equation for the sensitivity of beliefs that makes the quantity process (S.9) consistent with Bayesian updating, we make two observations.

1. It is easy to see that when $\zeta = 1$, then, as $\Delta \searrow 0$,

$$v_3^+ \rightarrow 2\lambda_0(\alpha + \beta) + (r + 2\phi) \pm \sqrt{[2\lambda_0(\alpha + \beta) + (r + 2\phi)]^2 - 4\lambda_0^2(\alpha + \beta)\Delta^2}$$

the right-hand side being the two positive roots for the equation that $v_3$ must satisfy in the continuous-time program.\(^1\) However, an equilibrium condition of the continuous-time model is $2\lambda v_3 = \alpha + 2\beta$ (last equation in (A.7)). As a result, either

$$2\lambda v_3^+ = \alpha + 2\beta \text{ or } 2\lambda v_3^- = \alpha + 2\beta$$

\(^1\)This equation can be obtained as follows: first, use (A.7) to solve for $(\alpha, \beta, \delta)$ as a function of $(v_2, v_3, v_3)$; second, insert the first-order condition (display preceding (A.7) in Appendix A) into (A.8); and, finally, equate the coefficient on $M$ to zero in the resulting equation.
must hold. However, the previous conditions reduce to
\[ r + 2\phi \pm \sqrt{(r + 2\phi)^2 + 4\lambda(\alpha + \beta)(r + 2\phi)} = 2\beta\lambda. \]

Since \( \beta < 0 \) in the equilibrium found, only \( v_3^- \) converges to the value of \( v_3 \) in the equilibrium studied.

2. In equilibrium, \( \hat{\zeta} = \zeta \). Using \( v_3^- \), straightforward algebra shows that this condition becomes
\[
e^{r\Delta}(\zeta - 1) = -\Gamma(\zeta) - \sqrt{\Gamma^2(\zeta) - 4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha + \beta)^2}
= -\Gamma(\zeta) + \sqrt{\Gamma^2(\zeta) - 4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha + \beta)^2},
\]
where the dependence of \( \Gamma \) on \( \zeta \) is being made explicit. For sufficiently small \( \Delta \), however, \( (1 - \Delta\phi)^2 - e^{r\Delta} < 0 \) and so \( -\Gamma(\zeta) > 0 \) for all \( \zeta \geq 1 \). The linearity of both \( 2(\zeta - 1) \) and \( \Gamma(\zeta) \) in \( \zeta \) then yields the existence of \( \zeta^* \) such that the previous equality holds. In particular, the convergence to of \( \zeta \) to 1 along a sequence of equilibria must be from above.

**Equation for \( \lambda \).** We conclude with the derivation of the equation that \( \lambda \) must satisfy for small \( \Delta \). For notational simplicity, we set \( \mu = \rho = \delta = 0 \), as the means and intercepts do not affect the sensitivity of beliefs.

Define the matrices
\[
X := \begin{bmatrix} \theta \\ Y \end{bmatrix}; \ A_\Delta := \begin{bmatrix} 1 - \kappa\Delta & 0 \\ \alpha\Delta & 1 - (\phi - \beta\lambda)\Delta \end{bmatrix}; \ B := \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_\xi \end{bmatrix}; \ \bar{\epsilon} := \begin{bmatrix} \epsilon^\theta \\ \epsilon^\xi \end{bmatrix}
\]
and notice that
\[
X_{(j+1)\Delta} = A_\Delta X_{j\Delta} + \sqrt{\Delta}B\bar{\epsilon}_{(j+1)\Delta}, \ j \in \mathbb{N}.
\]
The solution to this difference equation is given by
\[
X_{(j+1)\Delta} = A_\Delta^{j+1}X_0 + \sqrt{\Delta}A_\Delta^{j+1}\sum_{i=0}^{j} A_\Delta^{-(j+1-i)}B\bar{\epsilon}_{(j+1-i)\Delta}.
\]
To obtain a stationary Gaussian process, therefore, we impose first that \( X_0 \) is Gaussian and independent of \( (\bar{\epsilon}_j\Delta)_{j \in \mathbb{N}} \). Moreover, stationary requires that \( \bar{\mu} := \mathbb{E}[X_0] = 0 \), so as to obtain \( \mathbb{E}[X_{j\Delta}] = 0 \) for all \( j \in \mathbb{N} \). In addition, omitting the dependence on \( \Delta \), let \( \Gamma \) denote the candidate covariance matrix of \( (X_{j\Delta})_{j \in \mathbb{N}} \). It follows that
\[
\Gamma = A_\Delta^{j+1}\Gamma(A_\Delta^{j+1})^\top + \Delta A_\Delta^{j+1}\left[ \sum_{i=0}^{j} A_\Delta^{-(j+1-i)}B^2(A_\Delta^{-(j+1-i)})^\top \right] (A_\Delta^{j+1})^\top, \ \forall j \in \mathbb{N}.
\]

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Moreover, taking consecutive differences leads to
\[
0 = A^j_\Delta \{ A_\Delta \Gamma A_\Delta^\top - \Gamma \\ + \Delta A_\Delta \left[ \sum_{i=0}^{j-1} A_\Delta^{-(j+1-i)} B^2 (A_\Delta^{-(j+1-i)})^\top \right] A_\Delta^\top - \Delta \left[ \sum_{i=0}^{j-1} A_\Delta^{-(j-i)} B^2 (A_\Delta^{-(j-i)})^\top \right] \} \right\} (A^j_\Delta)^\top,
\]
and thus, \( \Gamma \) is defined by the equation
\[
A_\Delta \Gamma A_\Delta^\top - \Gamma + \Delta B^2 = 0.
\]
Straightforward algebra leads to the following equations for the unknowns \( \Gamma_{11} = \text{Var}[\theta_{j\Delta}] \), \( \Gamma_{12} = \Gamma_{21} = \text{Cov}[\theta_{j\Delta}, Y_{j\Delta}] \), and \( \Gamma_{22} = \text{Var}[Y_{j\Delta}] \), \( j \in \mathbb{N} \):
\[
\Gamma_{11} (1 - \kappa \Delta)^2 - \Gamma_{11} + \Delta \sigma^2_\theta = 0
\]
\[
\Gamma_{11} \alpha \Delta (1 - \kappa \Delta) + \Gamma_{12} (1 - (\phi - \beta \lambda) \Delta) (1 - \kappa \Delta) - \Gamma_{12} = 0
\]
\[
\Gamma_{11} (\alpha \Delta)^2 + 2 \Gamma_{12} (1 - (\phi - \beta \lambda) \Delta) \alpha \Delta + \Gamma_{22} (1 - (\phi - \beta \lambda) \Delta)^2 - \Gamma_{22} + \Delta \sigma^2_\xi = 0.
\]
This system has as a solution
\[
\Gamma_{11} = \frac{\sigma^2_\theta}{2 \kappa - \kappa^2 \Delta}
\]
\[
\Gamma_{12} = \frac{\alpha \sigma^2_\theta (1 - \kappa \Delta)}{[2 \kappa - \kappa^2 \Delta] [\phi - \beta \lambda + \kappa - (\phi - \beta \lambda) \kappa \Delta]}
\]
\[
\Gamma_{22} = \frac{1}{2(\phi - \beta \lambda) - (\phi - \beta \lambda)^2 \Delta} \left[ \sigma^2_\xi + \frac{\sigma^2_\theta \alpha^2 \Delta}{2 \kappa - \kappa^2 \Delta} + \frac{2 \alpha [1 - (\phi - \beta \lambda) \Delta] \sigma^2_\theta \alpha (1 - \kappa \Delta)}{[2 \kappa - \kappa^2 \Delta] [\phi - \beta \lambda + \kappa - (\phi - \beta \lambda) \kappa \Delta]} \right].
\]
(In particular, observe that we recover the expression for \( \Gamma \) in continuous time by letting \( \Delta \to 0 \) and replacing \( \lambda \) by \( \lambda_0 \).) To conclude, because
\[
\lambda = \frac{\text{Cov}[\theta_{j\Delta}, Y_{j\Delta}]}{\text{Var}[Y_{j\Delta}]} = \frac{\Gamma_{12}(\Delta, \lambda)}{\Gamma_{22}(\Delta, \lambda)},
\]
straightforward algebra yields
\[
\lambda = \frac{\sigma^2_\theta \alpha (1 - \kappa \Delta) [2(\phi - \beta \lambda) - (\phi - \beta \lambda)^2 \Delta]}{\sigma^2_\xi [2 \kappa - \kappa^2 \Delta] [\phi - \beta \lambda (1 - \kappa \Delta) + \kappa] + \sigma^2_\theta \alpha^2 2 - \kappa \Delta - (\phi - \beta \lambda) (1 - \kappa \Delta) \Delta}.\]
This concludes the proof. \( \square \)

References