Shock Persistence, Endogenous Skills and Career Concerns*

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May 2012

Abstract

I develop a continuous-time model of career concerns that incorporates human capital accumulation throughout the working life. In this model a worker is able to generate an output in diffusion form, with a drift that is the sum of the worker’s effort and skills. Skills are modeled as Gaussian diffusions with an endogenous drift component reflecting on-the-job experience accumulation. I find that workers’ incentives are crucially determined by the persistence of innovations to productivity. In this line, both under and over-provision of effort are robust steady-state equilibrium outcomes in settings where skills are exogenous. I also show that the value which reputation-driven agents attach to investing in human capital is always below its social counterpart. This is despite workers internalizing the full benefits and costs from acquiring skills. Finally, wages have both an effort and a reputational component. The latter always mean-revert towards current skills, which act as a moving-trend. As a result, human capital accumulation is able to generate increasing and concave profiles of wages.

Keywords: career concerns, human capital, incentives, learning, reputation.

JEL codes: D82, D83, J24, L14, M53.

*Email: gcistern@princeton.edu. A preliminary version of this work was under the title “A Continuous-time Model of Career Concerns and Human Capital Accumulation”. I am most grateful to Yuliy Sannikov for his guidance and valuable feedback throughout writing this paper, and to Hector Chade for very helpful discussions. I am also grateful to Benjamin Brooks, Sylvain Chassang, Henry Farber, Leandro Gorno, Jay Lu, Alex Mas and Stephen Morris for their valuable feedback.
1 Introduction

Environmental uncertainty plays an important role in the evolution of workers’ perceived skills. While firms can influence an employee’s productivity through tailored programs such as compensation schemes, on-the-job training or learning-by-doing, exogenous forces that affect the work environment can also have an important impact on performance. For instance, a worker’s productivity may vary because unforeseen events force him to be assigned to a different task at which his productivity changes, or because tasks itself evolve due to technological progress. These changes also affect an employer’s inference process about a worker’s ability, as current performance can become a poor predictor of future one. In settings where wages are based on perceived skills, the degree of randomness of the environment is thus expected to influence the strategic behavior of a worker whose ultimate goal is to affect his future income stream by building a good reputation.

In this paper I study how career concerns are shaped by the degree of randomness in the job environment. More specifically, I build on Holmstrom’s (1982, 1999) seminal paper of career concerns in order to construct a continuous-time model of reputation that extends his work along two dimensions. First, I allow skills to be any process within the class of Gaussian diffusion (the continuous-time analog of an AR(1) process), with the persistence of shocks to productivity being the measure of environmental uncertainty. Second, I allow the worker to take actions that directly affect productivity. Since these actions have persistent effects on skills, they have the flavor of investments in human capital. The outcome is a very flexible and general framework that provides particularly clean insights on how belief-distortion mechanisms operate, on how wages and effort levels evolve over time, and on the extent to which reputation motives generate socially efficient outcomes.

Traditional career concerns models have focused on issues such as the extent of markets’ efficiency (Holmstrom’s paper), on how different information structures and multitasking affect incentives (Dewatripont et al. 1999a,b), on the interplay between implicit incentives and short-term contracts (Gibbons and Murphy 1992) and even on herding behavior (Scharfstein and Stein 1990). In this paper, in turn, I provide a detailed analysis of how different degrees of environmental uncertainty influence the reputational motives faced by individuals in dynamic settings. It is widely understood that forward-looking agents evaluate not only the immediate reputational benefits from their actions, but also their long-run consequences. Such analysis is particularly relevant in settings where market participants have precise estimates of a worker’s skills, and hence, convergence to a neighborhood of the stationary learning level occurs relatively fast. I show that the persistence of shocks to productivity (which is what really determines the long-run uncertainty associated to
skills) crucially affects the size of the incentives created by career concerns. Moreover, I discover that Holmstrom’s classic efficiency result (skills evolving as a random walk and an infinitely patient worker) is truly an exception, with both under- and over-provision of effort as robust equilibrium outcomes for more general skills processes. Inefficiencies within career concerns models with exogenous Gaussian skills are a pervasive phenomena, going beyond discounting and transient-learning considerations.

The choice of a continuous-time framework is largely motivated by the intention to provide clean insights on how the dynamics of learning determine the gains that arise from “signal-jamming”. In continuous-time settings of learning with Gaussian processes (Liptser and Shiryaev 1977), the evolution of posterior beliefs reduce to a stochastic differential equation for the posterior mean and an ordinary differential equation for the posterior second moment. While the latter is completely exogenous, the former depends on the observed path of output and thus is controlled by the worker through his effort decision. In the absence of human capital accumulation, workers evaluate how much effort to exert, taking into account both how responsive beliefs are to new output observations, and how fast these beliefs subsequently decay over time. The strength of these forces are measured by what I call the sensitivity of beliefs to new information and the rate at which beliefs discount past output observations, respectively. While higher values of the sensitivity process increase the short-term benefits from belief-distortion, higher values of the discounting process make these distortions less persistent, and thus less attractive. In a stationary-learning setting the sensitivity-discount ratio corresponds to a measure of the overall responsiveness of beliefs to aggregate information, and this is what determines the benefits from belief-distortion. Most interestingly, this ratio is strictly increasing in the degree of persistence of shocks to skills, allowing us to draw a simple connection between the degree of randomness in the environment and the corresponding incentives created by career concerns: as shocks become more persistent, beliefs are gradually more responsive to overall information, and thus higher effort levels are induced.

I study human capital accumulation in order to understand how the reputational incentives that workers face are influenced by the possibility of the workers becoming endogenously more productive. In such a context, the market’s inability to observe skills creates belief-manipulation motives on the workers’ side that can be exploited through hidden investment decisions which boost productivity. Most importantly, I argue that investments in human capital are, in general, inefficiently low. This is despite the facts that labor markets are competitive, that there is no limited liability and that workers bear the full cost of training. Distorted incentives to invest in skills arise because, in reputation-driven markets, workers value the option to invest in human capital if and only if it can
be used to influence the market’s beliefs about skills. Market participants know that due to the persistent effects that human capital has on skills, a temporary additional unit of it today maps into an additional output stream that a more skilled worker is able to produce. Nevertheless, I show that the market is able to anticipate only a fraction of the flow actually realized. Competition then forces the market to pay the expected value of this anticipated stream as an ex-ante premium, and thus it ceases to have any reputational value for the worker. The unanticipated component of this additional output stream is thus attributed to non-observable skills improvement, and it is what determines the worker’s marginal private benefit from a temporary additional unit of human capital. In a couple of examples I show how this discrepancy in marginal values actually generates inefficiently low investments in human capital.

The reason why the market is not able to anticipate the entire additional output flow coming from human capital accumulation is purely due to discounting. In fact, the market’s belief process discounts past information at rates always higher than the rates at which skills keep track of past productivity shocks. This is because market participants have the history output not explained by effort as their only source of information. Such a process is the sum of both signal noise and current skills, which cannot be disentangled. Therefore, given that this process is not truly a martingale, the way in which optimal beliefs filter the information conveyed by this signal is by instantaneously reacting to new output observations, but making these reactions decay relatively quickly. This in turn can be strategically used by the worker to extract additional rents from belief-distortion. Even though it is this discounting wedge which makes human capital accumulation valuable for the worker, learning on the market’s side is never diffuse enough to induce efficiency.

Following this, I give a detailed study of two types of human capital accumulation technologies that differ in the degree of irreversibility of the investment technology mapping investments into skills. In the weak complementarity case investment occurs through deviating effort to an alternative, but related, activity. Moreover, investments are perfectly reversible such that temporary ones have low persistence effects on output. I show that, under some circumstances, the option to acquire training is delayed: in those environments, the signaling incentives generated by career concerns are so strong that workers initially focus on influencing the market’s perception about themselves, and then on investing in skills. Finally, in the strong complementarity case I study the incentives that are created when human capital accumulation arises as a byproduct of final goods production. Furthermore, these investments are more irreversible than in the previous case, so temporary investments have more persistent effects on output. In such a setting I show that effort profiles are always larger than predicted in career concern models with exogenous skills.
In addition, as long as there is long-run residual uncertainty concerning beliefs, the effort component associated with human capital accumulation never vanishes.

Wages in the model have a reputational component and, if the worker can directly influence output through the choice of an action, an effort component as well. The latter monotonically decreases, as a consequence of beliefs becoming less responsive to new information over time. Nevertheless, human capital accumulation introduces a positive drift in the beliefs process and, therefore, wages can present increasing and concave profiles on average. That is, the model is able to generate the observed life-cycle pattern of wages through the traditional channel of human capital accumulation (Becker 1964) and returns to experience (Mincer 1974 and Ben-Porath 1967). Even more interesting is the result that, because of learning, the posterior mean always locally mean-reverts toward the current true value of skills. In numerical examples I show how reversion towards an stochastically-evolving trend can generate positive autocorrelation of changes in wages under transient learning, and negative autocorrelation in steady state.

This paper is related to various strands in the economic literature. Regarding career concern models with Gaussian skills, Holmstrom (1982, 1999) provided a formal framework to analyze Fama’s (1980) conjecture that competitive markets are sufficient for inducing efficient incentives. He found that stable environments (fixed skills), discounting and transient-learning effects, among other things, can invalidate Fama’s claim. Nonetheless, he showed that efficiency is achieved in the random walk case provided the previous conditions are not met. In a static setting Dewatripont et al. (1999b) analyzed the effects of additional tasks on career concerns motives, where they stress the importance that focusing has on incentives. Gibbons and Murphy (1992) in turn studied the effects of short-term linear contracting on incentives in the presence of career concerns. They show (theoretically and empirically) that the sensitivity of optimal wages to performance increases with tenure.

The literature on human capital accumulation is extensive, with Becker (1964) and Mincer (1974) as classic references. Rosen (1972) has emphasized the role of jobs as investment opportunities where workers improve their skills and thus increase their productivity. The idea that workers, through performing tasks, can acquire skills which in turn are valued by the rest of the market can be understood as task-specific human capital (see Gibbons and Waldman 2004, 2005). Regarding wages’ structure, Abowd and Card (1988) found negative first-order autocorrelation in changes of wages using longitudinal date on earnings. They also documented no significant autocorrelation for changes in wages separated for more than two periods. Farber and Gibbons (1996) rejected their pure learning model’s prediction that the residuals of wages should evolve as a martingale. In a similar vein, the
error measure in my model— the gap between beliefs and true skills— is not a martingale, but instead a mean-reverting process around zero. Closely related to my paper is the work of Kahn and Lange (2011). They find that combining learning and evolving productivity does a better job at matching the covariance structure of wages in the data used by Baker, Gibbs and Holmstrom (1994a,b) than a pure learning or pure productivity model by themselves would. Finally, this paper is to some extent related to continuous-time techniques for addressing dynamic incentive problems. In particular, Sannikov (2008) developed a continuous-time framework to analyze a principal-agent interaction from a dynamic programming perspective. In his recursive formulation of the problem, the sensitivity of the agent’s continuation-value to new output observations plays a crucial role in shaping the agent’s incentives. Similarly, the sensitivity of beliefs to new information determines an important part of the gains from belief-distortion in the model presented here.1

In the Section 2 I present the general model. In Section 3 I study how learning in Gaussian settings takes place, I analyze the forces behind belief-distortion and show that the existence of equilibria in deterministic strategies is reduced to a simple optimization problem. In Section 4 I analyze career concerns models with exogenous skills. In Section 5 I add human capital accumulation and discuss the on-equilibrium evolution of wages. I conclude in Section 6. All proofs are relegated to Appendix A.

2 The Model

2.1 Output Technology, Skills Process and Human Capital

Consider a worker who is able to produce an output \( \xi := (\xi_t)_{t \geq 0} \) continuously over time. I assume it obeys the following dynamic

\[
d\xi_t = (a_t + \theta_t)dt + \sigma_\xi dZ^\xi_t, \quad t \geq 0, \tag{1}
\]

where \( Z^\xi := (Z^\xi_t)_{t \geq 0} \) is a one-dimensional Brownian motion, \( \sigma_\xi > 0 \) represents the volatility of the signal’s noise component and \( a_t \) is the worker’s effort choice at time \( t \), which is subject to moral hazard. The term \( \theta_t \) is a random variable representing some measure of the worker’s current skills, \( t \geq 0 \). Equivalently, it could be interpreted as the value of the worker’s ability in a changing environment.2 The stochastic process \( \theta := (\theta_t)_{t \geq 0} \) is not

1 An important difference between both models is that in Sannikov’s problem the principal controls the sensitivity process, while in my model it is completely exogenous.

2 In Jovanovic (1979) market participants learn about the quality of a firm-specific match between a worker and a firm. A similar matching interpretation could be applied to the model presented here if the match is understood as task-specific within a changing environment.
observable by the market participants, and I refer to it as the worker’s \textit{skills process}. The output process $\xi$ is a public signal in the economy.\footnote{The choice of this technology greatly simplifies the analysis and is standard in career concerns papers. The combination of complementarities in the production function, moral hazard and learning, within fully dynamic models is still an open question in the literature.}

I assume that skills evolve according to the stochastic differential equation (SDE)

$$d\theta_t = (\bar{\theta}_t(a) + \kappa \theta_t)dt + \sigma_\theta dZ^\theta_t, \ t \geq 0. \tag{2}$$

Here $Z^\theta := (Z^\theta_t)_{t \geq 0}$ is a one-dimensional Brownian motion independent from $Z^\xi$ representing shocks that affect true skills (or alternatively, that affect the value of a worker’s ability in a changing environment). I refer to these shocks as \textit{productivity shocks} and the parameter $\sigma_\theta > 0$ measures their volatility. The family $(\bar{\theta}_t(\cdot))_{t \geq 0}$ is an endogenous trend affecting the way in which skills grow over time (see Assumption 1 below for more details). Finally, the parameter $\kappa \in \mathbb{R}$ will be referred as the \textit{slope} of the skills process. This general specification is able to encompass a wide variety of situations: workers suffering productivity shocks of different persistence levels, environments evolving at different speeds, and workers endogenously adapting to the environment, among others.

To understand the power of this specification, suppose first that skills evolve in a completely exogenous fashion, i.e. $\bar{\theta}_t(\cdot) \equiv \bar{\theta} \in \mathbb{R}$, for all $t \geq 0$. In this case, by varying $\bar{\theta}$, $\kappa$ and $\sigma_\theta$ the entire class of time-homogeneous Gaussian diffusions is covered.\footnote{A time-homogeneous Gaussian diffusion $X := (X_t)_{t \geq 0}$ corresponds to an Ito process such that its volatility is constant and its drift is affine with constant coefficients, i.e. it satisfies the SDE $dX_t = (\alpha + \beta X_t)dt + \sigma dZ_t$ for some $\sigma, \alpha, \beta \in \mathbb{R}$, where $Z := (Z_t)_{t \geq 0}$ is a Brownian motion.} Particular examples are:

(i) Constant skills: $\sigma_\theta = \kappa = \bar{\theta} = 0$;

(ii) Skills evolving as a martingale: $\kappa = \bar{\theta} = 0$, $\sigma_\theta \neq 0$;

(iii) Mean-reverting skills: $\kappa < 0$, $\sigma_\theta \neq 0$;

(iv) Skills growing at a positive rate: $\kappa > 0$, $\sigma_\theta \neq 0$;

(v) I.i.d. skills: $\sigma_\theta = \sigma \sqrt{|\kappa|}$, $\sigma \neq 0$, and $\kappa \to -\infty$. In this case $\theta_t \sim N(0, \sigma^2/2)$ for all $t \geq 0$.

Situations with non-zero volatility can be interpreted as environments subject to important technological changes, or settings in which workers suffer from non-negligible productivity shocks. The relative strength of these shocks is represented by the drift of the skills process. The martingale specification represents rapidly changing environments and
workers who easily adjust to new scenarios. For example, technological changes that put pressure on the worker falling into obsolescence are, on average, immediately canceled out by (unmodeled) skills accumulation. As a consequence, whether the worker will become more or less productive relative to the environment cannot be anticipated, as captured by the zero-drift condition. In the mean-reverting specification (the continuous-time analog of an AR(1) process with root less than 1), the parameter $\frac{\bar{\sigma}}{|\kappa|}$ corresponds to a value towards which skills are expected to converge in the long-run. Since skills are driven back to this mean-trend whenever away from it, there is some short-run predictability on the value of the worker’s ability. In such a setting, productivity shocks are less persistent than in the martingale formulation and the rate at which they decay is measured by $\kappa$. In fact, as this parameter decreases, shocks tend to be less and less persistent, disappearing almost instantaneously in the limit as $\kappa \to -\infty$. A mean-reverting specification of skills may represent a worker subject to daily productivity shocks that temporarily push him away from his human capital level (the mean trend), but which tend to disappear on average as the time horizon expands. Alternatively, it could represent the value of a worker’s ability in environments where unmodeled frictions prevent immediate adjustments. In this sense, any advantage or disadvantage relative to the environment is expected to persist in the near future, but is also expected to disappear gradually. Finally, in (iv) (the continuous-time analog of AR(1) processes with root larger than 1), shocks to productivity exhibit higher persistence than in the martingale formulation. This could represent situations in which initial experiences have important consequences on the long-run value of skills. For example, a recently hired worker who is assigned to a negligent mentor may not develop an appropriate understanding of his assigned task, becoming permanently disadvantaged relative to other workers in the same cohort but assigned to more competent instructors. Alternatively, it could represent highly unstable environments characterized by great dynamism and randomness. For instance, in trying to adapt to the rapid changes in the tech industry a worker may either become permanently obsolete or he may develop the exact set of skills which will become essential for a long period of time.

In all these formulations different workers can be identified with different sample paths of $Z^\theta$. That is, unmodeled workers’ characteristics that affect skills are summarized in the realization of productivity shocks. This in turn will generate considerable cross-sectional dispersion of skills within each specification (i)-(v). Yet, the likelihood at which highly productive, average or unproductive workers arise will depend on the particular model.

In describing each model, I have emphasized how the persistence of shocks to productivity varies across specifications. The degree of this persistence is captured by the slope
κ. In fact, the solution to (2) has the form

$$\theta_t = e^{\kappa t} \theta_0 + \int_0^t e^{\kappa (t-s)} \bar{\theta}_s(a) ds + \sigma \theta \int_0^t e^{\kappa (t-s)} dZ_s^\theta, \quad t \geq 0,$$

from where it can be observed that κ measures the weight given to past productivity shocks. When κ = 0, all past productivity shocks are given the same weight, so two shocks of the same size at different points in time have the same impact on future skills. When κ < 0, i.e. skills are mean-reverting, productivity shocks are in fact discounted at a rate |κ|, so their impact on skills tends to disappear as time passes. Finally, when skills grow at a rate κ > 0, old productivity shocks have more influence on current ability than the most recent ones.

Regarding human capital accumulation, I assume that workers enter the labor market with a human capital stock denoted by θ₀ ∈ ℝ representing, for instance, different educational levels. Once in a firm, workers can learn from their experience and thus become endogenously more productive. For instance, researchers may become more skilled at their fields of expertise as a consequence of permanent attempts to solve similar problems. Similarly, by constantly monitoring the prospects of the companies they invest in, young traders accumulate experience on how to interpret information coming from markets. This in turn allows them to improve their trading strategies. In environments like the ones just described, acquiring skills is more a result of permanent costly-effort decisions than a result of a choice of investment in its traditional form (upfront payment in exchange for a stream of payoffs). From this perspective, the current effort history of efforts (aₙ : 0 ≤ s ≤ t), t ≥ 0 can be understood as a measure of the worker’s experience at that instant. This experience in turn maps into a value ̄θₚ(a), which I interpret as the worker’s human capital stock at time t ≥ 0. This value is, at any point in time, an aggregate measure of both the worker’s experience acquired on the job and of past investment in human capital made before entering the labor market. The family of functionals (̄θₚ(·))₉≥₀ captures the technology behind human capital accumulation and satisfies the following conditions:

**Assumption 1.** (i) For each t ≥ 0, ̄θₚ : M([0, t], ℝ⁺) → ℝ where M([0, t], ℝ⁺) is the set of measurable functions from [0, t] to ℝ⁺.

(ii) For every y ∈ M([0, t], ℝ⁺), the mapping s → ̄θₛ(y), 0 ≤ s ≤ t is Borel-measurable.

Part (i) in the previous assumption states that the mapping between experience– as measured by the past history of efforts– and human capital occurs in a deterministic way. Part (ii) simply ensures that integrals are well-defined. This is a very general formulation in which the only restriction I impose is that all the human capital technology is non-stochastic. Later in sections 5.1 and 5.2 I study particular examples. Observe also that
because of moral hazard, human capital is private information of the worker at any point in time.

The performance of a worker depends on both his experience and on how he adapts to the environment. Hence, it is skills—the interaction between human capital and productivity shocks—the relevant process for production purposes. My model is particularly interesting in its mean-reverting specification. In such a setting skills evolve around a mean-trend that is permanently changing over time, reflecting the knowledge that workers acquire from their working experience.

The next figure illustrates how a particular realization of productivity shocks (a fixed worker) varies across environments:

![Figure 1: Skills models: Random walk, mean-reverting around zero, mean-reverting around endogenous trend, and positive growth rate.](image)

In the figure, the lowest non-divergent path corresponds to a particular realization of a standard Wiener process of volatility $\sigma_\theta = 0.4$. In such a specification, the worker receives, on average, negative productivity shocks throughout the horizon studied. Adding mean-reversion around zero ($\kappa = -2$), instead forces the same sequence of shocks to fluctuate around zero. This may represent unmodeled forces that drive the worker to an average productivity level. By adding human capital accumulation according to an ordinary differential equation governed by $f(t, \bar{\theta}, a) = \alpha_t a_t - \phi \bar{\theta}$, $\phi = 0.2$ and $\alpha_t a_t \equiv 1$, skills now fluctuate around the plotted human capital trend. Finally, the diverging path...
corresponds to the case in which skills change at a rate $\kappa = 2$. Observe that in this case a short sequence of negative shocks in the beginning of the worker’s career generates a completely different behavior.

### 2.2 Connection with the Literature

**Career Concerns:** Because of their tractability in discrete-time frameworks, traditional career concerns models involving Gaussian skills (Holmstrom 1982, 1999; Dewatripont et al. 1999; Gibbons and Murphy 1982) have analyzed only two particular specifications: when skills are fixed over time and when they evolve as a random walk. Although interesting in their own, both models are probably too limited to capture a wide variety of real-life features related to the evolution of workers’ skills over long horizons. On the one hand, I have argued that workers may suffer non-negligible productivity shocks that influence short-term performance. On the other hand, the random walk specification implicitly assumes both workers and the environment freely adjusting to constant change, which allows for no predictability on the short-run evolution of the value of a worker’s skills. The paper presented here offers a more robust approach to modeling how skills– or their value in a changing environment– can potentially evolve over time. This allows me to recover both specifications as special cases of my general Gaussian-diffusion model in the absence of human capital accumulation.

Another important feature of these classic specifications is that they constitute landmarks within the class of Gaussian diffusions. First, observe that when skills correspond to a time-zero draw from a normal distribution, the underlying uncertainty in the model is realized at time zero. To the contrary, in any Gaussian specification with non-zero volatility ($\sigma_\theta > 0$) the underlying uncertainty is gradually revealed as the stochastic process of skills unfolds over time. As shown in Holmstrom’s paper, the way in which uncertainty is resolved has important effects on the long-run reputational motives that arise from career concerns. In fact, he showed that in the fixed “talent” case infinitely long series of observations reveal the true value of skills in the long-run, and thus the career concerns incentives asymptotically disappear. Yet, when skills evolve as a random walk, their inherent unpredictability generates non-negligible long-run residual uncertainty. This maps into career concerns incentives that survive in a steady-state learning level. Second, notice that the martingale specification is at the boundary between the family of Gaussian diffusions that admit a long-run stationary distribution ($\kappa < 0$) and the ones that do not ($\kappa \geq 0$). With respect to learning, the latter family offers a more uncertain environment than the former, even after arbitrarily long sequences of noisy observations of skills. This paper contributes to the previous literature not only by characterizing incentives for the entire
class of Gaussian diffusions, but also by providing a particularly clean characterization on how incentives are shaped by the persistence of shocks to skills. I address this in Section 4.

**General Training:** Becker (1964) argued that efficient investments in general skills take place when markets are competitive and workers can contribute to their training. Yet, this conclusion relies heavily upon the contractibility– hence, upon the ex-post verifiability– of these investments. This assumption is suitable in the context of formal training methods, since in those programs monitoring problems are typically not an important issue. However, it is less appropriate in settings where training programs are less rigid and where output is noisy. Moreover, if these investments affect output-relevant worker’s characteristics that are unobservable to potential employers, it is not clear that such an efficiency result would hold.

With this in mind, I study the reputational incentives that workers face when their investments in acquiring skills are imperfectly monitored. As mentioned earlier, these investments are in the form of costly-effort decisions. Jobs have become much more specific over the last decades and, consequently it is sometimes the task itself (or similar alternative ones) rather than formal training programs which provides the worker with the necessary on-the-job training. This is done in Section 5.

### 2.3 Market Structure and Equilibrium Concept

I maintain the assumptions on preferences and market structure imposed by Holmstrom (1982, 1999). The worker is assumed to be risk neutral and his utility function is separable in consumption and effort.\(^5\) The latter is costly according to a non-negative function \(g: \mathbb{R}_+ \to \mathbb{R}_+\), which is strictly increasing, strictly convex, and satisfying \(g(0) = 0\) and \(g'(0) = 0\). As a consequence, if at time \(t\) the manager is paid \(w_t\) and exerts effort \(a_t\), he will get a utility flow of \(w_t - g(a_t), \ t \geq 0\).

Regarding market structure, no output-based contracts can be written and the market for workers is perfectly competitive, so firms earn zero expected discounted profits from production. Moreover, no long-term contracting is possible and hence firms earn zero profits at any point in time. Since in this model it is the market that ultimately sets wages, it corresponds to the “principal” in this market-agent interaction.

Because the market cannot observe the worker’s skills, it will create estimates of it based on the public signal \(\xi\). The zero-profit condition at every point in time implies that,

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\(^5\)All the results shown in this paper are also valid for effort strategies taking multi-dimensional values, but with only one component of effort affecting output \(\xi\). Unless otherwise stated, effort is uni-dimensional.
if the market expects the agent to follow the effort strategy \( a^* := (a^*_t)_{t \geq 0} \), it will pay him a flow payoff corresponding to the rate at which current production is expected to change:

\[
w_t := \lim_{h \to 0} \frac{\mathbb{E}^{a^*}[\xi_{t+h}|\mathcal{F}^\xi_t] - \xi_t}{h} = \mathbb{E}^{a^*}[\theta_t|\mathcal{F}^\xi_t] + a^*_t, \quad t \geq 0,
\]

where \( \mathcal{F}^\xi_t \) denotes the public information at time \( t \), and \( \mathbb{E}^{a^*}[\cdot|\mathcal{F}^\xi_t] \) is the market’s conditional expectation under the assumption that the worker is following the strategy \( a^* \).\(^6\) This shows that wages have a reputational component and an effort component, both depending on what the market conjectures is the strategy that the worker will follow.

I allow for the worker to have potentially much more information about himself than the one provided by the public signal \( \xi \). The only restriction I impose is that his posterior beliefs remain Gaussian (in the next section I study the details of the learning process).\(^7\)

Let \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \) denote the worker’s information structure and observe that, in particular, the human capital trend \( (\theta_t(a))_{t \geq 0} \) belongs to it.

**Definition 2.** A feasible strategy for the worker corresponds to any \( \mathbb{F} \)-progressively measurable process taking values in \( \mathbb{R}_+ \). Denote this set by \( \mathcal{A} \).\(^8\)

Since \( \mathbb{F}^\xi := (\mathcal{F}^\xi_t)_{t \geq 0} \subset \mathbb{F} \), a class of strategies of particular interest is one of feasible strategies adapted to the public signal \( \xi \).

**Definition 3.** A strategy is public if it corresponds to a feasible strategy that is \( \mathbb{F}^\xi \)-progressively measurable.

The equilibrium concept corresponds to perfect public equilibria, as defined next: \(^9\)

**Definition 4.** A perfect public equilibrium corresponds to a public strategy \( a^* := (a^*_t)_{t \geq 0} \) and a wage process \( w := (w_t)_{t \geq 0} \), such that:

(i) Given \( a^* \), the market sets a wage of the form \( w_t = \mathbb{E}^{a^*}[\theta_t|\mathcal{F}^\xi_t] + a^*_t \) for all \( t \geq 0 \);

\(^6\)Different effort strategies generate different probability measures on the set of paths of \( \xi \). Therefore, if the market conjectures that the worker is following \( a^* \) when he has actually chosen the process \( a \), their beliefs will differ.

\(^7\)In fact, as long as screening contracts are not allowed, all the results presented below hold if the skills process is actually observed by the worker.

\(^8\)This definition is up to mild integrability conditions on \( \mathcal{A} \) such that the filtering equations exist. Of course, the equilibrium strategy will satisfy them. Hence, I do not constrain \( \mathcal{A} \) ex-ante.

\(^9\)More formally, the equilibrium notion corresponds to a sequential equilibrium. That is: (i) given a law of motion of beliefs (derived in the next subsection), the equilibrium strategy must be optimal for the agent; (ii) given the equilibrium action profile, the law of motion of beliefs is obtained via Bayes rule.
(ii) For any $t \geq 0$ and after any private history $\mathcal{F}_t$, the continuation strategy $(a^*_s)_{s \geq t}$ is optimal for the worker given the wage process in (i):

$$(a^*_s)_{s \geq t} \in \arg\max_{a \in A_t} \mathbb{E}^a\left[ \int_t^\infty e^{-r(s-t)}(w_s - g(a_s))ds \left| \mathcal{F}_t \right. \right]$$

s.t. $w_s = \mathbb{E}^a[\cdot | \mathcal{F}_s] + a^*_s$, $\forall s \geq t$ (5)

where $A_t$ is the set of feasible strategies at time $t$.

Note that this definition emphasizes the fact that, by choosing an effort strategy $a \neq a^*$, the worker induces a distribution over outcome paths that differs from the one anticipated by the market ($\mathbb{E}^a[\cdot]$ operator). This in turn will affect the market’s beliefs about how skilled the worker is. For instance, if the latter deviates from $a^*$ at some instant by exerting more effort, this will generate, in expectation, higher output observations and the market will revise its expectations upwards. As a consequence, an increase in effort today will generate, on average, a boost in the reputational component of future wages.

3 Preliminary Results

In this section I first explain how learning takes place and I shed light on which forces determine the agent’s benefits from distorting the market’s beliefs. Next, I show that the existence of deterministic equilibria—a particular sub-class of perfect public equilibria based only on the evolution of the posterior second moment—is reduced to the existence of a deterministic solution to a simple optimization problem. This simplification is helpful for understanding the differences between the traditional benefits from “signal-jamming” and the new gains from human capital accumulation.

3.1 Learning: Filtering Equations

This section characterizes the market’s evolution of beliefs under the conjecture that the worker is following a public strategy $a^*$.$^{10}$ Under this assumption, output (1) and skills (2) are given by

$$d\xi_t = (a^*_t + \theta_t)dt + \sigma_\xi dZ^\xi_t,$$
$$d\theta_t = (\bar{\theta}_t(a^*) + \kappa \theta_t)dt + \sigma_\theta dZ^\theta_t.$$  

Since the market does not observe $\theta$, it creates beliefs about the worker’s skills based on the observation of $\xi$. The first relevant question is how are the $\mathbb{F}^\xi-$Brownian motions,

$^{10}$Observe that reducing the analysis of beliefs to public strategies is an equilibrium restriction, but not a constraint on the agent’s set of feasible strategies.
i.e. the subjective source of uncertainty under the publicly available information. This question is important because this process will characterize the on-equilibrium evolution of beliefs.

**Lemma 5.** Suppose the market conjectures that the manager follows a public strategy $a^*$. Then, the process

$$Z_t^{a^*} := \frac{1}{\sigma_\xi} \left( \xi_t - \int_0^t (a_s^{a^*} + \mathbb{E}_{a^*}^{a^*}[\theta_s|\mathcal{F}_s^\xi])ds \right), \quad t \geq 0,$$

is an $\mathbb{F}^\xi-$Brownian motion from the market’s perspective. In particular, $\xi$ admits a diffusion representation of the form

$$d\xi_t = (a_t^{a^*} + \mathbb{E}_{a^*}^{a^*}[\theta_t|\mathcal{F}_t^\xi])dt + \sigma_\xi dZ_t^{a^*}, \quad t \geq 0.$$  

The process $Z_t^{a^*} := (Z_t^{a^*})_{t \geq 0}$ is called an innovation process under the public strategy $a^*$.

**Proof:** See Appendix A. \hfill \Box

The intuition for this result is straightforward. At any time $t \geq 0$, the expected rate of change in output corresponds to $\mathbb{E}_{a^*}^{a^*}[a_t^{a^*} + \theta_t|\mathcal{F}_t^\xi] = a_t^{a^*} + \mathbb{E}_{a^*}^{a^*}[\theta_t|\mathcal{F}_t^\xi]$, where the last equality comes from the fact that $a^*$ is $\mathbb{F}^\xi-$adapted. As a consequence, the difference $\xi_t - X_t, \quad t \geq 0,$ must be a martingale under the public information structure. Given $\xi$’s Gaussian form, this translates into the fact that $Z_t^{a^*}$ has to be a Brownian motion.

Suppose that the market’s initial belief is such that $\theta_0|\mathcal{F}_0 \sim \mathcal{N}(m_0, \gamma_0)$ and that it is conjectured that the agent follows a public strategy $a^* \in \mathcal{A}$. Given the above assumptions, the conditional distribution of $\theta_t$ given the information $\mathcal{F}_t^\xi$ is also a Gaussian for all $t \geq 0$ (Theorem 11.1. in Liptser and Shiryaev 1977). Denote by $m_t^* := \mathbb{E}_{a^*}^{a^*}[\theta_t|\mathcal{F}_t^\xi]$ and $\gamma_t := \mathbb{E}_{a^*}^{a^*}[(\theta_t - m_t^*)^2|\mathcal{F}_t^\xi]$ the market’s posterior mean and variance using the public information up to $t$, respectively, under the assumption that the agent follows $a^*$. The following result is a standard one in filtering theory (Theorem 12.1. in Liptser and Shiryaev 1977):

\[\text{Standard results in probability theory show that given any } \mathbb{F}^\xi-\text{progressively measurable strategy } a^*, \text{ there is measurable map } b: \mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R}) \to \mathbb{R}_+ \text{ such that } a^*_t = b(t, \xi) \text{ and, for each } t, b(t, \xi) \text{ is } \mathcal{F}_t^\xi-\text{measurable. As a consequence, } \tilde{\theta}_t(a^*) = \tilde{\theta}_t(\xi) \text{ for some } \tilde{\theta}_t(\xi), \text{ which is also } \mathcal{F}_t^\xi-\text{measurable, } t \geq 0. \text{ This functional representation is needed for applying the filtering techniques from Liptser and Shiryaev (1977).}\]
Lemma 6. The market’s posterior mean and posterior variance processes, \( m_t^* := \mathbb{E}^{a^*}[\theta_t | \mathcal{F}_t^\xi] \) and \( \gamma_t := \mathbb{E}^{a^*}[(\theta_t - m_t^*)^2 | \mathcal{F}_t^\xi] \), \( t \geq 0 \), respectively, satisfy the equations

\[
\begin{align*}
\mathrm{d}m_t^* &= (\bar{\theta}_t(a^*) + \kappa m_t^*)\mathrm{d}t + \gamma_t \sigma_\xi \mathrm{d}Z_t^{a^*}, \quad (8) \\
\dot{\gamma}_t &= 2\kappa \gamma_t + \sigma_\xi^2 - \left(\frac{\gamma_t}{\sigma_\xi}\right)^2, \quad (9)
\end{align*}
\]

where the \( Z_t^{a^*} \) is defined in Lemma 5.

Proof: See Liptser and Shiryaev 1977. \( \square \)

Three interesting features of (8) and (9) are worth noting. First, the evolution of the posterior mean preserves the stochastic structure of the evolution of skills: since \( \bar{\theta}_t(a^*) \geq 0 \) is adapted to \( \mathcal{F}_t^\xi \), the drift of the posterior mean is also affine with the same slope and intercept. Second, the posterior mean’s response to unexpected output observations (captured by the innovation process) increases with the size of the mean-square error and decreases with the signal’s volatility \( \sigma_\xi \). This implies that beliefs react more strongly in settings where either less information has been accumulated, or where signals are more accurate. Finally, the mean-square error evolves in a deterministic fashion, so, provided that \( \gamma_0 \) is common-knowledge, the entire trajectory of the posterior variance is perfectly anticipated by both parties. This motivates the following equilibrium definition:

Definition 7. A public perfect equilibrium \( a^* \) is deterministic if it depends only on the evolution of the market’s posterior mean-square error.

The reason for looking at equilibria with this characteristic is twofold. Firstly, and from a purely technical perspective, the task of finding output-dependent PPE is hard since the worker would also have to take into account the future benefits from distorting the market’s conjectured action profile—finding such a fixed-point can become a particularly hard task. Secondly, the main goal of this paper is to study both how the career concerns motives of workers are affected by the uncertainty in the environment and how these motives evolve throughout the workers’ life-cycle. By introducing an additional endogenous output-dependent process, the worker’s incentives may be pushed away from the purely reputational ones. Finally, observe that in any deterministic equilibrium the market’s anticipated human capital path \( \bar{\theta}_t(a^*) \) is exogenously fixed from the worker’s perspective.

The speed of learning is measured by the evolution of \( \gamma := (\gamma_t)_{t \geq 0} \). Suppose for the moment that learning is constant, that is, the posterior variance is at the stationary level.
\( \gamma^* \) given by the solution to the equation \( 0 = 2\kappa \gamma^* + \sigma_\theta^2 - (\gamma^*/\sigma_\xi)^2 \). It is easy to see that, given a fixed slope \( \kappa \in \mathbb{R} \), the unique stationary-learning level is given by

\[
\gamma^* = \sigma_\xi^2 \left( \sqrt{\kappa^2 + \sigma_\theta^2 / \sigma_\xi^2} + \kappa \right), \quad \kappa \in \mathbb{R}.
\]  

(10)

Thus \( \gamma^* \) corresponds to the market’s long-run residual uncertainty regarding the worker’s skills. Observe that a necessary condition for \( \gamma^* \) to be equal to zero is the absence of unobservable productivity shocks (\( \sigma_\theta = 0 \)). However, as one can see from the case in which skills grow at a strictly positive rate (\( \kappa > 0 \)), this is not sufficient. This may seem counterintuitive since when skills evolve in a deterministic way, a huge part of the model’s uncertainty is eliminated. Nonetheless, what really matters in terms of asymptotic learning is the value that skills take in the long-run, and either in the presence of productivity shocks or in a setting without them but in which skills grow at a strictly positive rate, no such a value exists. In these cases, arbitrarily long sequences of noise observations are never accurate enough to fully reveal the true value of skills. The steady-state learning levels for the specifications (i)-(v) are characterized in the following

**Proposition 8.** In a stochastic environment (\( \sigma_\theta > 0 \)), there is always a strictly positive long-run residual level of uncertainty (\( \gamma^* > 0 \)). Moreover, \( \gamma^* \) is strictly increasing in \( \kappa \) and \( \lim_{\kappa \to -\infty} \gamma^* = 0 \). In the absence of productivity shocks (\( \sigma_\theta = 0 \)), the long-run residual uncertainty is zero if and only if \( \kappa \leq 0 \), and equals \( 2\kappa > 0 \) otherwise. Finally, for the i.i.d. case (\( \bar{\theta}(\cdot) \equiv \overline{\theta} \in \mathbb{R}, \sigma_\theta = \sigma \sqrt{|\kappa|}, \sigma > 0 \text{ and } \kappa \to -\infty \)), \( \gamma^*_{\text{iid}} := \lim_{\kappa \to -\infty} \gamma^* = \frac{\sigma^2}{2} > 0 \).

**Proof:** Straightforward.

Observe first that for stochastic environments (\( \sigma_\theta \neq 0 \)), the long-run residual uncertainty is strictly increasing in the slope \( \kappa \) (and weakly increasing in deterministic settings). That is, environments in which productivity shocks have a higher persistence generate higher levels of long-run residual uncertainty. This is consistent with the discussion regarding stationary distributions. Second, \( \gamma^*_{\text{iid}} \) coincides with the prior variance of the skills process in the i.i.d. case. This is quite intuitive given that in this specification productivity shocks have no persistence and hence, no learning takes place. In the sequel, I assume that the market’s initial variance \( \gamma_0 \) is larger or equal than \( \gamma^* \), so the quality of information improves over time.

### 3.2 Belief-distortion

Because skills are not observable, the market cannot distinguish between output changes coming from the signal noise and output changes that are the consequence of skills’ variation
across time. Therefore, once the market has conjectured that the worker will follow a particular public strategy, say $a^*$, the only information that it has available to construct statistics about the worker’s skills is the process

$$Y_t := \xi_t - \int_0^t a^*_s \, ds, \quad t \geq 0,$$

that is, the component of output not explained by effort. Observe that the market’s posterior mean (8) admits the following representation with respect to $Y := (Y_t)_{t \geq 0}$

$$dm_t^* = (\bar{\theta}_t(a^*) + [\kappa - \beta_t] m_t^*) \, dt + \beta_t \underbrace{[d\xi_t - a^*_t \, dt]}_{dY_t},$$

where $\beta_t := \gamma_t / \sigma_\xi^2$, $t \geq 0$, is the market’s beliefs sensitivity process—a measure of the immediate response of beliefs to new information—which is also exogenous. Representation (12) is important because it shows how the market’s beliefs evolve from the worker’s perspective. In fact, the agent is aware that, by deviating from $a^*$, he can affect the evolution of $Y$ and, as a consequence, neither $Z_a^*$ nor $m^*$ are exogenous from his standpoint.\(^\text{12}\) Equivalently, (12) shows how the market’s beliefs are constructed based on the output signal $\xi$ and the conjectured effort strategy $a^*$, which are both part of the market’s information structure.

Market participants know that the signal $Y$ has unknown but, most importantly, non-zero increments. As a consequence, they are aware this process is not truly a martingale and hence, that changes in it are not really a surprise. As a result, the optimal way in which beliefs filter the information conveyed by this signal is by instantaneously reacting to new output observations, but making the reactions decay relatively fast. Signal-jamming indeed carries some informational costs: even though the market reacts to the information provided by $Y$, past observations of it are discounted more heavily. To see this observe that the solution to (12) has the form

$$m_t^* = m_0 e^{\int_0^t (\kappa - \beta_s) \, ds} + \int_0^t e^{\int_s^t (\kappa - \beta_u) \, du} \bar{\theta}_u(a^*) \, ds + \int_0^t \int_0^s e^{\int_s^u (\kappa - \beta_v) \, dv} \beta_v \, dY_v, \quad t \geq 0.$$  \(^\text{13}\)

It is straightforward to see that while skills weigh past productivity shocks according to $\kappa$ (see eq. (3)), skills’ estimates weigh the information generated by $Y$ using $\kappa - \beta_t$, $t \geq 0$, which is strictly less that $\kappa$. That is, learning creates a wedge between the rate at which skills keep track of past productivity shocks, and the rate at which beliefs keep track of past output observations. I call this wedge the discounting wedge that arises from learning. Suppose for example that skills grow at a rate $\kappa \geq 0$. Then, while skills give more weight to old productivity shocks relative to most recent ones using a factor $\kappa$, beliefs now discount more heavily past information using a discount rate $\delta_t := \beta_t - \kappa > 0$, $t \geq 0$.\(^\text{13}\) If

\(^{12}\) Of course, $Z_a^*$ and, consequently, $m^*$, are exogenous from the market’s perspective. Thus, the market’s beliefs do admit a representation with respect to $Z_a^*$.

\(^{13}\) Recall that $\beta_t := \frac{\gamma_t}{\sigma_\xi^2} > \frac{\gamma^*_t}{\sigma_\xi^2} = \sqrt{\kappa^2 + \sigma^2_\theta / \sigma_\xi^2 + \kappa}$.  

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instead skills are mean-reverting ($\kappa < 0$), and hence past productivity shocks are discounted according to the fixed rate $|\kappa|$, beliefs will discount the information generated by $Y$ more heavily using the deterministic discount process $\delta := (\delta_t)_{t \geq 0} > |\kappa|$. The dynamics of the posterior mean are determined by the sensitivity process $\beta := (\beta_t)_{t \geq 0}$ and the discount rate process $\delta := (\delta_t)_{t \geq 0}$. While the first process measures the initial response of beliefs to new information, the second process measures how these initial reactions decay over time. They measure two different aspects of the overall responsiveness of beliefs to aggregate information.

Given that the degree of persistence of the Brownian shocks has important consequences on the long-run behavior of the processes analyzed here, a natural question that arises is how the slope $\kappa$ affects both $\beta$ and $\delta$ in the long-run. Suppose that learning is stationary, so $\gamma^*(\kappa) = \sigma^2(\sqrt{\kappa^2 + \sigma^2_\theta/\sigma^2_\xi} + \kappa)$ and, hence, $\delta^*(\kappa) = \sqrt{\kappa^2 + \sigma^2_\theta/\sigma^2_\xi}$. From this expression it can be seen that the rate at which beliefs keep track of past output observations is minimized when skills evolve as a martingale. Moreover, although it is intuitive that past output observations are discounted more heavily in the mean reverting specification than in the martingale model (given the less erratic of skills beliefs don’t need to keep track of too much information), it is somewhat surprising that instead past observations are given less weight when skills grow at positive rates. The reason is that because in these specifications skills have explosive paths, past output observations become worse predictors for assessing the current value of skills. More interesting is that the stationary discount rate $\delta^*(\kappa)$ depends only on the absolute value of the slope. This occurs because the Kalman-Bucy filter minimizes the distance to the underlying unobservable and thus the weight that estimates attach to past information should not depend on the particular sign of the slope of the skills process. Nonetheless, the way in which the filter differentiates between processes that admit a long-run stationary distribution and the ones that don’t is through the sensitivity process. In fact, from Proposition 8, $\beta^*(\kappa) := \frac{\gamma^*(\kappa)}{\sigma^2_\xi}$ is increasing in $\kappa$. Intuitively speaking, as shocks become more persistent it is more likely that an output surprise is due to skills improvement rather than the consequence of noise. This translates into beliefs reacting more strongly to new output observations. Graphically:

When learning is stationary, a natural measure of the overall responsiveness of beliefs to information is given by the sensitivity-discount ratio

$$w^*(\kappa) := \frac{\beta^*(\kappa)}{\delta^*(\kappa)} = 1 + \frac{\kappa}{\sqrt{\kappa^2 + \sigma^2_\theta/\sigma^2_\xi}}.$$  \hfill (14)

The sensitivity-discount ratio is monotonically increasing in $\kappa$, as a reflect of beliefs reacting more strongly to information precisely in environments where there is more underlying randomness. Take as a reference the martingale specification $\kappa = 0$. In this case, the
Figure 2: Steady-state levels of discount rate and sensitivity of beliefs. While the former does not distinguish between mean-reverting and growth rate processes with the same absolute value of the slope, the optimal filter makes a distinction between them through the long-run sensitivity parameter.

intensity at which beliefs react to new information has the same size as the rate at which initial responses decay over time. As a consequence, the sensitivity-discount ratio takes value 1. If skills are instead mean-reverting, productivity shocks will have less persistence than in the martingale model. Given this less erratic behavior, the overall responsiveness of beliefs to information should be lower than in the martingale specification. This results in \( w^*(\kappa) < 1 \) whenever \( \kappa < 0 \). Finally, when skills grow at a positive rate \( \kappa > 0 \) the higher persistence of productivity shocks increase the overall value of information relative to the martingale model. Therefore, the responsiveness of beliefs to information will be higher than in the reference case, which translates into \( w^*(\kappa) > 1 \) when \( \kappa > 0 \). It turns out that this measure of belief-responsiveness plays a crucial role in the stationary-learning incentives that arise from career concerns motives.

3.3 Problem Reduction: Main Lemma

In this section I first state the general problem the worker solves. Next, I present a lemma that reduces the existence of a PPE in deterministic strategies to the existence of a particular type of solution to a simple deterministic optimization problem. In fact, in order for a solution to this reduced problem to constitute a deterministic PPE, it needs to be independent of the current history of human capital at any point in time. This is because the equilibrium concept used here is quite strong: the market must correctly anticipate the worker’s effort strategy after any possible private history observed by the worker.

The worker is allowed to have access to additional sources of information than the one provided by output. This is a reasonable assumption given that skills are an inherent characteristic of the agent himself. I do impose that the stochastic structure of the additional
signals must lie within the Gaussian framework. Recall that \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \) denotes the agent’s information structure (containing, in particular, \( \mathbb{F}^\xi \)) and that \( \mathcal{A} \) denotes the set of feasible strategies for the worker (see Definition 2). Given \( a \in \mathcal{A} \), the worker’s posterior mean evolves as

\[
dm_t = (\bar{\theta}_t(a) + \kappa m_t)dt + \sigma_t dZ_t, \quad t \geq 0
\]  

(15)

where \( Z := (Z_t)_{t \geq 0} \) is a \( \mathbb{F} \)-Brownian motion and \( \sigma := (\sigma_t)_{t \geq 0} \) is a non-negative process.\(^{14}\)

Suppose that the worker follows a strategy \( a \in \mathcal{A} \). Then, from his own perspective, the process

\[
Z^a_t := \frac{1}{\sigma} \left( \xi_t - \int_0^t (m_t + a_t)dt \right), \quad t \geq 0
\]

(16)

is an \( \mathbb{F} \)-Brownian motion that is correlated with \( Z \). Therefore, as a direct application of Lemma 5 for \( \mathbb{F} \)-Brownian motions, output can be written as

\[
d\xi_t = (m_t + a_t)dt + \sigma_t dZ^a_t, \quad t \geq 0,
\]

(17)

from the worker’s standpoint.

With this in hand, the worker’s optimization problem is straightforward: given a deterministic conjecture \( a^* \), after any private history \( \mathcal{F}_t \) (which includes, in particular, the current output path \( \xi^* := (\xi_s)_{0 \leq s \leq t} \) and current human capital path \( (\bar{\theta}_s)_{0 \leq s \leq t} \)), the worker solves

\[
\max_{a \in \mathcal{A}} \mathbb{E}^a \left[ \int_0^\infty e^{-r(s-t)} (m^*_s - g(a_s))ds \bigg| \mathcal{F}_t \right]
\]

s.t. \( dm^*_s = (\bar{\theta}_s(a^*) - \delta_s m^*_s)ds + \beta_s (d\xi_s - a^*_s ds), \quad s > t, \quad m^*_s = m^{s,o}, \)

\( d\xi_s = (a_s + m_s)ds + \sigma_s dZ^*_s, \quad s \geq t, \)

\( dm_s = (\bar{\theta}_s(a) + \kappa m_s)ds + \sigma_s dZ_s, \quad s \geq t, \quad m_t = m^o, \)

\[
\bar{\theta}_s(a), \quad s \geq t
\]

(18)

where \( (a^*_s)_{s \geq t}, (\beta_s)_{s \geq t}, (\delta_s)_{s \geq t}, (\sigma_s)_{s \geq 0} \) and \( (\bar{\theta}_s(a^*))_{s \geq t} \) are functions of calendar time only, and \( m^{s,o}, m^o \) are given.

In the next lemma I show that looking for deterministic PPE is indeed not a bad guess as long as the family of functionals \( (\bar{\theta}_t(\cdot))_{t \geq 0} \) is deterministic, which is part of Assumption 1:

\(^{14}\)A process like this can be generated as follows: suppose the agent also observes \( d \) signals of the form

\[
d\xi^i_t = \theta^i_t dt + B^i_1 dZ^i_t + C^i_1 dZ^j_t \]

where \( \{Z^0, Z^i : i = 1, \ldots, d\} \) is a family of independent one dimensional Brownian motions. Then, if the coefficients \( \{B^i_t, C^i_1, \Sigma^i_t : i = 1, \ldots, n\} \) (which may depend on output) satisfy some measurability and integrability conditions, standard filtering techniques yield that the conditional mean \( (m_t)_{t \geq 0} \) evolves as \( dm_t = (\bar{\theta}_t(a) + \kappa m_t)dt + \Sigma_t dZ^o_t \) where \( \Sigma_t \in \mathbb{R}^d \) and \( Z^o_t \) is a \( d \)-dimensional innovation process. Letting \( dZ_t := \Sigma_t dZ^o_t / ||\Sigma_t|| \) and \( \sigma_t := ||\Sigma_t|| \), proves the claim.
Lemma 9. The existence of a deterministic equilibrium is reduced to the existence of a 
calendar-dependent solution to the following deterministic optimization problem

\[ P := \begin{cases} 
\max_{a \in A} \int_0^\infty e^{-rt} \left[ \beta_t \lambda_t a_t - g(a_t) + \rho_t \bar{\theta}_t(a) \right] dt \\
\text{s.t.} \quad \bar{\theta}_t(a), \ t \geq 0, \\
\gamma_0 \geq \gamma^*, 
\end{cases} \]

where \( \gamma_t, \beta_t, \rho_t := \frac{1}{r - \kappa} - \lambda_t \) and

\[ \lambda_t = \int_t^\infty e^{-\int_s^t (r + \delta_u) du} ds, \quad t \geq 0, \]

are all deterministic functions.\(^{15}\)

Proof: See the Appendix.

\[ \square \]

As a consequence, if the starting value of human capital \( \bar{\theta}_t(\cdot) \) is common knowledge, the 
market can perfectly anticipate the worker’s optimal actions.

This section connects the model to the standard literature of career concerns without 
human capital accumulation. I complement the classic inefficiency results of excessive effort 
in the early stages of the workers’ life with the new finding that there are also long-run 
inefficiencies which are a robust phenomena within the class of Gaussian diffusions. Human 
capital is analyzed in Section 5.

4 Inefficiencies in Career Concerns Models

Suppose that there is no human capital accumulation, that is, \( \bar{\theta}_t(\cdot) \equiv \bar{\theta} \in \mathbb{R} \), for all \( t \geq 0 \). 
From Lemma 9, the existence of a PPE in deterministic strategies is reduced to finding 
a solution to \( P \). In this particular case such a solution can be found through pointwise 
optimization:

**Proposition 10.** The unique PPE in deterministic strategies \((a^*_t)_{t \geq 0}\) is characterized by 
the first order condition

\[ g'(a^*_t) = \beta_t \int_t^\infty e^{-\int_s^t (r + \delta_u) du} ds, \forall t \geq 0 \]  \tag{19} \]

\(^{15}\)Implicit in this result is the assumption that \( r > \kappa \) when there is human capital accumulation (for 
otherwise, the benefits from experience are unbounded). This assumption is not needed when \((\bar{\theta}_t(\cdot))_{t \geq 0}\) is 
exogenous, allowing us to perform comparative statics as the discount rate \( r \) approaches zero in standard 
career concerns models. Finally, observe that this constraint has a bite only when \( \kappa > 0 \), so the random 
walk and mean-reverting models admit arbitrarily low discount rates even in the presence of partially 
endogenous skills.
where $\beta_t := \gamma_t/\sigma^2$ and $\delta_t := \beta_t - \kappa$, for all $t \geq 0$. Moreover,

$$\frac{da_t^*}{dt} \leq 0 \iff \frac{d\gamma_t}{dt} \leq 0.$$ 

**Proof:** See the Appendix.

To understand this result observe that, from the worker’s standpoint, the market’s beliefs take the form

$$m_t^* = m_0 e^{-\int_0^t \delta_u ds} + \int_0^t e^{-\int_s^t \delta_u du} \psi_s(a^*) ds + \int_0^t e^{-\int_s^t \delta_u du} \beta_s \left[ m_s ds + \sigma_s dZ_s^a \right]$$

where I have plugged into (13) the evolution of output from the worker’s perspective (17). From (*) it is extremely clear how incentives are determined in the model: a marginal increase in effort over $[t, t+1]$ generates the additional wage flow $d_{sa} := \beta_t e^{-\int_s^t \delta_u du}$, $s \geq t$.

Standing at $t$, this reputational dividend flow has a net present value of size $\beta_t \lambda_t$, $t \geq 0$, and the worker just equates the marginal cost from exerting effort to the marginal benefit from affecting future wages. As it is clearly seen from this analysis, $\beta$ is what really determines the short-term gains of belief-distortion while $\delta$ determines the long-run benefits from it.

The last part of the proposition states that, as information improves, the incentives to exert effort decay over time. This goes in line with the traditional idea that career concerns motives generate higher returns in environments with more uncertainty. However, there are two opposing forces that make this conclusion non-trivial: when $\gamma_t$ decreases over time, both the sensitivity of beliefs to new information, $\beta_t$, and the rate at which the market’s beliefs decay, $\delta_t$, decrease. The first force reduces the short-term benefits from signal-jamming, while the second force makes any reputational gain more persistent over time. Therefore, the result says that the short-term losses from increased precision always outweigh the long-term benefits from more permanent distortions.

The rest of the section is devoted to the analysis of incentives in the long-run. This is relevant for two reasons. First, it is not irrational to think of situations in which falling in a neighborhood of the steady-state level of learning $\gamma^*$ occurs relatively fast. For instance, some industries suffer from considerable turnover of workers and, therefore, past experiences with former employees may allow firms to have relatively accurate priors about the skills of incoming workers. Second, a stationary-learning environment corresponds to the natural setting in where we can evaluate the degree of efficiency of the incentives created.
by career concerns motives. To see why this is the case, observe that the first-best effort process $a^e := (a^e_t)_{t \geq 0}$ maximizes the surplus generated by the interaction between the worker and the firm

$$E \left[ \int_0^\infty e^{-rt} (d\xi_t - g(a_t)dt) \right] = E \left[ \int_0^\infty e^{-rt} (\theta_t - g(a_t))dt \right].$$

In the absence of human capital accumulation, efficiency yields constant rule characterized by the condition $g'(a^e) = 1$. As a consequence, given the transient effects that learning has on incentives, a permanent efficient provision of incentives cannot be expected away from steady state.

**Theorem 11.** Assume that skills evolve according to time-homogeneous Gaussian diffusion of slope $\kappa \in \mathbb{R}$. In a stationary-learning setting optimal effort is constant, say $a^*$, and characterized by the first order condition

$$g'(a^*) = \beta^*(\kappa) = \frac{\sqrt{\kappa^2 + \sigma_\theta^2 / \sigma_\xi^2} + \kappa}{\sqrt{\kappa^2 + \sigma_\theta^2 / \sigma_\xi^2} + r}$$

As a consequence:

- When $\kappa < 0$: $g'(a^*(r)) < 1$, for all $r \geq 0$;
- When $\kappa = 0$: $g'(a^*(r)) \nearrow 1$ as $r \to 0$;
- When $\kappa > 0$: $g'(a^*(r)) > 1$ for all $r \in [0, \kappa)$ and $g'(a^*(r)) \leq 1$ for all $r \geq \kappa$ (with equality if and only if $r = \kappa$),

where I made explicit the dependence of effort on the discount rate $r \geq 0$.

**Proof:** Direct from Proposition 10 when learning is stationary.

The question of whether or not markets induce efficient incentives captured special attention in the past (see Fama 1980). In particular, Holmstrom showed that efficiency is achieved when skills evolve as a random walk, learning is stationary and the worker is infinitely patient. The above result proves that the random walk model is truly an exception, and long-run inefficiencies are robust to the entire class of Gaussian diffusions. In fact, observe that the introduction of a non-zero slope in the skills process moves the sensitivity-discount ratio $w^*(\kappa)$ away from 1 (see eq. (14)). If for instance skills are mean-reverting, the overall responsiveness of beliefs to new information is too low, and effort levels will be below efficiency. If skills instead grow at a positive rate $\kappa > 0$, the overall
responsiveness of beliefs is too high, and effort will be inefficiently high for a patient worker. It is only in the martingale model where the sensitivity-discount ratio has the right size to induce the appropriate incentives. The main message of this analysis is that although the sensitivity-discount ratio is, for any given degree of shock persistence, optimal from a statistical perspective, it is almost never optimal from a social-standpoint.

Given the crucial role that the slope $\kappa$ plays in the model, I illustrate next how the steady-state effort depends on it, along with its dependence on the ratio of volatilities $\sigma_\theta/\sigma_\xi$:

**Corollary 12.** For $\kappa \in \mathbb{R}$ and $\sigma_\theta, \sigma_\xi > 0$, denote by $a^*(\kappa, \sigma_\theta/\sigma_\xi)$, the stationary-learning effort level that arises in equilibrium. Then, for fixed $r > 0$:

(i) Incentives increase with the randomness of the environment: $\frac{da^*(\kappa)}{d\kappa} > 0$, for all $\kappa \in \mathbb{R}$.
   Also, $\lim_{\kappa \to -\infty} a^*(\kappa, \sigma_\theta/\sigma_\xi) = 0$ and $\lim_{\kappa \to \infty} g'(a^*(\kappa, \sigma_\theta/\sigma_\xi)) = 2$;

(ii) Asymptotic efficiency obtains as skills become infinitely volatile relative to signal noise: $\lim_{\sigma_\theta/\sigma_\xi \to \infty} g'(a^*(\kappa, \sigma_\theta/\sigma_\xi)) = 1$. For $r > \kappa$ the convergence is from below, $(a^*(\kappa, \sigma_\theta/\sigma_\xi)$ is strictly increasing in $\sigma_\theta/\sigma_\xi$) and from above otherwise.

(iii) Long-run incentives may survive in settings of deterministic skills or infinitely noisy signals: For $\kappa > 0$ $\lim_{\sigma_\theta/\sigma_\xi \to 0} g'(a^*(\kappa, \sigma_\theta/\sigma_\xi)) = \frac{2\kappa}{r+\kappa}$. Otherwise, the limit is zero;

(iv) Incentives disappear for i.i.d. skills: If $\sigma_\theta = \sigma\sqrt{|\kappa|}$, $\sigma > 0$, then $\lim_{\kappa \to -\infty} a^*(\kappa, \sigma_\theta(\kappa)) = 0$.

*Proof:* See the Appendix.

Part (i) in the previous corollary is the consequence of a sensitivity-discount ratio that is increasing in $\kappa$. Since beliefs are more responsive to overall information when skills are more unstable, there are more benefits from signal-jamming in such environments. Part (ii) states that by expanding the class of stochastic processes for skills, two traditional ideas in career concerns models are now only partially true: that incentives increase both with the uncertainty of the environment (as measured by the volatility of shocks to productivity), and with the precision of the output signal. In fact, in environments where shocks have a high persistence ($\kappa > 0$), a relatively patient agent may find it optimal to decrease effort as the ratio of volatilities $\sigma_\theta/\sigma_\xi$ increases. This is because the sensitivity-discount ratio is, for very unstable processes, decreasing in the ratio of volatilities: for low values of $\sigma_\theta/\sigma_\xi$ what really matter is shock persistence, whereas for large values of it the martingale
component of skills is the most relevant. Part (iii) comes from Proposition 8, which states that the long-run residual variance is non-zero for deterministic skills as long as they grow at strictly positive rates. Finally, part (iv) is just the consequence of the fact that no real learning takes place under i.i.d. skills. I illustrate next how the worker’s effort varies with the persistence of the shocks to productivity (part (i) in the previous corollary):

![Figure 3: Optimal effort as a function of shock persistence.](image)

5 Human Capital Accumulation

Recall that, from Lemma 9, the existence of a deterministic equilibrium is reduced to the existence of a deterministic solution to $\mathcal{P}$:

$$
\mathcal{P} := \left\{ \max_{a \in A} \int_0^\infty e^{-rt} \left[ \beta_t \lambda_t a_t - g(a_t) + \rho_t \bar{\theta}_t(a) \right] dt \right\}
\begin{align*}
\text{s.t.} & \quad \bar{\theta}_t(a), & t & \geq 0, \\
& \quad \gamma_0 \geq \gamma^*,
\end{align*}
$$

where $\gamma_t, \beta_t, \rho_t := \frac{1}{1+r} - \lambda_t$ and $\lambda_t = \int_t^\infty e^{-\int_s^t (r+\delta_u) du} ds, \ t \geq 0,$ are all deterministic.

The human capital model differs from the standard career concerns setting in that there is a new term in $\mathcal{P}$, $\rho := (\rho_t)_{t \geq 0}$, capturing the reputational benefits from human capital accumulation. This process is deterministic and strictly positive. Furthermore, $\rho$ is independent the particular technology at hand. More specifically, $\rho_t$ corresponds to the marginal private benefit from a temporary additional unit of human capital.
To understand how these rents operate, recall that the true value of skills is given by

$$\theta_t = \theta_0 e^{\kappa t} + \int_0^t e^{\kappa(t-s)} \bar{\theta}_s(a) ds + \sigma \int_0^t e^{\kappa(t-s)} dZ_s, \ t \geq 0. \quad (21)$$

It is easy to see that a temporary marginal increase in human capital at time $t$ boosts skills by an amount $e^{\kappa(s-t)}$ at time $s \geq t$ on average. This in turn translates into an additional output flow of expected net present value $\frac{1}{\rho} > 0$ that a more skilled worker is able to produce. However, since the market’s beliefs decay according to $\delta > \kappa$, market participants anticipate an additional output stream of value $\lambda_t$ only, which is strictly below the one actually generated. The difference in value between these two flows, $\rho_t$, corresponds to the expected value of a persistent abnormal output from the market’s perspective. As a result, it is attributed to skills improvement and, hence, it determines the private value from a temporary additional unit of human capital.

The appearance of an unanticipated output flows arising from the worker’s hidden investment decision is a phenomenon that takes place in equilibrium. It does not occur in standard signal-jamming models with exogenous skills (Section 4). In those models the component of output explained by effort is perfectly anticipated in equilibrium. Yet, the worker still trapped in exerting effort because the market’s high expectations about output force him to do so. When workers can instead secretly invest in acquiring skills, they can make strategic use of the discounting wedge to generate abnormal returns from the market’s perspective. These in turn determine the reputational benefits from human capital accumulation.

Even though a fraction of the gains arising from human capital indeed have reputational value, the discounting wedge is never large enough to align marginal private benefits with social ones. As a matter of fact, the efficient effort allocation $a^e := (a^e_t)_{t \geq 0}$ corresponds to the solution to

$$\mathcal{P}^e := \left\{ \begin{array}{l} \max_{a \in A} \int_0^\infty e^{-rt} \left[ a_t - g(a_t) + \frac{1}{\rho} \bar{\theta}_t(a) \right] dt \\ s.t. \quad \bar{\theta}_t(a), \ t \geq 0, \end{array} \right.$$ 

Therefore, the social benefit from a temporary additional unit of human capital stock at any time is in fact $\frac{1}{\rho}$, which is always larger than $\rho$. In a competitive setting the worker internalizes all the benefits and costs from production. This implies that the value of the anticipated output component $\lambda := (\lambda_t)_{t \geq 0}$ is also included in the worker’s payments. Yet, competition among firms forces them to incorporate this anticipated value in the form of an ex-ante premium on the worker’s wage process. Given this exogenous up-front payment, the associated additional output stream arising from human capital accumulation ceases to have any reputational value for the worker. This type of inefficiency is captured in the following:
**Theorem 13.** Suppose that skills are unobservable and that they evolve according to (2), that is, $d\theta_t = (\bar{\theta}_t(a) + \kappa \theta_t)dt + \sigma \theta dZ^\theta_t$, $t \geq 0$, where the family $(\bar{\theta}_t(\cdot))_{t \geq 0}$ satisfies Assumption 1. Suppose instead that output evolves according to $d\xi_t = \theta_t dt + \sigma \xi dZ^\xi_t$, $t \geq 0$, that is, the worker now solves a pure investment problem. Then, the existence of a deterministic equilibrium is reduced to finding a solution to

$$
P_I := \left\{ \max_{a \in A} \int_0^\infty e^{-rt} \left[ \rho_t \bar{\theta}_t(a) - g(a_t) \right] dt \right\}_{t \geq 0},$$

where $\rho_t = \frac{1}{r - \kappa} - \lambda_t \geq 0$, $t \geq 0$. However, the efficient investment allocation is given by the solution to

$$
P_E := \left\{ \max_{a \in A} \int_0^\infty e^{-rt} \left[ \frac{1}{r - \kappa} \bar{\theta}_t(a) - g(a_t) \right] dt \right\}_{t \geq 0},$$

That is, it is generally the case that inefficient training is an equilibrium outcome of Gaussian models of reputation.

**Proof:** See the Appendix.

In order to ascertain that inefficiencies are truly a robust phenomenon it is necessary to study different subclasses of non-stochastic human capital accumulation technologies. Nevertheless, given any particular class, it is highly unlikely both $P_I$ and $P_E$ have the same solution. Consider the two following examples.

**Example 14.** Perfectly reversible human capital technologies: Assume that $\theta_t(a_t) = a_t$, $t \geq 0$. Although the human capital trend is potentially discontinuous, these investments have persistent and continuous effects on skills that decay over time, as it can be seen from (3)

$$
\theta_t = e^{rt} \theta_0 + \int_0^t e^{r(t-s)} a_s ds + \sigma \int_0^t e^{r(t-s)} dZ^\theta_s, \quad t \geq 0
$$

Within this class $g'(a^*_t) = \rho_t < \frac{1}{r - \kappa} = g'(a^*)$. Moreover, $a^* := (a^*_t)_{t \geq 0}$ is decreasing. Finally, the same inequality would hold for more general perfectly-reversible technologies of the form $\bar{\theta}_t(a) = h(t, a_t)$ with $h$ increasing in its second argument.

**Example 15.** Physical-Capital Technologies: Assume that $(\bar{\theta}_t(a))_{t \geq 0}$, $t \geq 0$, corresponds to the solution to an ordinary differential equation (ODE) of the form

$$
d\bar{\theta}_t = (\alpha_t a_t - \phi \bar{\theta}_t)dt, \quad t \geq 0, \quad \bar{\theta}_0 = \bar{\theta}^0 \geq 0.
$$

28
Here $\alpha := (\alpha_t)_{t \geq 0}$ is a positive and uniformly bounded deterministic process representing life-cycle effects from learning-by-doing (typically non-increasing). $\phi \in (0, 1)$ is a depreciation coefficient. Dynamic-programming arguments used Section 5.3 allow us to conclude that
\[
g'(a_t) = \alpha_t \int_0^\infty e^{-(r+\phi)(s-t)} \rho_s ds < \frac{\alpha_t}{(r + \phi)(r - \kappa)} = g'(a^*)
\]
Moreover, $a^* := (a^*_t)_{t \geq 0}$ is decreasing.

The marginal private benefit from a temporary additional unit of human capital, $\rho_t$, decays as time goes by. This is because reputation-driven workers value the option to invest in human capital if and only it can be used to distort the market’s beliefs. As information becomes more precise, the market’s beliefs decay less strongly. This in turn yields abnormal outputs that decrease in size over time, reducing the reputational value that human capital has. Therefore, when the market learns about workers’ skills, human capital inefficiencies are expected to worsen as information improves. The previous examples show that this is indeed the case.

Instead, when skills are observable by all market participants a competitive market would set a wage flow process of the form $w_t = \theta_t$, $t \geq 0$. Observe that since it is the worker who actually controls $\theta := (\theta_t)_{t \geq 0}$, the market’s conjectured strategy $a^*$ plays no role in the way wages are set. Given this wage process, the worker’s problem coincides with the one that maximizes the surplus. That is, the standard efficiency result for general training is recovered.

The reason behind this important discontinuity lies in the fact that, when skills are observable, both parties can implicitly contract on future values of the skills process. Although the worker receives a fixed wage $\theta_t$ over the interval $[t, t + dt)$, by investing in human capital at time $t$ the worker affects $\theta_{t+dt}$. The latter random variable in turn determines the worker’s flow wage over the next interval of time. A worker standing at time $t$ knows that competition induces the market to implicitly offer a contract that is contingent on all the possible values that $\theta_{t+dt}$ can take at time $t + dt$. No ex-ante premia on skills are paid (if they are higher than the ones realized, firms make losses; if they are lower, the worker switches to a different firm). Since skills are observable, this contract is verifiable. Because of competition, this contract is enforceable. As a consequence, efficiency is obtained due to Becker’s classic argument.\footnote{A similar discontinuity occurs in comparative statics with respect to $\sigma_\xi$ in career concerns settings with exogenous skills when learning is stationary. As $\sigma_\xi$ decreases to zero, efficient incentives are induced in the limit (Corollary 12, part (ii)). Yet, when $\sigma_\xi = 0$, skills are observable and thus no incentives to exert}
observable, and the investment action monitored but not contractible. In this case, private and public beliefs would be always aligned, and controlled by the worker through his investment decision. Since beliefs would evolve in the same way as skills do (the slope of both processes coincide), incentives are determined by the same optimality conditions as in the skills-observability case, which in turn coincides with the efficient investment allocation.

Before moving on to the next section I would like to summarize the results found here. When skills are observable, hidden investments in human capital do not generate inefficiencies. That is, competitive markets induce workers to take socially efficient actions. However, the market’s inability to observe workers’ skills creates belief-manipulation motives on the workers’ side. These motives can be exploited through hidden investment decisions that boost productivity. Nevertheless, these reputation-driven incentives are never efficient, and their degree of inefficiency is expected to worsen over time.

5.1 Weak Complementarity

I assume that human capital accumulation and final-goods production are independent decisions. The worker is allowed to choose an unobservable action profile \( a := (a^1_t, a^2_t)_{t \geq 0} \), with the first component affecting output and the second one temporarily boosting skills. Moreover, the impact of the agent’s actions on human capital is additively separable:

\[
\bar{\theta}_t(a) := \bar{\theta} + a^2_t, \quad \bar{\theta} \in \mathbb{R}, \quad t \geq 0.
\]

Without loss of generality, \( \bar{\theta} = 0 \), so skills evolve according to

\[
d\theta_t = (a^2_t + \kappa \theta_t) dt + \sigma_0 dZ^\theta_t, \quad t \geq 0.
\]

When \( \kappa \geq 0 \), \( a^2_t \) boosts the rate at which skills grow. If in turn \( \kappa < 0 \), skills locally mean-revert towards \( a^2_t / |\kappa| \). The weak complementarity between human capital accumulation and final-goods production is understood as follows: because of their perfect reversibility, temporary investments in human capital have a low impact on future skills. Therefore, the impact of these investments on output disappears relatively quickly when compared to more irreversible technologies.

Using Lemma 9, the worker solves

\[
\max_{a \in A} \quad \int_0^\infty e^{-rt} \left[ \beta_t \lambda_t a^1_t + \rho_t a^2_t - g(a^1_t, a^2_t) \right] dt
\]

s.t.

\[
(a^1_t, a^2_t) \in C \subset \mathbb{R}^2_+, \quad t \geq 0,
\]

\[
\gamma_0 \geq \gamma^*,
\]

where \( g : C \to \mathbb{R}_+ \) represents the disutility of effort and \( C \) is the set of feasible values that effort can take.

The same comparative static can be performed in Examples 14 and 15 when learning is stationary. In fact the stationary value of \( \rho \) is

\[
\rho^*(\sigma_\xi) = \frac{1}{1 - \kappa + \beta^*(\sigma_\xi)^2} - \frac{1}{1 - \kappa + \beta^*(\sigma_\xi)^2}.
\]

But

\[
\beta^*(\sigma_\xi) = \gamma^*(\sigma_\xi) / \sigma_\xi^2 \to 0
\]

as \( \sigma_\xi \to 0 \). As a consequence, \( \rho^*(\sigma_\xi) \to 0 \) as \( \sigma_\xi \to 0 \).
A case of particular interest is when \( g(x, y) = \tilde{g}(x + y) \), some function \( \tilde{g} : \mathbb{R}_+ \to \mathbb{R}_+ \) strictly increasing and convex, since in such a setting strategic effects coming effort substitutability across tasks are eliminated. A natural questions that arises is whether the agent will actually decide to invest in human capital accumulation, since by affecting output directly, the choice of \( a^1 \) probably biases the worker’s preferences towards using the traditional signal-jamming channel. As I show next, this may not be the case and, furthermore, delayed training is sometimes optimal.

**Proposition 16.** Suppose that effort is perfectly substitutable in the cost-of-effort function and that \( C = \{(x, y) \in \mathbb{R}_+^2 | x + y \leq R\} \) for some \( R > 0 \). Then \( \min\{a^1_t, a^2_t\} = 0 \) for all \( t \geq 0 \) and:

(i) If \( r - \kappa \geq 1 \), \( a^1_t > 0 \) for all \( t \geq 0 \). That is, the worker never invests in human capital;

(ii) If \( r - \kappa < 1 \) and there is non-zero long-run residual uncertainty \( (\gamma^* > 0) \), then there exists \( T(\gamma_0) < \infty \) such that the worker invests in human capital \( (a^2_t > 0) \) from \( T(\gamma_0) \) on. Moreover, given any \( \gamma_0 > \gamma^* \), there exists \( \epsilon > 0 \) such that if \( 1 - \epsilon < r - \kappa < 1 \), then \( T(\gamma_0) > 0 \). That is, the worker delays training.

**Proof:** See the Appendix.

Part (i) eliminates any chance for endogenous accumulation of human capital when skills mean-revert at sufficiently high rates \( (\kappa < -1) \). In such environments, any investment in human capital vanishes too fast and, hence, the value of the abnormal output generated from it is too low relative to the benefits associated to directly boosting output. In contrast, as part (ii) shows, for relatively low values of the mean reversion coefficient or under a positive grow rate of skills, endogenous experience accumulation may prevail over the standard signal-jamming channel. More interestingly, it is plausible to observe agents that delay human capital accumulation. In those cases the worker favors signaling early in his working life since this is actually the fastest channel to quickly build up a reputation. Later on, once information has improved and the market’s beliefs are less responsive to new signals, the worker switches to invest in acquiring more skills. Although stylized on its own, this model shows that the decision to delay training is sometimes optimal for relatively impatient agents in environments where productivity shocks have enough persistence. Such a behavior is consistent with the career paths observed, for example, in the banking sector.
5.2 Strong Complementarity

Finally, I address the case in which human capital accumulation arises as a byproduct of final goods production. In this setting the worker chooses a unique action profile \( a := (a_t)_{t \geq 0} \) affecting both output and human capital. The latter arises as solution to the ODE (22) in Example 15, that is, \( d\theta_t = (\alpha_t a_t - \phi \theta_t) dt \). Again, by Lemma (9), the agent’s problem reduces to an optimal control problem

\[
P^c := \left\{ \max_{a \in A} \int_0^\infty e^{-rt} \left[ \beta_t \lambda_t a_t - g(a_t) + \rho_t \theta_t(a) \right] dt \right. \\
\text{s.t.} \quad d\theta_t = (\alpha_t a_t - \phi \theta_t) dt, \quad t > 0, \\
\theta_0 = \theta^* \geq 0.
\]

I assume that the worker enters the market with a non-negative human capital stock so as to prevent future levels of it from taking negative values and thus from growing even without the exertion of effort. Finally, in order for the worker’s problem to be finite, \( \alpha \geq 0 \) is assumed to be uniformly bounded and effort can take values in a bounded set \([0, \ell]\), with \( \ell \) large enough. As a consequence, the set of feasible strategies \( A \) corresponds to the set of measurable functions from \( \mathbb{R}_+ \) to \([0, \ell]\).

Strong complementarity is captured by two features. First, human capital is task-specific: workers become more productive as a consequence of leaning-by-doing. Second, since the mean-trend evolves continuously over time, human capital investments have now a higher degree of irreversibility when compared to the weak complementarity case. This in turn maps into investments having more persistent effects on output relative to perfectly reversible ones.

When human capital evolves as in (22) the model becomes fully separable in effort and the hidden variables (skills and human capital) so a deterministic public equilibrium exists. For this purpose, define the continuation value

\[
V(t, x) := \sup_{a \in A} \left. \int_0^\infty e^{-rs} \left[ a_s \beta_s \lambda_s - g(a_s) + \rho_s \theta^t,x_s(a) \right] ds, \quad (t, x) \in \mathbb{R}^2_+ \right)
\]

where \( (\theta^t,x_s(a))_{s \geq t} \) is the solution of the ODE (22) with initial condition \( \theta_t = x \). Classic results in dynamic programming state that if \( V \) is smooth enough (of class \( C^1(\mathbb{R}_+^2) \)) and satisfies a growth condition (see Pham (2009)), then it is the unique solution to the Hamilton-Jacobi equation (HJ)

\[
0 = \sup_{u \in A} \left\{ e^{-rt} \left[ \beta_t \lambda_t u - g(u) + \rho_t x \right] + \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)|\alpha_t u - \phi x| \right\}, \quad (t, x) \in \mathbb{R}_+^2,
\]

satisfying the following transversality condition: \( \lim_{t \to -\infty} V(t, \theta^0,x_t(a)) = 0 \) for every \( x \in \mathbb{R}_+ \) and any control \( a \in A \). Moreover, the optimal effort strategy is given by the maximizer \( u^*(\cdot) \) of the right-hand side in (HJ), i.e. \( a^*_t = u^*(t, \theta_t) \), \( t \geq 0 \).
The above problem, although linear in its dynamic, it is not straightforward to solve. First, the cost function \( g \) corresponds to any strictly increasing and convex function and not necessarily of quadratic form. Second the problem is a non-stationary one. Nevertheless, it has a particularly clean solution:

**Proposition 17.** In the strong complementarity case a deterministic equilibrium exists. It is characterized by the following first order condition:

\[
g'(a_\ast^t) = \beta_t \lambda_t + \alpha_t \mu_t
\]

(24)

where \( \mu_t := \int_t^\infty e^{-(r+\phi)(s-t)} \rho_s ds \) is a decreasing function and \( \rho_t = \frac{1}{r+\kappa} - \lambda_t, \ t \geq 0 \). The continuation-value function takes the form

\[
\mathcal{V}(t, x) = e^{-rt} [\eta_t + \mu_t x], \ (t, x) \in \mathbb{R}_+^2,
\]

(25)

where \( \eta_t := \int_t^\infty e^{-r(s-t)} [g'(a_\ast^s) a_\ast^s - g(a_\ast^s)] ds \) for all \( t \geq 0 \).

**Proof:** See the Appendix.

As it can be seen from the result, the optimal effort allocation is larger than what predicted by traditional career concerns with exogenous skills. The difference is given by the discounted benefits associated to a marginal increase in human capital \( \mu \), adjusted by the rate at which the worker learns from experience, \( \alpha \). The discrepancy between \( \rho \) and \( \mu \) comes from the observation that the former corresponds to the benefit associated to a temporary marginal increase in the stock of human capital. But in this model of such a marginal increase disappears only gradually, and thus the shadow value of human capital is given by \( \mu \). Since the informational rent that the worker acquires through the human capital channel \( \rho \) decreases as information improves, \( \mu \) is also decreasing. However, as long as there is non-zero residual uncertainty and the marginal productivity of effort in the human capital technology is bounded away from zero, \( \mu \) will never vanish.

Finally, a comment on the incentives to acquire human capital before entering the job market. The worker’s expected discounted benefits from entering it with a human capital level of size \( \tilde{\theta} \) are given by \( \mathcal{V}(0, \tilde{\theta}) = \eta_0(\alpha, \phi) + \mu_0(\phi) \tilde{\theta}, \) where I made explicit the dependence of the initial values \( \eta_0 \) and \( \mu_0 \) on \( \alpha = (\alpha_t)_{t \geq 0} \) and \( \phi \). Given the long-term effects that it has on the worker’s skills, the human capital depreciation rate \( \phi \) indeed affects the worker’s schooling choice (the rate at which he learns from work experience is irrelevant at this stage). Suppose the “future worker” makes a static choice on how much education to acquire. The cost of education is given by an increasing and differentiable
function \( c(e) \) while the benefits from it by an increasing and concave function \( \bar{\theta}(e) \). The first-order condition of this problem yields \( \mu_0(\phi) \bar{\theta}(e^*(\phi)) - c'(e^*(\phi)) = 0 \), from where we see that \( e^*(\phi) \) is be decreasing. That is, individuals that suffer from less human capital depreciation would choose a higher level of education before entering the labor market.

5.3 Predicted Path of Wages

Recall that the wage process takes the form \( w_t = m_t^* + a_t^1 - a_t^2 \), \( t \geq 0 \), where effort either strictly decays over time (away from steady-state) or it remains constant (in a stationary-learning environment). In equilibrium, the posterior mean evolves as

\[
\frac{dm_t^*}{m_t^*} = (\theta_t(a^*) + \lambda t) dt + \frac{\gamma_t}{\sigma^2} dZ_t^a, \quad t \geq 0,
\]

where \( Z_t^a \) is a Brownian motion. As a consequence, and from both parties’ perspectives, wages will, on average, decay in the fixed-skills case and martingale model \((\kappa = \bar{\theta} = 0)\).

Some of the human capital accumulation models presented here are able to generate wages with endogenous positive drift. For example, as direct corollary of Proposition 16, if effort across tasks were not perfectly substitutable in the cost function, workers would focus both in human capital accumulation and in final-goods production. In such settings, the endogenous accumulation of skills may offset the negative effect that increasingly precise information has on incentives. Also, the strong complementarity case predicts increasing wages close to the stationary-learning level as long as the agent is indeed accumulating human capital. Moving away from steady-state the fast dynamics of the learning structure used here prevent the model from generating concave wages. Nevertheless, the presence of contractual rigidities that prevent the downward adjustment of wages may still give the model validity in explaining the path of wages over the life-cycle through the well-known human capital accumulation channel (Mincer 1974; Ben-Porath 1967). Moreover, the model is still able to explain the evolution of wages in environments where firms have relatively accurate estimates of workers’ skills, in settings where the reputational component of wages is the dominating one (see Dewatripont et al. (1999) for a multi-task analysis motivated by the case of government agencies), or in environments where signal manipulation is not possible.

It is important to emphasize that (26) does not correspond to the process actually generating observations from the reputational component of wages. In fact, because of

\[\footnotesize{\text{In a pure learning model Harris and Holmström (1982) provide an alternative explanation for wages that increase with tenure. When workers are risk averse, firms learn about workers’ abilities and long-term contracts can be written, workers pay a risk premium in order to insure themselves from future low-outcome realizations. This premium decreases with the precision of skills’ estimates, so senior workers have higher salaries on average.}}\]
learning, the distribution of wages under the subjective probability measure differs from the distribution of wages under the true data-generating process. More specifically, it is always the case that beliefs locally mean-revert towards the contemporaneous true value of skills at every point in time, regardless of the specification used for skills. To understand this, observe that under the true probability measure, output is driven by $d\xi = (a^*_t + \theta_t)dt + \sigma^*_t dZ^\xi_t$, so that $\Delta_t := m^*_t - \theta_t, t \geq 0$, evolves according to

$$d\Delta_t = -\delta_t \Delta_t + \beta_t dZ^\xi_t - \sigma^*_t dZ^\theta_t, \ t \geq 0,$$

implying that whenever the market is optimistic or pessimistic about the worker’s ability, the arrival of new information will tend to eliminate this bias.

Traditional pure-learning models (learning about a fixed unobservable) have predicted that increments in wages should be uncorrelated. However, this is only true from the agents’ perspective in the economy— that is, given their limited information sets. The econometrician would observe wages coming from the true data generating process, and, because of mean-reversion, some persistence in wages would be observed. This point is in fact not new. In an asset pricing context, Lewellen and Shanken (2002) show how learning about fundamentals can generate predictability in stock returns even when stock prices follow martingales from the market participants’ perspectives.

There is a vast literature studying the covariance structure of earnings using data coming from different sources. Relevant to my work are the results from Kahn and Lange (2011), who give empirical support to the idea of combining workers’ evolving productivity and employer learning as a way to explain some observed patterns in wages. In independent work, they show how such a model does a better job at matching second moments of wages than a pure-productivity model or a pure-learning model can do on their own. In this line, unobserved heterogeneity is probably the most accepted (and reasonable) theory to explain the persistence of changes in earnings observed cross-sectionally. Yet, learning models are still able to generate positive autocovariance in increments of earnings, especially during the early stages of the learning experience. In the next figures I show how, under specific parameters, beliefs do “chase” true skills until the stationary-learning level is reached (mean-reversion), and also show how autocorrelations evolve over time. I do this in the context of a pure investment model, in which different specifications for the growth-trend arising from human capital accumulation are taken as given:

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18 From a modeling perspective my setting differs from theirs along two lines. First, the model adds a strategic component to the combined framework of workers’ evolving productivity and employer learning. Second, I study AR(1) skills processes with a partially endogenous drift, while they focus on the random walk case with an exogenous growth trend.

19 That initial increments in beliefs are positively correlated appears to be very sensitive to the slope $\kappa$. 

35
Figure 4: Three different models of mean-reverting skills with parameters $\kappa = -2$, $\sigma_\theta = 0.2$ and $\sigma_\xi = 0.5$. The first row corresponds to a mean-reverting model around zero. The left panel shows a particular realization of productivity shocks. The middle panel shows different sample paths of beliefs under different realizations of the output signal $\xi$. Finally the right panel shows the autocorrelation of increments in wages separated 100 units apart from each other (10,000 points in the grid). The second and third row perform the same analysis under a linear trend and the physical-capital dynamics represented by (22) with $\alpha(t) \equiv 1$, $t \geq 0$, respectively. Shocks realizations (for skills and 100 of them for output noise) are independent across rows.

Finally, as any model involving Brownian shocks would, the model predicts that the variance of wages increases over time. More interestingly, by allowing skills to have residual uncertainty, the variance of changes in wages does not decay as fast as it would in a pure learning model with fixed skills (see Farber and Gibbons 1996). By varying the shock persistence parameter $\kappa$, the speed at which the variance of these increments decays over time changes.

For low degrees of mean-reversion (negative values, close to zero) or positive growth rates this effect tends to disappear. In fact, given that learning in general generates reversion to the mean, a necessary condition to obtain positive autocorrelation in beliefs is that the unobservable process suffers from strong changes in its behavior. The latter typically take place under relatively high levels of mean reversion.
6 Conclusions

This paper developed a flexible dynamic model of career concerns involving Gaussian skills and on-the-job experience accumulation. It contributes to the labor markets literature by studying the effects that learning, evolving skills and human capital accumulation have on incentives. These issues interact significantly in defining the costs and benefits associated to workers’ careers. The use of continuous-time techniques is of major importance for elucidating the forces that shape reputation-driven incentives in competitive markets.

I emphasized the primary role that the environment plays for influencing the incentives that arise from career concerns. The persistence of shocks to productivity determines the size of the monetary gains that arise from belief-distortion in stationary-learning settings. This is because shock persistence dictates the overall responsiveness of beliefs to aggregate information. As productivity shocks become more persistent, beliefs become more responsive and, therefore, workers exert more effort. More interestingly, under-provision of effort is not the unique long-run outcome, since inefficiently high effort is an optimal strategy for patient workers in highly unstable environments.

The possibility to secretly invest in human capital creates a new belief-manipulation channel that workers can exploit. Even though workers internalize the full benefits from human capital accumulation, only a fraction of them actually have a reputational value. This is because of the interplay between learning and market competition. Learning about skills allow firms to anticipate part of the additional benefits associated with investments in human capital. Market competition in turn forces firms to incorporate the expected value of these gains as an ex-ante premium in the workers’ wage processes. As a result, only the unanticipated component of the monetary benefits associated with human capital accumulation determines the worker’s marginal private value for human capital. Inefficiently low investments are expected to be a robust finding and I confirm this in two classes of human capital accumulation technologies.

There are several tangential issues not presently addressed by the present model. With respect to turnover, building a model that incorporates learning, separations, endogenous skills accumulation and moral hazard may seem an attractive challenge, it is unclear whether new substantial insights can be obtained from such a complex structure. Also, I have avoided the analysis of career concerns in the presence of complementarities between skills and effort in the output signal. This has proven itself to be a particularly challenging question, especially because endogenous information asymmetries play a non-trivial role. These and other interesting questions are beyond the scope of this paper, and are left for future research.
References


7 Appendix A: Proofs

Proof of Proposition 5: Since \( a^* \) is an \( \mathbb{F} \)-progressively measurable process, the result is a direct application of Theorem 7.12. in Liptser and Shiryaev (1977).

\[
\square
\]

Proof of Proposition 9: Suppose that the market conjectures that the manager will follow a deterministic strategy \( a^* := (a^*_t)_{t \geq 0} \). Since wages take the form \( w_t = m^*_t + a^*_t \), only \( (m^*_t)_{t \geq 0} \) matter for incentives. Also, the fact that for each \( t \geq 0 \), \( \theta_t(\cdot) \) is a deterministic functional of paths of the form \( (y_s : 0 \leq s \leq t) \), implies that the trajectory of human capital conjectured by the market, \( (\overline{\theta}_t(a^*)) \), is fixed at time zero and unaffected by the worker’s effort choice. The market’s beliefs evolve according to

\[
dm^*_t = (\overline{\theta}_t(a^*) + \kappa m^*_t)dt + \frac{\gamma_t}{\sigma^2_{\xi}} d\xi_t - (m^*_t + a^*_t)dt
\]

where \( \gamma_t \) follows the dynamic (9) and \( Z^{a^*} \) is a Brownian motion from the market’s perspective. The solution to the above SDE is given by

\[
m^*_t = e^{-\int_0^t \delta_s ds} m_0 + \int_0^t e^{-\int_s^t \delta_u du} [\overline{\theta}_s(a^*) ds + \beta_s (d\xi_s - a^*_s ds)]
\]

where \( \beta_t := \frac{\gamma_t}{\sigma^2_{\xi}} \) and \( \delta_t := \beta_t - \kappa \) for all \( t \geq 0 \). Since from the worker’s perspective \( (\overline{\theta}_t(a^*))_{t \geq 0} \) and \( a^* \) are exogenously given, incentives are determined only by

\[
G_t := \int_0^t e^{-\int_0^s \delta_u du} \beta_s d\xi_s.
\]

Let \( (m_t)_{t \geq 0} \) denote the worker’s posterior belief process of his own talent when he follows any strategy \( a := (a_t)_{t \geq 0} \). Assume that it evolves according to an SDE of the form

\[
dm_t = (\overline{\theta}_t(a) + \kappa m_t)dt + \sigma_t dZ_t
\]

where \( Z := (Z_t)_{t \geq 0} \) is a Brownian motion from the worker’s standpoint. Moreover, the process \( Z^a_t := \frac{1}{\sigma^2_{\xi}} \left( \xi_t - \int_0^t (a_s + m_s) ds \right) \), \( t \geq 0 \) is also a Brownian motion from his perspective and is correlated to \( Z \). By Lemma 5 we can write output from the worker’s perspective as

\[
d\xi_t = (m_t + a_t)dt + \sigma_{\xi} dZ^a_t, \ t \geq 0.
\]
Inserting this into the expression for \( G_t \) gives us how the worker evaluates belief-distortions on the market’s side,

\[
G_t := \int_0^t e^{-\int_0^t \delta_u \, du} \beta_s [(m_s + a_s) \, ds + \sigma \xi \, dZ_s^{m}]
\]

\[
= \int_0^t e^{-\int_0^t \delta_u \, du} \beta_s \left[ e^{\kappa s} m_0 + \int_0^s e^{\kappa (s-u)} (\overline{\theta}_u(a) \, du + dZ_u) + a_s \, ds + \sigma \xi \, dZ_s^{a} \right]
\]

where I used that \( m_s = e^{\kappa s} m_0 + \int_0^s e^{\kappa(s-u)} (\overline{\theta}_u(a) \, du + dZ_u), \)  \( s \geq 0 \). The first term in \( G_t \) is unaffected by the effort decision so the worker’s optimization problem is reduced to

\[
\max_{a \in A} \mathbb{E}^a \left[ \int_0^\infty e^{-rt} \left( \int_0^t e^{-\int_0^t \delta_u \, du} \beta_s \left\{ \int_0^s e^{\kappa(s-u)} (\overline{\theta}_u(a) \, du + a_s \, ds) \right\} - g(a_t) \right) \, dt \right]
\]

For any strategy \( a \in A \) that the worker follows, \( Z^a \) and \( Z \) are exogenous Brownian motions. Moreover, since of any initial condition \( \gamma_0, (\beta_t)_{t \geq 0} \) and \( (\delta_t)_{t \geq 0} \) are uniformly bounded, all the stochastic integrals above will have zero expectation. As a consequence, the problem is reduced to

\[
\max_{a \in A} \mathbb{E}^a \left[ \int_0^\infty e^{-rt} \left( \int_0^t e^{-\int_0^t \delta_u \, du} \beta_s \left\{ \int_0^s e^{\kappa(s-u)} (\overline{\theta}_u(a) \, du + a_s \, ds) \right\} - g(a_t) \right) \, dt \right] \quad (33)
\]

Integration by parts and the fact that \( \delta_t = \beta_t - \kappa \) yield

\[
\int_0^t e^{-\int_0^t \delta_u \, du} \beta_s \int_0^s e^{\kappa(s-u)} (\overline{\theta}_u(a) \, du
\]

\[
= e^{rt} \int_0^t e^{-\kappa s} (\overline{\theta}_s(a)) \, ds - e^{-\int_0^t \delta_s \, ds} \int_0^t e^{-\int_0^s \delta_u \, du} (\overline{\theta}_s(a)) \, ds
\]

With this in hand, the manager’s objective function has 3 integrals of the form (up to multiplicative constants)

\[
I := \int_0^\infty e^{-rt} \left[ e^{-\int_0^t \tau_u \, du} \int_0^t e^{\int_0^s \tau_u \, du} \nu_s \right] \, dt
\]

where \( \tau = \delta \) or \( -\kappa \) and \( \nu = a \) or \( \overline{\theta}(a) \). Since in any case \( r + \tau > 0 \), a direct application of Fubini’s theorem implies that

\[
I = \int_0^\infty e^{\int_0^t \tau_u \, dt} \nu_t \int_t^\infty \exp(-\int_0^t (r+\tau_u) \, du) \, ds \, dt = \int_0^\infty e^{-rt} \nu_t \int_t^\infty e^{-\int_0^s (r+\tau_u) \, du} \, ds \, dt
\]

Defining \( \rho_t = \frac{1}{r-\kappa} - \lambda_t \) and \( \lambda_t := \int_t^\infty e^{-\int_0^s (r+\delta_u) \, du} \, ds \), the worker will solve

\[
\mathcal{P} := \left\{ \max_{a \in A} \int_0^\infty e^{-rt} \left[ \beta_t \lambda_t a_t - g(a_t) + \rho_t \overline{\theta}_t(a) \right] \, dt \quad s.t. \quad \overline{\theta}_t(a), \ t \geq 0, \gamma_0 \geq \gamma^*, \right\}
\]

concluding the proof of Lemma 9.
Proof of Proposition 10: We only need to show that $l_t := \beta_t \lambda_t$ is decreases over time, where $\beta_t = \gamma_t / \sigma^2$, $\lambda_t = \int_{t}^{\infty} e^{-\int_{t}^{u} (r + \delta_u) du} du$ and $\delta_t = \beta_t - \kappa$, $t \geq 0$. Observe that

$$\frac{d \log(l(t))}{dt} = \frac{\dot{\gamma}_t}{\gamma_t} + r + \frac{\gamma_t}{\sigma^2} - \frac{1}{\int_{t}^{\infty} e^{-\int_{t}^{u} (r + \gamma_u / \sigma^2 - \kappa) du} du}. \tag{34}$$

Suppose $\gamma_t > \gamma^*$, which occurs if and only if $\dot{\gamma}_t \leq 0$. Then,

$$\lambda_t = \int_{t}^{\infty} e^{-\int_{t}^{u} (r + \gamma_u / \sigma^2 - \kappa) du} du < \frac{1}{r + \gamma^*/\sigma^2 - \kappa},$$

implying that

$$\frac{d \log(l(t))}{dt} < \frac{\dot{\gamma}_t}{\gamma_t} + \frac{\gamma_t}{\sigma^2} - \frac{\gamma^*}{\sigma^2}.$$

Finally, from the ODE that governs $\gamma_t$, (see (9)), it can be observe that $\dot{\gamma}_t / \gamma_t + \gamma_t / \sigma^2 = 2\kappa + \sigma^2 / \gamma_t$, so

$$\frac{d \log(l(t))}{dt} < 2\kappa + \frac{\sigma^2}{\gamma_t} - \frac{\gamma^*}{\sigma^2} < 2\kappa + \frac{\sigma^2}{\gamma^*} - \frac{\gamma^*}{\sigma^2} = 0$$

by definition of $\gamma^*$. When $\dot{\gamma}_t \geq 0$ (and so $\gamma_t < \gamma^*$ for all $t \geq 0$) an analogous argument shows that $l_t$ increases over time (the above inequalities just reverse). This concludes the proof.

Proof of Corollary 12: The only non-trivial assertion in the corollary is the the first part of (i) when $\kappa > 0$. I will show in fact that the derivative is strictly positive for all $\kappa \in \mathbb{R}$. Recall that $\beta^* = \beta^*(\kappa)$, $\delta^*(\kappa) = \beta^*(\kappa) - \kappa = \sqrt{\kappa^2 - \sigma^2 / \sigma^2} > 0$. It is easy to see that

$$\frac{d a^*(\kappa)}{d \kappa} = \frac{(\delta^*(\kappa) + \kappa)(r + \delta^*(\kappa) - \kappa)}{\delta^*(\kappa)(r + \delta^*(\kappa))^2} > 0$$

where the last inequality comes from $\delta^*(\kappa) \pm \kappa > 0$ and $r > 0$.

Proof of Proposition 13: Comes from the same steps followed in the proof of Lemma 9.

Proof of Proposition 16: Omitting the dependence on time, the first order conditions of the worker’s problem correspond to

$$\tilde{g}'(a^{1,*} + a^{2,*}) - \beta \lambda - \mu_1 + \mu_3 = 0$$

$$\tilde{g}'(a^{1,*} + a^{2,*}) - \rho - \mu_2 + \mu_3 = 0$$

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where \( \mu_i \geq 0 \) is the lagrange multiplier associated to the constraint \( a_i \geq 0, \ i = 1, 2, \) and \( \mu_3 \geq 0 \) the one corresponding to \( a_1 + a_2 \leq R \). Therefore, the optimal effort allocation satisfies

\[
a^{1,*} = (\tilde{g}')^{-1}(\beta \lambda) > 0, \ a^{2,*} = 0 \ \iff \ \beta \lambda > \rho
\]

\[
a^{2,*} = (\tilde{g}')^{-1}(\rho) > 0, \ a^{1,*} = 0 \ \iff \ \beta \lambda < \rho
\]

and any \((a^1, a^2)\) s.t. \( a^1 + a^2 = (\tilde{g}')^{-1}(\rho) \) when \( \rho = \beta \lambda \).

(i) Suppose that \( r - \kappa \geq 1 \). This yields \( \frac{\beta_t + 1}{r + \beta_t - \kappa} \geq 1 \), for all \( t \geq 0 \). If \( \gamma_0 > \gamma^* \), \( \beta_t \) is strictly decreasing over time, implying that it is optimal to set \( a^*_t = 0 \). As a consequence, it is optimal for the worker to specialize in final-goods production. When \( \gamma_0 = \gamma^* \), \( \beta_t = \beta^* \) for all \( t \) and \((*)\) becomes a weak inequality, so the result still holds.

(ii) Now assume that \( r - \kappa < 1 \). In steady state \( \beta_t = \beta^* \) for all \( t \geq 0 \) and assume \( \beta^* > 0 \). Trivially, \((r - \kappa)\beta^* < \beta^* \), so

\[
\beta^* \lambda^* = \frac{\beta^* + 1}{r + \beta^* - \kappa} < \frac{1}{r - \kappa}.
\]

This is equivalent to \( \beta^* \lambda^* < \rho^* \), therefore showing that the agent specializes in human capital accumulation in steady-state. By continuity, there exists \( T(\gamma_0) \geq 0 \) sufficiently large \( t \) s.t. \( \beta_t \lambda_t - \rho_t < 0 \) and thus \( a^{2,*}_t > 0 \) for all \( t \geq T(\gamma_0) \).

To conclude, observe that when \( r - \kappa = 1 \), \( \beta_0 \lambda_0 - \rho_0 = (\beta_0 + 1)\lambda_0 - 1 \). But, for \( \gamma_0 > \gamma^* \), \((\beta_t)_{t \geq 0}\) is strictly decreasing, implying that

\[
(\beta_0 + 1)\lambda_0 = (\beta_0 + 1) \int_0^\infty e^{-f'^*(\beta_t + 1)du} ds > 1.
\]

As a consequence, given \( \gamma_0 > \gamma^* \), there exists \( \epsilon > 0 \) s.t. for \( 1 - \epsilon < r - \kappa < 1 \) we have

\[
(\beta_0 + 1)\lambda_0 - 1 = (\beta_0 + 1)\lambda_0 - \frac{1}{r - \kappa} > 0,
\]

implying that it is optimal to set \( a^{1,*}_t > 0 \) and \( a^{2,*}_t = 0 \) in the early stages of the agent’s working life. This concludes the proof.

\[\Box\]

**Proof of Proposition 17:** I will find a solution of the form \( \mathcal{V}(t, x) = b_t + c_t x \). Plugging this in \((HJ)\) we get

\[
\sup_{u \in A} \left\{ e^{-rt}[\beta_t \lambda_t u - g(u) + \rho_t x] + \frac{db_t}{dt} + \frac{dc_t}{dt} x + c_t[\alpha_t u - \phi x] \right\} = 0 \tag{35}
\]

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Impose that \((c_t)_{t \geq 0}\) satisfies the ODE \(dc_t + (e^{-rt} \rho_t - \phi c_t) dt = 0\) with transversality condition \(\lim_{t \to \infty} e^{-dt} c_t = 0\). Then,

\[
    c_t = e^{-rt} \int_t^\infty e^{-(r+\phi)(s-t)} \rho_s ds, \quad t \in \mathbb{R}_+
\]

In fact, since \(\rho_t\) is bounded we get a stronger transversality condition: \(c_t \to 0\) as \(t \to \infty\). The (HJ) equation then becomes \(\sup_{u \in A} \{ e^{-rt}[\beta_t \lambda_t u - g(u)] + \frac{db_t}{dt} + c_t \alpha_t u \} = 0\) which yields the first order condition

\[
    g'(u_t^*) = \beta_t \lambda_t + \alpha_t \mu_t, \quad t \in \mathbb{R}_+
\]

and that \(b_t\) must satisfy \(db_t = e^{-rt}[g(u_t^*) - g'(u_t^*)u_t^*] dt\). Observe that since \((u_t^*)_{t \geq 0}\) is bounded \((\rho_t)_{t \geq 0}\) and \((\alpha_t)_{t \geq 0}\) are bounded and \((\beta_t \lambda_t)_{t \geq 0}\) is decreasing and non-negative) the last condition has as a solution

\[
    b_t := e^{-rt} \int_t^\infty e^{-r(s-t)} [g'(u_s^*)u_s^* - g(u_s^*)] ds, \quad t \in \mathbb{R}_+
\]

and moreover, \(b_t \to 0\) as \(t \to \infty\). Therefore, we have found a function \(V(t, x) = b_t + c_t x\) of class \(C^1(\mathbb{R}_+ \times \mathbb{R})\) such that it satisfies (HJ). We now need to show that is satisfies the transversality condition. To see this, fix an initial condition for all \(\lambda\) over time \((\rho_t)\). Because of this, \(\beta_t \lambda_t\) bounded \((\rho_t)\) becomes \(\sup_{u \in A} \{ e^{-rt}[\beta_t \lambda_t u - g(u)] + \frac{db_t}{dt} + c_t \alpha_t u \} = 0\) which yields the first order condition

\[
    g'(u_t^*) = \beta_t \lambda_t + \alpha_t \mu_t, \quad t \in \mathbb{R}_+
\]

and that \(b_t\) must satisfy \(db_t = e^{-rt}[g(u_t^*) - g'(u_t^*)u_t^*] dt\). Observe that since \((u_t^*)_{t \geq 0}\) is bounded \((\rho_t)_{t \geq 0}\) and \((\alpha_t)_{t \geq 0}\) are bounded and \((\beta_t \lambda_t)_{t \geq 0}\) is decreasing and non-negative) the last condition has as a solution

\[
    b_t := e^{-rt} \int_t^\infty e^{-r(s-t)} [g'(u_s^*)u_s^* - g(u_s^*)] ds, \quad t \in \mathbb{R}_+
\]

and moreover, \(b_t \to 0\) as \(t \to \infty\). Therefore, we have found a function \(V(t, x) = b_t + c_t x\) of class \(C^1(\mathbb{R}_+ \times \mathbb{R})\) such that it satisfies (HJ). We now need to show that is satisfies the transversality condition. To see this, fix an initial condition \(x \geq 0\). For any feasible control \(a\) the path \(t \mapsto \overline{\theta}_t^{t,x}(a)\) takes values in the interval \([0, \max\{\overline{\theta}, \ell K/\phi\}]\), where \(K\) bounds \((\alpha_t)_{t \geq 0}\). As a consequence, the path of human capital remains bounded all the time. Because \(b_t, c_t \to 0\) as \(t \to \infty\), we trivially conclude that \(\lim_{t \to \infty} b_t + c_t \overline{\theta}_t^{t,x}(a) = 0\).

Recall that \(\rho_t = \frac{\kappa}{\sigma} - \kappa \lambda_t\) where

\[
    \lambda_t := \int_t^\infty \exp \left( - \int_t^s (r + \delta_u) du \right) ds, \quad t \geq 0.
\]

Direct calculations show that \(\lambda_t\) satisfies the ODE \(d \lambda_t = (|\gamma_t| \lambda_t - 1) dt\). If \(\gamma_t\) is decreasing over time \((\gamma_0 > \gamma^*)\), so will be \(\delta_t = \gamma^* / \sigma^2 + \kappa\), which implies that \(\lambda_t > 1 / (r + \delta_t)\) for all \(t \geq 0\). As a consequence \(\lambda_t\) is increasing and, furthermore, bounded above by \(1 / (r + \delta_t)\) (with \(\delta := \gamma^* / \sigma^2 + \kappa\)), so it converges. With this in hand, we conclude that the marginal benefit from an extra unit of human capital at time \(t\), \(\rho_t\), decreases over time and will also converge (it is bounded below by zero). Because of this,

\[
    \mu_t = \int_t^\infty e^{-(r+\phi)(s-t)} \rho_s ds < \frac{\rho_t}{r + \phi}
\]

for all \(t \geq 0\). Finally, observing that \((e^{rt} \mu_t)_{t \geq 0}\) satisfies the ordinary differential equation \(d \mu_t = (|r + \phi| \mu_t - \rho_t) dt\), we conclude.
8 Appendix B: General Human Capital Technologies

It is not unreasonable to think that there are complementarities in the technology connecting investments in skills and their current stock or level. The problem with studying more general markovian diffusions of the form
d\theta_t = \mu(a_t, \theta_t)dt + \sigma(a_t, \theta_t)dZ^\theta_t, \ t \geq 0
is that, even though the filtering equations associated to posterior moments of \theta given \xi may exist, such a system may not be closed. A tractable way to incorporate such complementarities is by doing so in a deterministic way through an additional state variable. In this section I assume that for any feasible effort strategy \(a \in A\) (a concept to be defined immediately), \((\bar{\theta}_t(a))_{t \geq 0}\) is the solution to the ODE
\[
d\bar{\theta}_t = f(t, \bar{\theta}_t, a_t)dt, \quad \bar{\theta}_0 = \bar{\theta}^0 \geq 0. \tag{37}
\]

Since in the above specification the agent’s optimal action will typically depend on his current human capital stock, we need to relax the equilibrium concept. Even though the worker’s effort strategy cannot be correctly guessed once a deviation has taken place, the fact that human capital evolves deterministically allows the market to perfectly anticipate the on-equilibrium effort strategy.

**Definition 18.** An effort strategy \(a := (a_t)_{t \geq 0}\) is of the feedback form is it corresponds to a function \(a_t = a(t, \bar{\theta})\), where \(\bar{\theta}\) is the agent’s stock of human capital at time \(t \geq 0\), for some function \(a : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+\).

Given \(\bar{\theta}^0\), which I assume common knowledge, and a conjectured feedback control \(a^*\), the market conjectures a human capital trend \((\bar{\theta}_t(a^*))_{t \geq 0}\). This trend is therefore fixed ex-ante and the question is whether a Bayesian nash equilibrium in feedback strategies exists:

**Definition 19.** An equilibrium of this economy is a feedback effort strategy \(a^*\) and a wage process \(w := (w_t)_{t \geq 0}\), such that:

(i) Given \(a^*\), the market sets a wage of the form \(w_t = E^{a^*}[\theta_t | \mathcal{F}_t^\xi] + a_t^*\) for all \(t \geq 0\);

(ii) \(a^*\) is optimal for the manager given the wage process in (i):

\[
a^* \in \arg\max_{a \in A} E^a \left[ \int_0^\infty e^{-rt}(w_t - g(a_t))dt \right] \tag{38}
s.t. \quad w_t = E^{a^*}[\theta_t | \mathcal{F}_t^\xi] + a_t^*, \ \forall t \geq 0.
\]

That is, I have eliminated the requirement that the market perfectly anticipates the worker’s strategy after all private histories.
Given that the family \((\bar{\theta}(a^*))_{t \geq 0}\) is deterministic, Lemma 9 applies and the existence of an equilibrium is reduced to the existence of a feedback control to the optimal control problem

\[
\max_{a \in A} \int_0^{\infty} e^{-rt} \left[ \beta_t \lambda_t a_t - g(a_t) + \rho_t \bar{\theta}_t(a_t) \right] dt
\]

s.t. \(d\bar{\theta}_t = f(t, a_t, \bar{\theta}_t) dt, \ t > 0, \ \bar{\theta}_0 = \bar{\theta}^0.\)

I will look for optimal strategies in the following class of controls:

**Definition 20.** (Feasible Control) A control is said to be feasible if it corresponds to a piecewise continuous\(^{20}\) function of time \(a : \mathbb{R}_+ \to A := [0, \ell].\) Denote the set of feasible controls by \(\mathcal{A}.\)

Since we want to capture that the agent accumulates more human capital as he becomes more experienced (as measured by how engaged in production the worker has been), I assume that \(a \mapsto f(t, \bar{\theta}, a)\) is strictly increasing in for all \(\bar{\theta} \geq 0.\) More generally, this function is required to satisfy this very weak conditions:

**Assumption 21.** It is assumed that

(i) \(f : \mathbb{R}_+ \times \mathbb{R} \times A \to \mathbb{R}\) is such that \(f(t, \cdot, a) \in C^1(\mathbb{R})\) for all \(a \in A\) and \(f(t, \bar{\theta}, \cdot)\) is differentiable for all \(\bar{\theta} \in \mathbb{R}.\)

(ii) For all \(\bar{\theta} \geq 0,\) the function \(a \mapsto f(t, \bar{\theta}, a)\) is strictly monotone.

The following result gives necessary conditions that the any optimal control must satisfy. It is an application of Pontryagin’s Maximum Principle for infinite horizon problems (see Halkin (1974)).\(^{21}\)

**Proposition 22.** Let \(a^* \in \mathcal{A}\) be an optimal control and suppose \(a \neq 0, \ell.\) Then, there exists a piecewise continuously differentiable function \(q : \mathbb{R}_+ \to \mathbb{R}\) such that

(i) For almost every \(t \in \mathbb{R}_+\)

\[
 dq_t = \left\{ q_t \left[ r - \frac{\partial f}{\partial \bar{\theta}}(t, \bar{\theta}_t, a^*_t) \right] - \rho_t \right\} dt; \tag{39}
\]

\(^{20}\) A function \(\phi : \mathbb{R} \to A \subseteq \mathbb{R}\) is piecewise continuous if for any interval \([a, b] \subseteq \mathbb{R}\) there exists a finite set of points \(a = t_0 < t_1 < \ldots < t_n = b\) such that \(\phi\) is continuous in \([t_0, t_1]\) and \((t_i, t_{i+1}],\) for \(i = 1, \ldots, n - 1\) and has a finite right hand limit for each \(t_i, \ i = 1, \ldots, n.\) This definition follows from Halkin 1974.

\(^{21}\) The characterization of Proposition 22 is also valid when \(f\) is decreasing in effort. That is, what we really require is \(f\) to be strictly monotone in effort.
(ii) For every \( t \geq 0 \) such that \( 0 < a_t^* < \ell \), \( a_t^* \) satisfies

\[
g'(a_t^*) = \beta_t \lambda_t + q_t \frac{\partial f}{\partial a}(t, \overline{\theta}_t(a^*), a_t^*)
\]

(40)

where \( \overline{\theta}_t(a^*) \) is the solution to \( d \overline{\theta}_t = f(t, \overline{\theta}_t, a_t^*) dt \), \( \overline{\theta}_0 = \overline{\theta} \geq 0 \).

Proof: By the Pontryagin Maximum Principle for infinite horizon (Halkin 1974), if \( a^* := (a_t^*)_{t \geq 0} \) is an optimal control then there exists \( \mu \geq 0 \) and a piecewise continuously differentiable function \( q: \mathbb{R}_+ \to \mathbb{R} \) s.t.

I. \( |(\mu, q_0)| \neq 0 \);

II. \( \dot{q}_t - r q_t = -\frac{\partial}{\partial x} \mathcal{H}(t, x, a_t^*, \mu, q_t)|_{x = \overline{\theta}_t}, \text{ a.s.} \)

III. \( \mathcal{H}(t, \overline{\theta}_t, a_t^*, \mu, q_t) \geq \mathcal{H}(t, \overline{\theta}_t, a, \mu, q_t) \), for all \( t \geq 0, a \in A \).

where the Hamiltonian \( \mathcal{H} \) is defined by

\[ \mathcal{H}(t, x, a, \mu, y) := \mu [a \beta_t \lambda_t - g(a) + \rho_t x] + y f(t, x, a). \]

with \( \rho_t, \beta_t \) and \( \lambda_t \) as in the Proposition.

Now I will prove that, under the hypothesis of the proposition, \( \mu \neq 0 \). Replacing the expression for the Hamiltonian in II yields the ODE

\[
\dot{q}_t = q_t \left[ r - \frac{\partial f}{\partial \theta}(t, \overline{\theta}_t(a^*), a_t^*) \right] - \mu \rho_t
\]

(41)

Recall that the set of times where the last ODE does not hold is the set of points at which \( a^* \) is discontinuous (at those points \( q \) is not differentiable). By definition of piece-wise continuity, for any \( T > 0 \) there is only a finite number of times less than \( T \) at which the optimal control is discontinuous. Therefore, II holds for intervals \([0, t_1], \{(t_i, t_{i+1}] \mid i \in \mathbb{N}\}\) such that their union is the real line. Moreover, since \( q \) is continuous, it must that the solution of the above ODE at any subinterval \((t_i, t_{i+1}]\) must satisfy

\[ q_{t_i^+} = q_{t_i} \]

where we understand that \( q_{t_i^+} \) is the limit as \( t \) decreases to \( t_i \) of the solution to the ODE (41) in \((t_i, t_{i+1}]\) with final condition \( q_{t_{i+1}}, i \geq 1 \). The proof is based on the following

**Lemma 23.** If \( \mu = 0 \), then either \( a^* \equiv 0 \) or \( a^* \equiv \overline{a} \).

\(^{22}\)That is, a differentiable function which derivative is piecewise continuous
Proof of the Lemma:

Suppose $\mu = 0$. Then it must be that the following relationship holds for $t \in [0, t_1^*]$: 

$$q_t = q_0 \exp \left( \int_0^t \left[ r - \frac{\partial f}{\partial \theta}(s, \bar{\theta}_s(a^*), a_s^*) \right] ds \right)$$

where $s \mapsto \bar{\theta}_s(a^*)$ is the trajectory generated by $(a_s^*)_{s \in [0, t_1^*]}$. If $q_{t_1^*} = 0$, then $q_0 = 0$, contradicting I. Thus, $q_{t_1^*} \neq 0$ and therefore $q$ cannot vanish in $[0, t_1^*]$. Suppose that $q > 0$ over this set. Then, the maximum condition II implies that the optimal control must satisfy

$$a_s^* \in \arg \max_{a \in A} q_s f(\bar{\theta}_s(a^*), a), \forall s \in [0, t_1^*]$$

But $q_s f(\bar{\theta}_s(a^*), \cdot)$ is increasing, and thus $a_s^* \equiv \bar{a}$ for all $s \in [0, t_1^*]$. As a consequence, whenever $\mu = 0$, if $q_{t_1^*} > 0$ then the optimal control takes the maximum possible value in the first interval. If in turn, $q_{t_1^*} < 0$ the same reasoning shows that the optimal control will take the minimum value over the same set, this because $q_s f(s, \bar{\theta}_s^*, \cdot)$ would be decreasing for all $s \in [0, t_1^*]$. In the remainder of the proof, I assume without loss of generality that $q_{t_1^*} > 0$ (the other case is analogous). If $q_{t_2} \leq 0$, then (41) in $(t_1, t_2]$ implies that $q_s \leq 0$ in the same interval. Therefore 

$$q_{t_1^*} := \lim_{s \searrow t_1^*} q_s \leq 0 < q_{t_1^*}$$

contradicting the fact that $q$ is continuous. Hence, $q_{t_2} > 0$ implying that $q$ is strictly positive in $(t_1, t_2]$ and thus the optimal control must take value $\bar{a}$ over that interval. Proceeding inductively, if $q_{t_i^*} > 0$ then $q_{t_i} > 0$ for all $i = 0, 2, 3...$ and by the maximum condition $\bar{a}$ is the optimal control. The same reasoning allows us to conclude that when $q_{t_1^*} < 0$, $a^* \equiv 0$ must be optimal. This concludes the proof.

The previous Lemma shows that when an optimal control exists and is neither identically zero nor equal to $\bar{a}$, then $\mu > 0$. When this is the case it is clear that we can assume $\mu = 1$ (equivalently, redefine $q$ as $q/\mu$ and note that $q/\mu$ satisfies all the conditions of the theorem). This proves part (i) in the proposition. Finally part (ii) is simply the necessary condition that an unconstrained optimum must satisfy. This concludes the proof.

The next result states that differentiability of the continuation-value function with respect to the state variable $\bar{\theta}$ ensures that effort is always above the career concerns benchmark. For this purpose define

$$V[t, \bar{\theta}] \equiv \sup_{a \in A} \int_t^\infty e^{-rs} \left[ a_s \beta_s \lambda_s - g(a_s) + \rho_s \bar{\theta}_s^t(a) \right] ds \quad (42)$$

where $(\bar{\theta}_s^t(a))_{s \geq t}$ is the solution of the ODE (37). We have the following
Proposition 24. Fix $t \geq 0$ and suppose $\bar{\theta} \in \mathbb{R}$ is such that $V(t, \bar{\theta}) < +\infty$. Let $(a^*_s)_{s \geq t}$ be the continuation strategy that attains this value. Then, the continuation-value function is increasing in the state variable. Moreover, if $V(t, \cdot)$ is differentiable in a neighborhood $\Theta$ of $\bar{\theta}$, then for any $s \geq t$ such that $\bar{\theta}_s(a) \in \Theta$, $q_s = e^{-rs} \frac{\partial V}{\partial \theta}(s, \bar{\theta}_s(a^*)) > 0$.

Proof of Proposition 24: If $V(t, \bar{\theta}^2) = \infty$ the first part of the Proposition is trivially true. Suppose that is finite. Let $a^* := (a^*_s)_{s \geq t}$ be the optimal control (a function of time) that attains value $V(t, \bar{\theta}^1)$ when starting from the level $\bar{\theta}^1 \geq 0$ at time $t$. In the same vein let $(\bar{\theta}_s(a^*; \bar{\theta}))_{s \geq t}$ denote the path of human capital generated by the feasible control $a^*$ when starting from point $\bar{\theta} \in \mathbb{R}_+$ at time $t$, that is, the solution to

$$d\bar{\theta}_s = f(s, \bar{\theta}_s, a^*_s), \; s > t, \; \bar{\theta}_t = \bar{\theta}$$

Since the solutions of these two ordinary differential equations cannot cross (they differ only in the initial condition), it must be the that

$$\bar{\theta}_s(a^*; \bar{\theta}^2) > \bar{\theta}_s(a^*; \bar{\theta}^1), \; \forall s \geq t$$

which implies that $V[t, \bar{\theta}^2; a^*] > V[t, \bar{\theta}^1; a^*] = V(t, \bar{\theta}^1)$, so $V(t, \cdot)$ is increasing. If, moreover, it turns out to be differentiable in a neighborhood $\Theta$ of $\bar{\theta}^1$ and standard perturbation analysis shows that $\frac{\partial V}{\partial \theta}(s, \bar{\theta}_s(a^*)) = \lambda_s$, as long as $\bar{\theta}_s(a^*) \in \Theta, s \geq t$, where $\lambda_s$ is the multiplier associated to the dynamic that governs the ODE of human capital accumulation. Implicit in the formulation of Proposition 22 is the fact that $q_t = e^{rt} \lambda_t, t \geq 0$. This concludes the proof.

$\square$