Signaling with Private Monitoring*

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Preliminary

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Abstract

We examine linear-quadratic signaling games between a long-run player that has a
normally distributed type and a myopic player who privately observes a noisy signal
of the long-run player’s actions. An imperfect signal of the myopic player’s behavior
is publicly observed, and thus there is two-sided signaling. Time is continuous over a
finite horizon, and the noise is Brownian. We construct linear-Markov equilibria using
the players’ beliefs up to the second order as states. In such equilibria, the long-run
player’s second-order belief is controlled, reflecting that past actions are used to fore-
cast the continuation game. Via this higher-order belief channel, the informational
content of the long-run player’s action is not only driven by the weight attached to her
type, but also by how aggressively she has signaled in the past. Applications to models
of leadership, reputation, and trading are examined.

Keywords: signaling; private monitoring; private beliefs; learning; Brownian motion.

JEL codes: C73, D82, D83.

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1 Introduction

Signaling—that is, the strategic transmission of private information through actions—plays a central role in many economic interactions. In labor markets, for instance, workers’ educational choices can credibly convey information about their hidden abilities (Spence, 1973). In financial markets, signaling influences how quickly market prices incorporate the information privately possessed by traders with market power (Kyle, 1985). In organizations, it offers a productivity-enhancing rationale for leadership by example (Hermalin, 1998).

In all these settings, “receivers” must extract the informational content of the signals generated by those who hold superior information: employers must interpret a college degree as a signal of ability; investors must interpret financial data to learn about others’ information; followers must interpret leaders’ actions to decide how to follow, etc. In most studies of signaling so far, however, the signals involved are either perfect and/or public, or private but restricted to static settings, thereby making any such an analysis public: at the moment they have to move, senders know their receivers’ beliefs. Crucially, while such public analyses can be appropriate in some settings (e.g., public academic credentials can be highly informative signals), they are less so in others.

Indeed, imperfect private signals are pervasive: they naturally appear when employers subjectively assess workers’ performances (MacLeod, 2003; Levin, 2003); when brokers handle orders of retail investors (Yang and Zhu, 2018); or when data brokers collect data from consumers’ online behavior (Bonatti and Cisternas, 2019), for instance. The presence of idiosyncratic noise then renders the inferences made by receivers as private, raising a fundamental question: how do learning and signaling in repeated interactions play out when those who hold payoff-relevant information do not know what others have seen?

In this paper, we make progress in this direction by examining a player’s signaling incentives in settings where her actions generate signals that are hidden to her. Specifically, a long-run player (she) of a normally distributed type interacts with a myopic player (he) over a finite horizon. The myopic player privately observes a noisy signal of the long-run player’s actions, while the long-run player can learn about the myopic player’s private inferences from an imperfect public signal of the myopic player’s behavior. The players’ preferences are linear-quadratic and the noise is Brownian. Using continuous-time methods, we construct linear Markov perfect equilibria (LME) using the players’ beliefs as states.

The games we study feature one-sided incomplete information and imperfect private monitoring. Consider a leader of an organization interacting with a follower (or many of them, acting in coordination). The organization’s payoff increases with both the proximity of the leader’s action to a newly realized state of the world (adaptation) and the proximity of
the follower’s actions to the leader’s action (coordination). Moreover, the follower attempts to match the leader’s action at all times. The environment is, however, complex, in the sense that the leader cannot immediately convey the state of the world to the follower: the latter learns it only gradually by subjectively evaluating the leader’s actions. In turn, the leader receives feedback of the follower’s inferences through a public signal of the follower’s actions.

Due to the private monitoring, the follower’s belief is private, and hence both parties have private information to signal to one other. In addition, the leader is forced to use her past actions to estimate the follower’s belief. This forecast—the leader’s second-order belief—is itself private, even along the path of play, as the leader conditions her actions on the state of the world. How does the leader then manage the transition of the organization to the new state of the world accounting for this higher-order uncertainty? What are the implications on learning and payoffs, and hence on the value of better information channels which reduce higher-order uncertainty?

**Economic forces.** We construct LME using beliefs up to the long-run player’s second order as states. This second-order belief is controlled by the long-run player, reflecting that she uses past play to forecast the continuation game. The well-known problem of the state space growing due to the myopic player attempting to forecast such private belief is then circumvented by a key *representation lemma* (Lemma 2) that expresses the (candidate, on path) second-order belief as a convex combination of the long-run player’s type and the belief about that type based on public information exclusively. This representation reflects how the long-run player calibrates her belief using the public information when learning about the myopic player’s belief. This “public” state is therefore part of the set of belief states, and is affected by the myopic player.

Because different types take different actions in equilibrium, and actions are used to forecast the myopic player’s belief, different types also perceive different continuation games as measured by their second-order beliefs. This creates a novel *history-inference effect*, whereby the sensitivity of the long-run player’s action to her type—which determines the myopic player’s learning—is comprised not only of the direct weight her strategy places on her type, but also by how aggressively she has signaled in the past via the myopic player’s inference of the long-run player’s private history. This effect compounds over time as the second-order belief reflects more the long-run player’s type, and its amplitude is decreasing in the quality of the public signal: shutting down the public signal (*no feedback* case) maximizes the potential reliance on the type, while making the public signal noiseless (*perfect feedback*) eliminates this dependence. These extreme cases are exploited in the applications we study.
Applications. In Section 2, we illustrate the main economic insights of the paper by examining a game in which a leader must adapt an organization to a new economic environment while controlling the coordination costs with a myopic follower who tries to match her action. To accommodate to the follower, the leader’s action is less sensitive to her type (i.e., achieves less adaptation) than in the full-information benchmark; but successful accommodation requires knowing the follower’s belief. Critically, because higher types take higher actions due to their stronger adaptation motives, they also expect their followers to have higher beliefs—the coordination motive leads higher types to take higher actions via the history-inference channel. In the absence of feedback, therefore, standard decreasing adaptation incentives that fully determine signaling and learning when beliefs are public are then offset by higher-order belief effects that make the leader’s signaling increasing over time.

This qualitatively different signaling behavior has important consequences on learning and payoffs, and hence on the value of better information channels within the organization. In the extreme cases of a myopic and a patient leader, we show that the follower’s overall learning from the interaction is always higher when the leader receives no feedback than when the public signal is perfect, a consequence of stronger overall adaptation in the former case. Learning is, however, a measure of the organization’s struggle: learning occurs only when the private signal is informative of the state of the world, and hence only when miscoordination occurs along the way. The stronger signaling that arises in the no-feedback case is then more decisive when the leader is impatient. Specifically, the history-inference effect substitutes for the lack of adaptation of a myopic leader, thereby reducing the added value of a noiseless public signal relative to the no-feedback case as the degree of impatience increases.

In Section 5 we explore two applications based on extensions of our model. In the first, the type is a bias, and the long-run player wants to preserve a reputation for neutrality, modeled via a terminal quadratic loss in the myopic player’s belief (e.g., a politician facing reelection). Clearly, eliminating the public signal has a negative direct effect on the long-run player’s payoff (increased uncertainty in a concave objective). Since higher types take higher actions due to their larger biases, however, those types must offset higher beliefs to appear unbiased; the history-inference effect is then negative, which weakens signaling and hence the sensitivity of the myopic player’s belief, potentially leading to higher payoffs.

Finally, we exploit the presence of public belief state in a trading model in which an informed trader faces both a myopic trader that privately monitors her orders and a competitive market maker who only observes the public total order flow. In this context, we show that there is no linear Markov equilibrium for any degree of noise of the private signal. Intuitively, the myopic player introduces momentum into the price process, as the information he obtains now gets distributed to the market maker through all future order flows.
This causes prices to move against the insider and creates urgency, leading the insider to trade away all information in the first instant.

**Technical contribution.** The setting we examine is asymmetric, in terms of the players’ preferences and their private information (a fixed state versus a changing one). In particular, the players can signal at substantially different rates, which is in stark contrast to a small literature on symmetric multi-sided learning (see the literature review section). With different rates of learning, however, the equilibrium analysis can become severely complicated.

Specifically, the belief states we construct depend on two functions of time: (1) the myopic player’s posterior variance, which determines the sensitivity of the myopic player’s belief to her private signal, and (2) the weight attached to the long-run player’s type in the representation result, which captures the contribution of the history-inference effect to the long-run player’s signaling. Standard dynamic-programming arguments reduce the problem of existence of LME to a boundary value problem (BVP) that these two functions, along with the weights in the long-run player’s linear strategy, must satisfy. The two learning ordinary differential equations (ODEs) endow the BVP with exogenous initial conditions, while the rest carry endogenous terminal conditions arising from myopic play at the end of the game.

Determining the existence of a solution to such a BVP is challenging because it involves multiple ODEs in both directions. For this reason, we establish two sets of results. In a private value environment, the myopic player’s best response does not directly depend on his belief about the long-run player’s type, but only indirectly via his expectation of the latter player’s action. In that context, we show that there is a one-to-one mapping between the solutions to the learning ODEs (Lemma 5), a consequence of the ratio of the signaling coefficients being constant. This, in turn, makes traditional shooting methods based on the continuity of the solutions applicable. Via this method, we show the existence of LME in the leadership model of Section 2 when the public signal is of intermediate quality for times that are decreasing in the variance of the prior about the state of the world (Theorem 1).

In common value settings, the multidimensionality issue seems unavoidable. Building on the literature on BVPs with intertemporal linear constraints (Keller, 1968), however, we can show the existence of LME via the use of fixed-point arguments applied to our BVP with intratemporal nonlinear (terminal) constraints. Specifically, the multidimensional shooting problem can be reformulated as one of the existence of a fixed point for a suitable function derived from the BVP, which we then tackle for a variation of the leadership model when the follower also cares about matching the state of the world (Theorem 2). Critically, the method is general—it applies to the whole class of games under study, and opens a way for examining other settings exhibiting learning and asymmetries.
Related Literature  Static noisy signaling was introduced by Matthews and Mirman (1983) in a limit pricing context, and further studied by (Carlsson and Dasgupta, 1997) as a refinement tool. Recent dynamic analyses involving Gaussian noise and public beliefs include Dilmé (2019), Gryglewicz and Kolb (2019), Kolb (2019) and Heinsalu (2018).1

Multisided signaling has been examined by Foster and Viswanathan (1996) and Bonatti et al. (2017) in symmetric settings with imperfect public monitoring and dispersed fixed private information. In those settings, beliefs are private, but the presence of a commonly observed public signal permits a representation of first-order beliefs that eliminates the need for higher-order ones.2 Bonatti and Cisternas (2019) in turn examine two-sided signaling in a setting where firms privately observe a summary statistic of a consumer’s past behavior to price discriminate. Via the prices they set, however, firms perfectly reveal their information to the consumer.

The literature on repeated games with private monitoring is extensive, and has largely focused on non-Markovian incentives—Ely et al. (2005) and Hörner and Lovo (2009) (the latter allowing for incomplete information) study equilibria in which inferences of others’ private histories are not needed, and Mailath and Morris (2002) and Hörner and Olszewski (2006) study almost public information structures. Levin (2003) and Fuchs (2007) examine one-sided private monitoring in repeated principal-agent interactions.

Regarding our applications, the stage game of our leadership model is a simplified version of Dessein and Santos (2006).3 In turn, the value of public information has been studied by Morris and Shin (2002), Angeletos and Pavan (2007), and Amador and Weill (2012) in settings with infinitesimal players, thus rendering signaling and inferences of individual private histories are absent. Regarding trading models, Yang and Zhu (2018) show, in a richer two-period version of our model, that a linear equilibrium ceases to exist if a signal of an informed player’s last trade is too precise and privately observed by another player.

To conclude, this paper contributes to a growing literature employing continuous-time techniques to the analysis of dynamic incentives. Sannikov (2007) examines two-player games of imperfect public monitoring; Faingold and Sannikov (2011) reputation effects with behavioral types; Cisternas (2018) off-path private beliefs in games of ex ante symmetric uncertainty; and Horner and Lambert (2019) information design in career concerns settings.

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1This last paper also displays a normally distributed type, but it lacks strategic interdependence between actions. Thus, behavior is unchanged under some information structures involving private monitoring.

2Likewise in He and Wang (1995), where infinitely many agents privately see dynamic exogenous signals.

3See Bolton and Dewatripont (2013) for such a static analysis with one round of pre-play communication. More generally, these are instances of the linear-quadratic team theory of Marschak and Radner (1972).
2 Application: Leading Coordination and Adaptation

Economists have long understood that adaptation to changes in the external economic environment is a key problem for organizations (e.g., Simon, 1951), and that successful adaptation often requires substantial coordination of activities within them.\(^4\) Since at least Radner (1962), however, it has been recognized that coordination is threatened by the presence of different decision-makers and sources of information. As Williamson (1996) further points out, “failures of coordination can arise because autonomous parties read and react to signals differently, even though their purpose is to achieve a timely and compatible combined response.” Private signals—a consequence of either private sources or subjective interpretations—therefore play an integral role in organizations’ ability to adapt to change.

The study of the adaptation-coordination trade-off has lead to important insights regarding the returns to specialization (Dessein and Santos, 2006), centralization (Alonso et al., 2008), and governance structures (Rantakari, 2008), but the great majority of these analyses have been static. In particular, the central question of how information about the economic environment is gradually transmitted and reflected in decision-making has been much less explored.\(^5\) The difficulty in analyzing learning dynamics in organizations while accounting for idiosyncratic shocks and/or private information is apparent: to appropriately coordinate their actions, individuals must forecast what other members know.

In this application, we examine how a leader—a member of an organization with crucial information about the economic environment and the opportunity to influence others—manages the dynamics of adaptation and coordination when a follower privately monitors her actions. For instance, top management wishes to adapt its strategy to a shift in the market fundamentals, but it suffers from imperfect control: the information about such changes trickles down the organization through various layers before reaching key productive divisions. Or consider an expert who leads by example to transmit a technique or skill—an intangible activity or knowledge—to an apprentice, and the latter subjectively evaluates the expert’s actions. In both cases, the economic ‘fundamentals’ are learned only gradually by the receiver, and the sender does not directly observe what the receiver has seen.

Specifically, consider the following game inspired by the team theory of Marschak and

\(^4\)The literature on the topic is extensive. Refer, for instance, to Chapter 4.2 in Williamson (1996) and Chapter 4 in Milgrom and Roberts (1992).

\(^5\)Marschak (1955), in discussing team theory as a framework for examining organizations, makes the case clear (p. 137): “A realistic theory of teams would be dynamic. It takes time to process and pass messages along a chain of team members; and messages must include not only information on external variables but also information on what has been done by [others...]. Knowledge about [probabilities and payoffs] is acquired gradually, while the team already proceeds with decisions. These facts make the dynamic team problem similar to those in cybernetics and in sequential statistical analysis.”
Radner (1972). A team consisting of a leader (she) and a follower (he) operates in an environment parametrized by a state of the world $\theta \sim \mathcal{N}(\mu, \gamma^o)$. The team’s payoff is

$$\int_0^T e^{-rt} \{-(a_t - \theta)^2 - (a_t - \hat{a}_t)\} dt,$$

where $a_t$ denotes the leader’s action at time $t$, $\hat{a}_t$ the follower’s counterpart, $r \geq 0$ is a discount rate, and $T < \infty$. Thus, performance increases with the proximity of the leader’s action to the state of the world (adaptation) and with the proximity of both players’ actions (coordination). Such actions can take values over the whole real line.

We depart from Marschak’s and Radner’s approach to modeling teams by allowing divergence in preferences. Specifically, we assume that the leader’s preferences coincide with the team’s payoff, while the follower is myopic trying to minimize $(a_t - \hat{a}_t)^2$ at all times $t \in [0, T]$. The leader knows the realized value of $\theta$, while the follower only knows its distribution. As time progresses, however, the follower privately observes an imperfect signal of the form

$$dY_t = a_t dt + \sigma_Y dZ_t^Y,$$

where $Z^Y$ is a Brownian motion and $\sigma_Y > 0$ a volatility parameter. In particular, immediate adaptation at no coordination costs via a perfectly revealing action is not possible.

In turn, the leader can learn about what the follower has done (and, hence, about what the follower has seen) from

$$dX_t = \hat{a}_t dt + \sigma_X dZ_t^X,$$

where $Z^X$ is independent of $Z^Y$. This signal is public; for instance, an output measure observed by both parties, or information prepared by the follower.

In this context, our goal is twofold. First, to understand how private monitoring affects the players’ behavior. Second, to assess the value of better information channels for the team, which is an issue of central importance for the performance of organizations. To achieve both objectives in a unified way, we thus fix $\sigma_Y > 0$ and focus on the cases of $\sigma_X = 0$ and $+\infty$.

Specifically, to understand how the presence of a noisy private signal affects the players’ behavior, it is useful to examine the benchmark in which $Y$—and hence, the follower’s belief—is public; an indirect approach for studying this case is by setting $\sigma_X = 0$, to the extent that the follower’s action reveals his belief at all times. On the other hand, to assess the value of better bottom-up information systems (as measured by lower values of $\sigma_X$), it is natural to consider the baseline case in which $X$ is absent, which is equivalent to setting $\sigma_X = \infty$. Reductions in $\sigma_X$ have the appealing interpretation of being the result of interventions intended to improve the information that leader’s receive from within the
organization (which can be important if leaders are busy with other activities). As it turns out, the extremes cases of $\sigma_X$ to be discussed deliver the sharpest economic intuitions.

2.1 Perfect Feedback (“Public”) Case: $\sigma_X = 0$

When the public signal has no noise, the observation of the follower’s action opens the possibility of the leader perfectly inferring the follower’s belief. Thus, we aim to characterize a linear Markov equilibrium (LME) of the form

$$a_t = \beta_0 t + \beta_1 \hat{M}_t + \beta_3 \theta \quad \text{and} \quad \hat{a}_t = \hat{E}_t[a_t] = \beta_0 t + (\beta_1 + \beta_3) \hat{M}_t$$

(2)

where $\hat{M}_t := \hat{E}_t[\theta]$, and $\beta_i$, $i = 0, 1, 3$ are functions of time satisfying $\beta_1 + \beta_3 \neq 0$, $t \in [0, T]$. The deterministic feature of the candidate equilibrium coefficients is explained shortly; the last condition permits identifying the follower’s belief from his observed action.

From standard results in filtering theory, if the follower expects $(a_t)_{t \geq 0}$ as in (2), then

$$d\hat{M}_t = \frac{\beta_3 \gamma_t}{\sigma^2_Y} [dY_t - \{\beta_0 t + (\beta_1 + \beta_3) \hat{M}_t\} dt], \quad \text{where} \quad \gamma_t = -\left(\frac{\gamma_t \beta_3}{\sigma^2_Y}\right)^2.$$ 

(3)

That is, the follower updates upwards whenever the observed increment, $dY_t$, is larger than the follower’s expectation of it, $\hat{E}_t[dY_t] = [\beta_0 t + (\beta_1 + \beta_3) \hat{M}_t] dt$. Moreover, the intensity of the reaction is larger the more aggressively the leader signals the state (i.e., a larger $\beta_3$), and the less is known about the latter (i.e., a larger $\gamma_t$). Finally, learning is deterministic due to the Gaussian structure, and is faster the stronger the intensity of the leader’s signaling.

The leader’s problem is to maximize (1) subject to (3), recognizing that she affects $\hat{M}$ via $dY_t = a_t dt + \sigma_Y dZ^Y_t$. The next result establishes the existence of a LME, along with some properties that any such an equilibrium should satisfy:

**Proposition 1** (LME—Public Case). For all $r \geq 0$ and $T > 0$:

(i) Existence: there exists a LME. In any such equilibrium $a_t = \beta_3 \theta + (1 - \beta_3) \hat{M}_t$.

(ii) Signaling coefficient: $\beta_3 \in (1/2, 1)$ for $t < T$, $\beta_3 T = 1/2$, and $\beta_3$ is strictly decreasing.

Recall that the full information benchmark is simply $a_t = \hat{a}_t = \theta$ at all times. From this perspective, the leader sacrifices adaptation (i.e., $\beta_3 < 1$) to be able to coordinate with the

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6We skip subindex 2 to be consistent with the general model presented in Section 3, where we complete the notion of linear Markov equilibrium with an additional state variable.

7More precisely, a LME as in (2) is perfect when $Y$ is public, but only Nash when $Y$ is private but $\sigma_X = 0$, as the continuity of the paths of $\hat{M}$ makes deviations by the myopic player observable in this latter case. Due to the full-support noise, however, this distinction is vacuous in discrete time.
follower; in particular, the coefficient on $\hat{M}$, $\beta_1 = 1 - \beta_3$, must be positive, as higher values of $\hat{M}$ then require higher actions by the leader.\footnote{That $\beta_0t = 0$ and $\beta_1t + \beta_3t = 1$ hold at all times in this dynamic setting can be understood from the leader’s incentives at $M_t = \theta$: in this case, there is no coordination loss ($a_t = \hat{a}_t$), and the objective then becomes, locally, one of minimizing the adaptation cost; but since $\mathbb{E}_t[dM_t] = 0$ (i.e., $\hat{M}$ is locally unpredictable), there are no incentives to move away from $a_t = \theta$. Thus, $\beta_0 + (\beta_1 + \beta_3)\theta = \theta$ for all types.}

The leader’s incentives to sacrifice adaptation are weaker the farther the team is from the end of the interaction. In fact, in equilibrium, $d\hat{M}_t \propto \gamma_t \beta_3 t [\theta - \hat{M}_t]$ at all times, and so stronger adaptation today brings—via signaling—more coordination tomorrow. This dynamic incentive decays (deterministically) because there is less time to enjoy future coordination and beliefs are less responsive as learning progresses. The terminal value simply reflects that static equilibrium behavior $(a, \hat{a}) = (\frac{1}{2} \theta + \frac{1}{2} \hat{M}, \hat{M})$ arises at the end of the interaction.

### 2.2 No Feedback Case: $\sigma_X = \infty$

When the public signal is uninformative, the leader must perform a non-trivial inference of the follower’s belief to correctly assess how her actions affect future payoffs (i.e., to correctly assess the continuation game). This is, in turn, an exercise of inference of private histories.

**Forecasting by input.** In the public case the leader’s past actions were immaterial for inferring the follower’s contemporaneous belief, as the latter was fully determined by the realized history $Y^t$—i.e., the leader forecasted by output. In fact, since (3) is linear, we have

$$\hat{M}_t = A_1(t) + \int_0^t A_2(t, s) dY_s,$$

for some deterministic functions $A_1$ and $A_2$. The ability to observe $Y^t$ implies that the leader always computes her forecast as above, with larger effort profiles only indicating that the corresponding shocks $Z^Y$ were lower, and vice-versa.

In the absence of feedback, the leader does not observe $\hat{M}$. Thus, her second-order belief, $M_t := \mathbb{E}_t[\hat{M}_t]$, is a payoff-relevant state. Further, as long as $\hat{M}$ is as above (for potentially different functions $A_1$ and $A_2$), and using that $\mathbb{E}_t[dY_t] = a_t dt$, the leader’s forecast reads

$$M_t = A_1(t) + \int_0^t A_2(t, s) a_s ds.$$  \hfill (4)

Unlike the public case, therefore, the leader now forecasts by input: the more effort she has exerted in, say, pushing the follower’s belief upwards towards $\theta$ from a low prior, the higher she thinks the current value of $\hat{M}$ is.
This contrast between the public and no-feedback cases is natural: in the absence of any additional information, an expert must rely on how much emphasis she has given to a particular idea or technique to assess how much the apprentice has assimilated the latter. (By contrast, in the public case, the apprentice’s output signal suffices for perfectly inferring his understanding of the topic.) This dependence of $M_t$ on the past history of play in the no-feedback case reflects the well-known idea that, in games of private monitoring, players must rely on their past behavior to forecast others’ private histories; yet, it has important effects on equilibrium outcomes.

**Representation of second-order belief and history-inference effect.** Observe that $M$ is hidden to the follower: off the path of play, because deviations go undetected; and in equilibrium, because the leader’s action carries her type. Along the path of play of a linear strategy, however, one would expect a linear relationship between $\theta$ and $M$, as the linearity of (4) suggests. When this is the case, the follower’s (third-order) inference of $M$ would then reduce to a function of $\hat{M}$, and the system of beliefs would “close.”

To this end, suppose that the follower expects that, in equilibrium, $M$ satisfies

$$M_t = \left(1 - \frac{\gamma_t}{\gamma^0}\right) \theta + \frac{\gamma_t}{\gamma^0} \mu \tag{5}$$

when the leader follows a strategy

$$a_t = \beta_0 t \mu + \beta_1 t M_t + \beta_3 t \theta, \tag{6}$$

for some deterministic coefficients $\beta_{it}$, $i = 0, 1, 3$ (potentially different from those in the public case). The representation (5) encodes two ideas. First, there is no second-order uncertainty at time zero, i.e., $M_0 = \mu = \hat{M}_0$; this is obtained by setting $\gamma_0 = \gamma^0$ in the right-hand side of (5). Second, if enough signaling has taken place, the leader would expect the follower to have learned the state: $\gamma_t \approx 0$ in the same expression leads to $M_t \approx \theta$.

How is the follower’s learning, $\gamma_t$, now determined? To simplify notation, let us use

$$\chi := 1 - \frac{\gamma}{\gamma^0}$$

to denote the weight on the type in (5). Inserting this into (6) yields $a_t = [\beta_0 t + \beta_1 t (1 - \chi_t)] \mu + [\beta_3 t + \beta_1 t \chi_t] \theta$, and so the new signaling coefficient is no longer given by the weight that the equilibrium strategy directly attaches to the type, $\beta_3$, but instead by

$$\alpha \equiv \beta_3 + \beta_1 \chi.$$
We refer to $\beta_1 \chi$ as the history-inference effect on signaling. In fact, because the leader uses her actions to forecast $\hat{M}$, the follower needs to infer the leader’s private histories to extract the correct informational content of the signal $Y$. However, since higher types take higher actions due to their static adaptation and future coordination motives—forces that fully determine behavior in the public case—those types also expect their followers to have higher beliefs. Alternatively, given a history $Y^t$, consider the impact that a marginal increase in $\theta$ has on the leader’s action: in the public case, the overall effect is $\beta_3$, as all types agree on the value that $\hat{M}$ takes; this is not the case when there is no feedback, as different types perceive different continuation games via $M$. We collect these ideas in the next result.

**Lemma 1** (Belief Representation). Suppose that the follower expects $a_t = [\beta_0 + \beta_1(1-\chi_t)]\mu + \alpha_t \theta$, where $\alpha = \beta_3 + \beta_1 \chi$, $\chi = 1 - \gamma^o$, and $\gamma_t := \hat{E}_t[(\theta_t - \hat{M}_t)^2]$. Then, $\dot{\gamma} = -\left(\frac{\gamma \alpha_t}{\sigma^2_Y}\right)^2$. Moreover, if the leader follows (6), $M_t = \chi_t \theta + (1-\chi_t)\mu$ holds at all times.

The representation of the second-order belief (5) holds only under the linear strategy (6). More generally, the leader controls $M$ as reflected by (4), and thus $(\theta, M, \mu, t)$ effectively summarizes all the payoff-relevant information for the leader’s decision-making.

**Proposition 2** (LME—No Feedback Case). For all $r \geq 0$ and $T > 0$:

(i) Existence: there exists a LME. In any such an equilibrium: $\beta_0 + \beta_1 + \beta_3 = 1$; $\beta_3 > 1/2$, $t \in [0, T]$; $\beta_3 T = 1/2$; and $\beta_1 > 0$ over $[0, T]$.

(ii) Signaling coefficient: $\alpha > 1/2$; $\alpha_T \rightarrow 1$ as $T \rightarrow \infty$, and $\alpha'_t \geq 0$, $t \in [0, T)$, with strict inequality if and only if $r > 0$.

Thus, in the no-feedback case, the signaling coefficient $\alpha$ has a radically different behavior relative to the public case counterpart: it is non-decreasing, and its right-end point becomes asymptotically close to 1 as the length of the interaction increases. See Figure 1.

![Figure 1](image-url)

Figure 1: Left $r = 0$; Right: $r = 1$. Other parameter values: $\gamma^o = 1$, $\sigma^2_Y = 1.5$, $T = 10$. 

12
The reason behind the discrepancy lies in the history-inference effect compounding over time. In fact, since the leader expects the follower to gradually learn the state as signaling progresses, $M$ attaches an increasingly higher weight $\chi$ to $\theta$ in (5). With a positive coordination motive ($\beta_1 > 0$), this implies that higher types take higher actions over time via this second-order belief channel. We conclude that private monitoring generates an interesting phenomenon whereby standard monotonically decreasing signaling effects under public beliefs are more than offset by an increasingly strong informational content in the leader’s past history of play (except for the $r = 0$ case, where both forces perfectly offset each other).

2.3 Learning, Coordination, and the Value of Public Information

The fact that the leader has to rely on her private information to coordinate with the follower, and that this force reinforces the direct signaling effect coming from $\beta_3$, opens the possibility for more information to be transmitted in the no-feedback case. At least, $\alpha_T > \beta_{3T}^{Pub} = 1/2$ for all $T > 0$, so more signaling indeed takes place by the end of the game.

To assess the validity of this conjecture, we take advantage of the model’s analytic solutions in the patient ($r = 0$) and myopic ($r = \infty$) cases. Let $\gamma_{Pub}$ and $\gamma^{NF}$ denote the follower’s posterior variance in the public and no-feedback case, respectively.

**Proposition 3** (Learning comparison). For every $T > 0$:

(i) Patient case: if $r = 0$, $\beta_{30}^{Pub} > \alpha_0$ and $\gamma_{T}^{Pub} > \gamma_{T}^{NF}$;

(ii) Large $r$ case: for every $\delta \in (0, T)$, $\gamma_t^{Pub} > \gamma_t^{NF}$ for $t \in [T - \delta, T]$ if $r$ large enough.

![Figure 2: Terminal values of $\gamma_{Pub}$ and $\gamma^{NF}$, Parameter values: $\gamma^o = \sigma_Y = 1$, and $T = 4$.](image)

Consequently, when the leader is either patient or very impatient, in the no-feedback case the follower always has a more precise knowledge of the state of the world by the end of the interaction. In this line, part (i) says that, if the leader is patient, this result is non-trivial due to an inter-temporal substitution effect: the leader, anticipating that the history-inference
effect will eventually take place, decides to reduce $\alpha_0 = \beta^{NF}_{30}$ below the public counterpart, $\beta^{Pub}_{30}$. Part (ii) then states that the fraction of time over which the follower has a more accurate belief can converge to 1 as $r$ grows large. Figure 2 shows, albeit numerically, that learning is higher in the no-feedback case for intermediate values of $r$.

One may feel tempted to conjecture that the organization can be better off by isolating the leader from any information about the follower, as this fosters the latter’s learning. The caveat is that information transmission happens through actions: a more precise belief that is the result of more aggressive signaling is necessarily the reflection of more transient miscoordination, as learning occurs only when $Y$ is informative about the state of the world.

**Proposition 4** (Team’s ex ante payoffs—public vs. no-feedback).

(i) Patient case: if $r = 0$, the team’s ex ante payoff is larger in public case for all $T > 0$.

(ii) Large $r$ case: there is $\bar{T} > 0$ such that, for all $T > \bar{T}$, the team’s ex ante flow payoffs are larger in the no feedback case over $[\bar{T}, T]$ for $r$ sufficiently large.

In the patient case, the team is unequivocally better off in the public case. In particular, one can show that ex ante coordination costs satisfy

$$0 < \mathbb{E}_\theta \int_0^T [\beta^{Pub}_{3t}(\theta - \hat{M}_t)]^2 dt = -\sigma^2_Y \log \left( \frac{\gamma^{Pub}_T}{\gamma^o} \right) < -\sigma^2_Y \log \left( \frac{\gamma^{NF}_T}{\gamma^o} \right) = \mathbb{E}_\theta \int_0^T [\alpha_t(\theta - \hat{M}_t)]^2 dt.$$

Thus, the extent of the follower’s learning is effectively a measure of the total coordination costs incurred by the team, which are larger in the no-feedback case. Consequently, an important takeaway of our analysis is that not accounting for the specific features of the information channels within firms, but instead just focusing on an outcome measure such terminal learning $\gamma_T$, can be very misleading in terms of assessing past (or even future) performance: a better understanding of the economic environment can in fact be the reflection of a painful struggle to coordinate actions. Alternatively, our analysis uncovers how affecting an information channel that does not feed a follower can affect his learning via the strategic response of other members in the organization.

We conclude our analysis by discussing the second part in the proposition, which allows us to begin talking about the value of better public information and its dependence on parameters of the model, such as the leader’s degree of patience.

Part (ii) states that, for sufficiently large horizons and discount rates, the organization’s continuation payoffs at a time $\bar{T}$ (independent of $r$) are ranked in favor of the no-feedback case. This is in fact the result of a stronger adaptation by the leader. To see why, observe

---

Note that $-\log(\gamma_T/\gamma^o)$ is the entropy reduction in the follower’s belief over $[0, T]$. 

---
first that a sufficiently long horizon is needed for the history-inference effect to gain strength. In this line, the first row in Figure 3 plots total ex ante coordination and adaptation losses when \( r = 0 \): the latter are lower in the no-feedback case for \( T \) beyond a threshold.

![Figure 3](image)

Figure 3: First row: total adaptation and coordination losses for \( T \in [0, 10] \) for \( r = 0 \). Second row: flow losses for \( T = 30, r \in \{0, 0.5, 1, +\infty\} \). Other parameters: \( \gamma^0 = \sigma_Y = 1 \).

The second row plots differences of ex ante coordination and adaptation flow losses between cases in a large horizon context. In particular, adaptation losses are eventually lower in the no-feedback case (right panel), and this is more pronounced as \( r \) grows. In fact, recall that a myopic leader attaches a weight of 1/2 to her type at all times in the public case. By contrast, in the no-feedback case, the history-inference effect continues to operate if the leader is myopic, allowing \( \alpha \) to become arbitrarily close to 1 as \( T \) lengthens. Thus, an impatient leader in the public case simply adapts too little after a threshold.

This analysis of discounting and time-horizon effects has two implications. First, from (ii), initializing the game with second-order uncertainty can improve upon a public counterpart with potentially higher uncertainty yet a perfect ability to coordinate; for instance, intensive one-sided communication by a leader, and subsequent leadership by example without feedback, can dominate two-sided communication followed by a perfect feedback channel.

Second, the additional value that a noiseless feedback channel brings to an organization relative to the no-feedback case is expected to fall with discounting due to the leader being weakly adapted to the environment when beliefs are public. See Figure 4:
We have examined how a leader gradually adapts a team to a new economic environment while controlling the team’s coordination costs, uncovering three sets of results. First, higher-order uncertainty arising from private monitoring radically affects the way in which a leader’s actions transmit her private information: the signaling coefficient is non-decreasing, which is in stark contrast with its public counterpart. Second, the history-inference effect driving the previous result is sufficiently strong to generate more learning on behalf of the follower—learning is however, a measure of the coordination costs incurred by the organization. Third, the value of interventions aimed at improving the information flow to leaders depends critically on both horizon effects and discounting: longer interactions combined with leader myopia reduce the value of noiseless information structures.

Critically, this example is just a first attempt at understanding organizations as dynamic enterprises, where decision makers can signal and learn information at the same time that decisions are being made. From this standpoint, it is important to recognize that public signals rarely are perfectly informative or pure noise. Away from those cases, we would expect informed parties’ forecasts to lie somewhere in between the two extreme cases just analyzed: i.e., to rely both on input and output measures. This is done in the next section, where the key will be to derive a generalization of the representation \( M_t = \chi_t \theta + (1 - \chi) \mu \) for \( 0 < \sigma_X \leq \infty \).

## 3 General Model

We consider two-player linear-quadratic-Gaussian games with one-sided private information and one-sided private monitoring in continuous time. The baseline model considered is introduced next, and extensions of it are presented in Section 5 via two further applications.

**Players, Actions and Payoffs.** A forward looking long-run player (she) and a myopic counterpart (he) interact in a repeated game that is played continuously over a time interval
[0, T], T < ∞. At each t ∈ [0, T], the long-run player chooses an action $a_t$, while the myopic player chooses $\hat{a}_t$, both taking values over the real line. Given a profile of realized actions, $(a_t, \hat{a}_t)_{t \in [0, T]}$, the long-run player’s total payoff is

$$\int_0^T e^{-rt}U(a_t, \hat{a}_t, \theta)dt.$$  \hspace{1cm} (7)

In this specification, $r \geq 0$ is the long-run player’s discount rate, $U : \mathbb{R}^3 \to \mathbb{R}$ is quadratic capturing her flow (i.e., stage-game) utility function, and $\theta$ denotes the value of a normally distributed random variable with mean $\mu$ and variance $\gamma_o > 0$ that parametrizes the economic environment. In turn, the myopic player’s stage-game payoff at any time $t \geq 0$ is given by

$$\hat{U}(a_t, \hat{a}_t, \theta)$$  \hspace{1cm} (8)

if $(a_t, \hat{a}_t)$ was chosen at that time, where $\hat{U} : \mathbb{R}^3 \to \mathbb{R}$ is also quadratic.

In what follows, we assume the following properties on the quadratic functions $U$ and $\hat{U}$ (partial derivatives are denoted by subindices):

**Assumption 1.**

(i) *Strict concavity:* $U_{aa} = \hat{U}_{\hat{a}\hat{a}} = -1$;

(ii) *Non-trivial signaling:* $U_{a\theta}(U_{a\theta} + U_{a\hat{a}}\hat{U}_{a\hat{a}}) > 0$;

(iii) *Second-order inferences:* $U_{a\hat{a}} \neq 0$ and $|\hat{U}_{a\theta}| + |\hat{U}_{a\hat{a}}| \neq 0$.

(iv) *Myopic best-replies intersect:* $U_{aa}\hat{U}_{a\hat{a}} < 1$;

We first require that the players’ objectives are concave in their respective choice variables; from this perspective (i) is simply a normalization. A second minimal requirement is that the long-run player strategically care about $\theta$, which is implied by (ii). Equipped with this, part (iii) says that second-order inferences are relevant for play: the myopic player’s first-order belief matters for his behavior—either directly because he cares about $\theta$, or because he wants to predict the long-run player’s action—and in turn the long-run player wants to predict the myopic player’s action, invoking a second-order belief.\(^{10}\)

The remaining parts are technical conditions pertaining to the one-shot game that arises at the end of the interaction via the static game with higher-order uncertainty that takes place at $T$. Specifically, part (iv) ensures that a static Nash equilibrium always exists, and part (ii) ensures that any such equilibrium involves non-trivial signaling. We elaborate more on these conditions when we explain how to find equilibria of the type we are interested in.

\(^{10}\)Of course, (iii) is not really a restriction to our analysis, but instead a choice.
**Information.** The long-run player observes $\theta$ before play begins, while the myopic player only knows the distribution $\theta \sim \mathcal{N}(\mu, \gamma^o)$ from which it is drawn (and this is common knowledge). In addition, there are two signals $X$ and $Y$ that convey noisy information about the players’ actions according to

$$
\begin{align*}
    dX_t &= a_t dt + \sigma_X dZ^X_t, \\
    dY_t &= \hat{a}_t dt + \sigma_Y dZ^Y_t,
\end{align*}
$$

where $Z^X$ and $Z^Y$ are independent Brownian motions, and $\sigma_Y$ and $\sigma_X$ are strictly positive volatility parameters. In this linear product-structure specification, the signal $Y$ is only observed by the myopic player, while the signal $X$ is public.

Let $E_t[\cdot]$ denote the long-run player’s conditional expectation operator, which can condition on the histories $(\theta, a_s, X_s : 0 \leq s \leq t)$, $t > 0$, and on her conjecture of the myopic player’s play. Likewise, $\hat{E}_t[\cdot]$ denotes the myopic player’s analog, which conditions on $(\hat{a}_s, X_s, Y_s : 0 \leq s \leq t)$ and on her belief about the long-run player’s strategy.

**Strategies and Equilibrium Concept.** To characterize equilibrium outcomes, we focus on Nash equilibria. From a time-zero perspective, an admissible strategy for the long-run player is any square-integrable real-valued process $(a_t)_{t \in [0, T]}$ that is progressively measurable with respect to the filtration generated by $(\theta, X)$. Similarly, an admissible strategy $(\hat{a}_t)_{t \in [0, T]}$ for the myopic player satisfies identical integrability conditions, but the measurability restriction is with respect to $(X, Y)$.

**Definition 1** (Nash equilibrium.). An admissible pair $(a_t, \hat{a}_t)_{t \geq 0}$ is a Nash equilibrium if,

(i) given $(\hat{a}_t)_{t \geq 0}$, the process $(a_t)_{t \geq 0}$ maximizes

$$
E_0 \left[ \int_0^T e^{-rt} U(a_t, \hat{a}_t, \theta) dt \right]
$$

among all admissible processes, and

(ii) $\hat{a}_t$ solves $\max_{\hat{a}' \in \mathbb{R}} \hat{E}_t[\hat{U}(a_t, \hat{a}', \theta)]$ for all $t \in [0, T]$.

In the next section, we characterize Nash equilibria that are supported by linear Markov strategies that are sub-game perfect, i.e., that sequentially rational on and off the path.

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11Thus, flow payoffs do not convey any additional information to the players (i.e., they are either realized after time $T$, or they can be written in terms of the actions and signals observed by each player).

12Square integrability is in the sense of $E_0[\int_0^T a_t^2 dt] < +\infty$ for the long-run player. Such condition ensures that a strong solution to 9 exists, and thus that the outcome of the game is well defined.
Remark 1 (Extensions). The baseline model can be generalized along two dimensions:

(i) Terminal payoffs: terminal payoffs of the form $e^{-rT}\Psi(a_T)$, with $\Psi$ quadratic, can be added to (7). A reputation model with this property is studied in 5.1.

(ii) Long-run player affecting the public signal $X$: the drift of (9) can be generalized to $\dot{a}_t + \nu a_t$, where $\nu \in [0, 1]$ is a scalar. An insider trading model involving $\nu = 1$, as well as $U_{aa} = 0$, (i.e., linear utility) is explored in 5.2.

4 Equilibrium Analysis: Linear Markov Equilibria

To construct linear Markov perfect equilibria (henceforth, LME), we first postulate a minimal set of belief states up to the second order to be used by the players in any equilibrium of this kind. We then derive a representation of the long-run player’s second-order belief as a linear function of a subset of such belief states, when the players use the candidate belief states in a linear fashion. This result generalizes the representation (5) obtained in section 2.2, and it circumvents the problem of the set of states growing without bound (Section 4.1).

In Section 4.2 we then turn to setting up the long-run player’s best-response problem, and elaborate on how the problem of existence of LME reduces to finding solutions to a boundary-value problem. In Section 4.3, we illustrate two proof techniques that depend on whether the myopic player’s best response explicitly depends on his belief about the state of the world or not (common vs. private values environments, respectively). Finally, we obtain two existence results for LME, each for a variation of the coordination game from Section 2.

4.1 Belief States and Representation Lemma

With linear-quadratic payoffs and signals that are linear in the players’ actions, it is natural to examine equilibria in which the long-run player conditions on her type $\theta$ linearly.

The logic is then analogous to that in Section 2.2. Specifically, since the myopic player cares about the long-run player’s action (and/or her type) to determine his best response, he will use $Y$ to learn about $\theta$. Because $Y$ is privately observed, however, the myopic player’s (first-order) belief about $\theta$ is private. The strategic interdependence of the players’ actions in the long-run player’s payoff then forces the latter agent to forecast the myopic player’s belief, which leads her second-order belief to become a relevant state. As we demonstrate shortly, such second-order belief is also private due to its dependence on the long-run player’s type.
via her past actions. Thus, the myopic player is forced to perform a non-trivial inference about such hidden second-order belief, and so forth.

Along the path of play of any pure strategy, however, the outcome of the game should depend only on the tuple \((\theta, X, Y)\). Intuitively, given any rule that specifies behavior as a function of past actions and information, the dependence on past play must disappear when such a rule is followed, thus leading to realized outcomes that depend on the exogenous elements of the model. In particular, the long-run player’s second-order belief should be a function of \((\theta, X)\) exclusively, which is the only source of information available to her. Moreover, in this Gaussian environment, one would expect the relationship between \(M\) and \((\theta, X)\) to be linear if the rule that drives behavior is linear in some belief states.

Let \(\hat{M}_t := \hat{E}_t[\theta]\) denote the mean of the myopic player’s belief, and \(M_t := E_t[\hat{M}_t]\) denotes the long-run player’s second-order counterpart. The previous discussion then suggests the existence of a deterministic function \(\chi\) and a process \((L_t)_{t \in [0,T]}\) that depends on the paths of the public signal \(X\), such that \(M\) admits the representation

\[
M_t = \chi_t \theta + (1 - \chi_t)L_t \tag{11}
\]

when the players follow linear Markov strategies

\[
a_t = \beta_{0t} + \beta_{1t} M_t + \beta_{2t} L_t + \beta_{3t} \theta \tag{12}
\]

\[
\hat{a}_t = \delta_{0t} + \delta_{1t} \hat{M}_t + \delta_{2t} L_t, \tag{13}
\]

where the coefficients \(\beta_{it}\) and \(\delta_{jt}\), \(i = 0, 1, 2, 3\) and \(j = 0, 1, 2\), are deterministic. (We occasionally use \(\vec{\beta} := (\beta_0, \beta_1, \beta_2, \beta_3)\) and \(\vec{\delta} := (\delta_0, \delta_1, \delta_2)\) for convenience.) The reason for augmenting the strategies of 2.2 by the public state \(L\) is apparent: if true, the myopic player uses (11) to forecast \(M\), which means that \(L\) becomes a payoff-relevant state for both players.

Lemma 2 below characterizes the pair \((\chi, L)\) that validates (11)–(13). Before stating the result, it is instructive to explain its derivation and introduce some notation. When the myopic player conjectures that (11)–(12) hold for some “public” process \(L\), he therefore expects the long-run player’s realized actions to follow

\[
a_t = \underbrace{\beta_{0t}}_{\equiv \alpha_{0t}} + \underbrace{(\beta_{2t} + \beta_{1t}(1 - \chi_t))}_{\equiv \alpha_{2t}} L_t + \underbrace{(\beta_{3t} + \beta_{1t} \chi_t) \theta}_{\equiv \alpha_{3t}}. \tag{14}
\]

Because \(L\) is public, the myopic player can then filter \(\theta\) from \((X, Y)\) when \(Y\) is driven by (14). This learning problem is (conditionally) Gaussian, and hence the myopic player’s posterior
beliefs is fully characterized by a mean process \((\hat{M}_t)_{t \geq 0}\), and a deterministic variance
\[
\gamma_t := \bar{\text{Var}}_t = \hat{\mathbb{E}}_t[(\theta_t - \hat{M}_t)^2],
\]
where we have omitted the hat symbol in \(\gamma_t\) for notational convenience. As in Section 2, this posterior variance will be determined by the signaling coefficient
\[
\alpha_{3t} := \beta_{3t} \chi_t,
\]
with \(\beta_1 \chi_t\) encoding the history-inference effect: different types are expected to take different actions not only because of their direct signaling incentives (captured by \(\beta_3\)) but also because their past actions have lead them to hold different beliefs today.

Critically, while the long-run player does not observe \(\hat{M}\), she recognizes that deviations from (14) affect its evolution via \(Y\)—thus, her problem is one of stochastic control of an unobserved state. Given the linear-quadratic payoffs, linear dynamics, and Gaussian noise, this problem can be recast as one of controlling the long-run player’s estimate of \(\hat{M}\)—namely, \(M_t := \mathbb{E}_t[\hat{M}_t]\)—after appropriately adjusting her flow payoffs via the use of conditional expectations.\(^{13}\) Inserting the general linear Markov strategy (12) into the law of motion of \((M_t)_{t \geq 0}\), and solving for \(M_t\) as a function of \(\{\theta, (X_s)_{s < t}\}\) allows us to pin down \((\chi, L)\) given the coefficients in the strategies.

**Lemma 2 (Representation of second-order belief).** Suppose that \((X, Y)\) is driven by (12)–(13) and that the myopic player believes that (11) holds. Then, (11) holds at all times (path-by-path of \(X\)), if and only if
\[
\begin{align*}
\dot{\gamma}_t &= -\frac{\gamma_t^2(\beta_{3t} + \beta_{1t} \chi_t)^2}{\sigma_Y^2}, \ t > 0, \ \gamma_0 = \gamma^0, \\
\dot{\chi}_t &= \frac{\gamma_t(\beta_{3t} + \beta_{1t} \chi_t)^2(1 - \chi_t)}{\sigma_Y^2} - \frac{\gamma_t \chi_t^2 \delta_{1t}^2}{\sigma_Y^2}, \ t > 0, \ \chi_0 = 0, \\
dL_t &= (l_{0t} + l_{1t} L_t)dt + B_t dX_t, \ t > 0, \ L_0 = \mu,
\end{align*}
\]
where \(l_{0t}\) and \(l_{1t}\), and \(B_t\) are given in (B.6)-(B.8). Moreover, \(L_t = \mathbb{E}[\hat{M}_t | \mathcal{F}_t^X] = \mathbb{E}[-\theta_t | \mathcal{F}_t^X]\) and \(\gamma_t \chi_t = \text{Var}_t = \mathbb{E}_t[(M_t - \hat{M}_t)^2].\)

The long-run player uses the public signal to learn about the myopic player’s belief. By

\(^{13}\)This is the so-called separation principle, which allows one to filter first, and optimize afterwards using belief states, in these types of problems. We elaborate more on this topic in the proof of Lemma 4, where we derive the laws of motion of the Markov belief states.
the lemma, along the path of (12)–(13), we have that

\[ M_t = \frac{\text{Var}_t \theta}{\text{Var}_t} + \left( 1 - \frac{\text{Var}_t}{\text{Var}_t} \right) \mathbb{E}[\theta | \mathcal{F}_t^X]. \]

Indeed, while learning about \( \hat{M} \) from \( X \), the only informational advantage that the long-run player has relative to an outsider who observes \( X \) exclusively is that she knows her type. Due to the Gaussian structure of the model, therefore, (i) \( M_t \) is a linear combination of \( \theta \) and \( \mathbb{E}[\hat{M}_t | \mathcal{F}_t^X] \), and (ii) the weights are deterministic. By the law of iterated expectations, \( \mathbb{E}[\hat{M}_t | \mathcal{F}_t^X] = \mathbb{E}[\theta | \mathcal{F}_t^X] \), and the representation follows.

Let us now elaborate on the structure of the \( \chi \)-ODE (16). Recall that the common prior assumption implies that the long-run player knows that \( \hat{M} = \mu \) at time zero: \( \text{Var}_0 = 0 \) then implies \( M_0 = \mu \) in the previous expression, and so the \( \chi \)-ODE starts at zero. As signaling progresses, however, second-order uncertainty arises due to the long-run player losing track of \( \hat{M} \) (i.e., \( \text{Var}_t > 0 \)): this is captured in \( \dot{\chi} > 0 \) as soon as \( \alpha_3 > 0 \) in (16). In other words, the long-run player expects \( \hat{M} \) to gradually reflect her type \( \theta \), and so \( \chi_t > 0 \).

Observe that if \( \sigma_X = \infty \) (the public signal is infinitely noisy) or \( \delta_1 \equiv 0 \) (the myopic player does not signal back) the public signal is uninformative, so we would expect the long-run player to forecast \( \hat{M} \) solely by input: in fact, \( L_t = L_0 = \mu \) and \( \chi_t = 1 - \gamma_t/\gamma_0 \) hold in this case, exactly as in Section 2.\(^{14}\) Otherwise, she also forecasts by “output,” as reflected in the dependence of \( L \) on \( X \). Conversely, as \( \delta_1^2/\sigma_X^2 \) grows, there is more downward pressure on the growth of \( \chi \): as the signal-to-noise ratio in \( X \) improves, the long-run player relies less on her past actions to forecast \( \hat{M} \), everything else being held constant. Thus, the no-feedback maximizes the potential impact of second-order belief effects on behavior.

Our subsequent analysis takes the system (15)–(16) as an input. Thus, we require it to have a unique solution over \( [0, T] \) so as to ensure that the ODE-characterization is valid. To this end, notice that the myopic player’s best reply can be written as \( \delta_1 := \dot{\gamma}_t + \dot{\chi}_t + \delta_2 \), where \( \delta_2 = \hat{U}_a \hat{a} \) and \( \dot{\gamma}_t = \hat{U}_a \hat{a} \) are real numbers.

**Lemma 3.** Suppose that \( \beta_1 \) and \( \beta_3 \) are continuous, \( \beta_3 \neq 0 \) and \( \delta_1 := \dot{\gamma}_t + \dot{\chi}_t + \delta_2 \). Then, there is a unique solution to (15)–(16). Such solution satisfies \( 0 < \gamma_t \leq \gamma_0 \) and \( 0 < \chi_t < 1, \: t \in (0, T] \).

The idea is that, under the conditions in the lemma, \( (\gamma, \chi) \) is bounded, and hence a solution to the system exists over \( [0, T] \) (as solutions to ODE systems either exist or explode over unrestricted domains). Since the system is locally Lipschitz continuous, uniqueness

\(^{14}\)Setting \( \delta_1/\sigma_X \equiv 0 \) in (16) leads to the same ODE that \( \chi \) satisfies in the no-feedback case. By uniqueness, such solution is \( \chi = 1 - \gamma_t/\gamma_0 \). See the proof of Lemma 1.
ensues; in particular, $\gamma_t = \mathbb{E}_t[(\theta_t - \hat{\mathbb{E}}_t[\theta])^2]$ and $\chi_t = \text{Var}_t/\text{Var}_t = \mathbb{E}_t[(M_t - \hat{M}_t)^2]/\gamma_t$.

The belief representation (11) relies on the long-run player following the linear strategy (12); i.e., it does not hold off the path of play. In fact, as argued earlier, $(M_t)_{t \geq 0}$ is controlled by the long-run player, a phenomenon that is the consequence of the private monitoring present in the model: past play is used for forecasting the myopic player’s private histories, and so different actions yield different perceptions of the continuation game as measured by $M$. Moreover, because such deviations are hidden, from the long-run player’s perspective, the myopic player is always assuming that (11) holds—thus, the pair $(\gamma, \chi)$ affects the evolution of $\hat{M}$ in the myopic player’s learning process, and hence the evolution of $M$. The next result introduces the law of motion of $M$ and $L$ for an arbitrary strategy of the long-run player, which will allow us to state her best-response problem.

**Lemma 4.** Suppose that the long-run player follows $(a'_t)_{t \geq 0}$ while the myopic player follows (13) and believes (11)–(12). Then, from the long-run player’s perspective

\[
\begin{align*}
dM_t &= \frac{\gamma_t \alpha_3 t}{\sigma_Y^2}(a'_t - [\alpha_0 t + \alpha_2 L_t + \alpha_3 \hat{M}_t])dt + \frac{\chi_t \gamma_t \delta_1 t}{\sigma_X}dZ_t \\
dL_t &= \frac{\chi_t \gamma_t \delta_1 t}{\sigma_X^2(1 - \chi_t)}[\delta_1 t(M_t - L_t)dt + \sigma_X dZ_t]
\end{align*}
\]

where $(\gamma, \chi)$ solves (15)–(16) and $(Z_t)_{t \geq 0}$ is a Brownian motion from her standpoint.

The dynamic (18) shows that long-run player’s choice of strategy $a'$ affects $M$. In particular, she will update her belief upward when $a'_t > \mathbb{E}_t[\alpha_0 t + \alpha_2 L_t + \alpha_3 \hat{M}_t]$, i.e., when she exceeds her own expectation of the myopic player’s belief about her behavior. The intensity of such a reaction is given by $\gamma_t \alpha_3 t/\sigma_Y^2$: more uncertainty (higher $\gamma$) and stronger signaling (larger $\alpha_3$) makes the long-run player’s belief more sensitive to her own actions. Further, $M$ evolves deterministically when $\delta_1 / \sigma_X \equiv 0$.\(^{15}\)

The drift of (19) demonstrates that the long-run player affects $L$ only indirectly via changes in $M$, due to her action not entering the public signal directly. Further, the drift captures that the belief of an outsider who only observes $X$ always moves in the direction of $M$ on average, reflecting that such an outsider learns the long-run player’s type. From this perspective, by leading to $L_t = \mu$ at all times, the no-feedback case ($\sigma_X = \infty$) misses a mild signal-jamming effect—the ability to influence a public belief, albeit only indirectly.

Finally, the full-support monitoring and linear-quadratic structure, along with the (equilibrium) representation (11), make it clear that $(t, \theta, L_t, M_t)$ and $(t, L_t, \hat{M}_t)$ summarize all

\(^{15}\)It is worth noting that $(M_t)_{t \geq 0}$ corresponds to a player’s non-trivial belief that is controlled by the same player. Unless there are experimentation effects, players’ own beliefs are usually affected by other players.
the payoff-relevant information our players. In this line, the time variable captures both time-horizon effects and the learning effects via \( \gamma \) and \( \chi \).

### 4.2 Dynamic Programming and the Boundary-Value Problem

**The long-run player’s best-response problem.** Given a conjecture \( \tilde{\beta} \) by the myopic player, the coefficients \( \tilde{\delta} \) will be such that

\[
\hat{a}_t := \delta_{0t} + \delta_{1t}\hat{M}_t + \delta_{2t}L_t = \arg\max_{a'} \tilde{E}_t[U(\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\theta, a', \theta)].
\]

Because \( U \) is quadratic, the long-run player’s best-response problem is, up to a constant,

\[
\max_{(a_t)_{t \in [0, T]}} \mathbb{E}_0 \left[ \int_0^T e^{-rt}U(a_t, \delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t, \theta)dt \right]
\]

s.t. \((18)\) and \((19)\)

and where \( \tilde{\delta} \) satisfies \((20)\). Observe that we have replaced \( \hat{M} \) by \( M \) in the flow by means of \( \mathbb{E}_t[\hat{M}^2_t] = M^2_t + \chi_t\gamma_t \), and then using that \( \chi_t\gamma_t \) is deterministic.

We can now define the notion of a *linear Markov perfect equilibrium* (LME).

**Definition 2 (Linear Markov Perfect Equilibrium).** A Nash equilibrium \((a_t, \hat{a}_t)_{t \geq 0}\) is a Linear Markov Equilibrium (LME) if there are deterministic coefficients \((\tilde{\beta}, \tilde{\delta})\) such that \( \hat{a}_t \) satisfies \((20)\) and \( a_t = \alpha_0 + \alpha_{2t}L_t + \alpha_{3t}\theta, \) where: (i) \((L_t)_{t \geq 0}\) evolves as in \((17)\), (iii) \( \tilde{\alpha} \) satisfies \((14)\), and (iii) \( \beta_0 + \beta_1M + \beta_2L + \beta_3\theta \) is an optimal policy for the long-run player.

The natural approach for establishing the existence of LME is via dynamic programming. Specifically, we postulate a quadratic value function

\[
V(\theta, m, \ell, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\ell + v_{4t}\theta^2 + v_{5t}m^2 + v_{6t}\ell^2 + v_{7t}\theta m + v_{8t}\theta \ell + v_{9t}m\ell,
\]

where \( v_i, i = 0, ..., 9 \) depend on time only. In turn, the HJB equation is

\[
rV = \sup_{a'} \left\{ U(a', \hat{a}_t, \theta) + V_t + \mu_M(a')V_m + \mu_LV_\ell + \frac{\sigma^2_M}{2}V_{mm} + \sigma_M\sigma_LV_{m\ell} + \frac{\sigma^2_L}{2}V_{\ell\ell} \right\},
\]

where \( \mu_M(a') \) and \( \mu_L \) (respectively, \( \sigma_M \) and \( \sigma_L \)) are the drifts (respectively, volatilities) in the laws of motion for \( M \) and \( L \) given in Lemma 4, and where \( \hat{a}_t \) is determined by \( \tilde{\beta} \) and

---

16We have focused on the long-run player exclusively. While deviations by the myopic player do affect \( L \), the same assumptions (i.e., linear-quadratic structure and undetectable deviations) make his flow payoff fully determined by the current value of \((t, L, \hat{M})\) after all private histories.
\( \chi \) via (20). A LME with coefficients \( \bar{\beta} \) for the long-run player is obtained when the linear Markov strategy (12) is an optimal policy for the previous HJB equation.

**The boundary-value problem.** We briefly explain how to obtain a system of ordinary differential equations (ODEs) for \( \bar{\beta} \). Letting \( a(\theta, m, \ell, t) \) denote the maximizer of the right-hand side in the HJB equation, the first-order condition (FOC) reads

\[
U_a(a(\theta, m, \ell, t), \delta_0 t + \delta_1 m + \delta_2 \ell, \theta) + \gamma_t \frac{\alpha_3 t}{\sigma_Y^2} \left[ v_{2t} + 2v_{5t} m + v_{7t} \theta + v_{9t} \ell \right] = 0. \tag{21}
\]

where \( \gamma_t \alpha_3 t / \sigma_Y^2 \) in the second term captures the sensitivity of \( M \) to the long-run player’s action at time \( t \). Solving for \( a(\theta, m, \ell, t) \) in the previous FOC, the equilibrium condition becomes \( a(\theta, m, \ell, t) = \beta_0 t + \beta_1 m + \beta_2 \ell + \beta_3 \theta \).

Because the latter condition is a linear equation, we can solve for \((v_2, v_5, v_7, v_9)\) as a function of the coefficients \( \bar{\beta} \). Inserting these into the HJB equation along with \( a(\theta, m, \ell, t) = \beta_0 t + \beta_1 m + \beta_2 \ell + \beta_3 \theta \) in turn allows us to obtain a system of ODEs that the \( \bar{\beta} \) coefficients must satisfy. The resulting system is coupled with the ODEs that \( v_6 \) and \( v_8 \) satisfy (and that are obtained from the HJB equation): since \( M \) feeds into \( L \), the envelope condition with respect to \( M \) is not enough to determine equations for the candidate equilibrium coefficients. Finally, since the pair \((\gamma, \chi)\) affects the law of motion of \((M, L)\), it also affects the evolution of \((\bar{\beta}, v_6, v_8)\), and so the ODEs (15)–(16) must be considered.

The boundary conditions for the system of ODEs that \((\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8, \gamma, \chi)\) satisfies are as follows. First, there are the exogenous initial conditions that \( \gamma \) and \( \chi \) satisfy, i.e., \( \gamma_0 = \gamma^o > 0 \) and \( \chi_0 = 0 \). Second, there are terminal conditions \( v_6 T = v_8 T = 0 \) due to the absence of a lump-sum terminal payoff in the long-run player’s problem. Third, more interestingly, there are *endogenous* terminal conditions that are determined by the static Nash equilibrium that arises from myopic play at time \( T \). In fact, letting

\[
U_{a\theta} = u_\theta, \ U_{a\dot{a}} = u_{\dot{a}} \text{ and } U_a(0, 0, 0) = u_0,
\]

and analogously for the myopic player via the substitution \((\cdot) \leftrightarrow (\dot{\cdot})\), we obtain

\[
\beta_{0T} = \frac{u_0 + u_{\dot{a}} \dot{u}_0}{1 - u_{\dot{a}} \dot{u}_a}, \quad \beta_{1T} = \frac{u_{\dot{a}} [u_\theta \ddot{u}_a + \ddot{u}_\theta]}{1 - u_{\dot{a}} \dot{u}_a \chi T}, \quad \beta_{2T} = \frac{u_{\dot{a}}^2 \dddot{u}_a [u_\theta \ddot{u}_a + \ddot{u}_\theta] (1 - \chi T)}{(1 - u_{\dot{a}} \dot{u}_a)(1 - u_{\dot{a}} \dot{u}_a \chi T)}, \quad \beta_{3T} = u_\theta \tag{22}
\]

which are well-defined thanks to (iv) in Assumption 1 and \( \chi \in (0, 1).^{17} \)

We conclude that \( b := (\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8, \gamma, \chi)' \) satisfies a boundary-value problem

\[ \delta_{0T} = \ddot{u}_0 + \dddot{u}_a \beta_{0T}, \ \delta_{1T} = \ddot{u}_\theta + \dddot{u}_a [\beta_{3T} + \beta_{1T} \chi T] \text{ and } \delta_{2T} = \dddot{u}_a [\beta_{2T} + \beta_{1T} (1 - \chi T)]. \]

\[17\text{From here, } \delta_{0T} = \ddot{u}_0 + \dddot{u}_a \beta_{0T}, \ \delta_{1T} = \ddot{u}_\theta + \dddot{u}_a [\beta_{3T} + \beta_{1T} \chi T] \text{ and } \delta_{2T} = \dddot{u}_a [\beta_{2T} + \beta_{1T} (1 - \chi T)]. \]
(BVP) of the form

\[ \dot{b}_t = f(b_t), \quad \text{s.t. } D_0 b_0 + D_T b_T = (B(\chi_T)', \gamma^o, 0)' \]  

(23)

where (i) \( f: \mathbb{R}^6 \times \mathbb{R}_+ \times [0, 1) \to \mathbb{R}^8 \), (ii) \( D_0 \) and \( D_T \) are the diagonal matrices

\[ D_0 = \text{diag}(0, 0, 0, 0, 0, 0, 1) \quad \text{and} \quad D_T = \text{diag}(1, 1, 1, 1, 1, 1, 0, 0) \]

and

(iii) \( \chi \mapsto B(\chi) := \left( \frac{u_0 + u_\pi}{1 - u_\pi}, \frac{u_a[u_\pi u_\pi + u_\pi]}{1 - u_\pi u_\pi}, \frac{u_2^2 u_\pi u_\pi + u_\pi}{(1 - u_\pi u_\pi)(1 - u_\pi u_\pi)}, u_\theta, 0, 0 \right) \in \mathbb{R}^6. \)  

(24)

The general expression for \( f(\cdot) \) given any pair \((U, \hat{U})\) satisfying Assumption 1 is tedious and long, and can be found in \texttt{spm.nb} on our websites. In the next section, we provide examples that exhibit all the relevant properties that any such \( f(\cdot) \) can satisfy.

The question of finding LME is then reduced to finding solutions to the BVP (23) (subject to the rest of the coefficients of the value function being well defined). We turn to this issue in the next section.

4.3 Existence of Linear Markov Equilibria: Interior Case

In this section, we present two existence results for LME in the case \( \sigma_X \in (0, \infty) \): one for the application introduced in Section 2, and the second for a variation of it in which the follower (i.e., the myopic player) cares about both matching the leader’s action and matching the her type. We accomplish this via proving the existence of a solution to the BVP that arises in each setting, for the case in which the leader is patient (i.e., \( r = 0 \))—the applicability of the methods is, however, more general (both in terms of the flow payoffs and time preferences).

The problem of finding a solution to any instance of the BVP (23) is complex because there are multiple ODEs in either direction: \((\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8)\) are traced backward from their (endogenous) terminal values, while \((\gamma, \chi)\) are traced forward using their initial (exogenous) ones—see Figure 5. In practice, this implies that the traditional “shooting methods” can become severely complicated. Specifically, when constructing, say, a modified backward initial value problem (IVP) in which \((\gamma, \chi)\) has a parametrized initial condition at \( T \), the requirement becomes that the chosen parameters induce terminal values at 0 that exactly match \((\gamma^o, 0)\). With more than one variable, however, this method essentially requires having an accurate knowledge of the relationship between \( \gamma \) and \( \chi \) at \( T \) for all possible coefficients \( \bar{\beta} \): it is only then that we can find a way to trace the parametrized values over a region of
initial (time-$T$) values in a way that it is possible to ensure that the target is hit.

\[
\begin{align*}
\gamma_0 & \quad \chi_0 \\
\gamma_t & \quad \chi_t
\end{align*}
\]

\[
\begin{align*}
B(\chi, \gamma)
\end{align*}
\]

Figure 5: In the BVP, $(\gamma, \chi)$ has initial conditions, while $(\vec{\beta}, v_6, v_8)$ has terminal ones.

The reason behind this dimensionality problem is the asymmetry in the environment: the rate at which the long-run player signals her private information, $\alpha_3 := \beta_3 + \chi \beta_1$, can be substantially different than rate at which the myopic player signals his private belief, $\delta_1$. This, in turn, potentially introduces a non-trivial history dependence between $\gamma$ and $\chi$, reflected in the coupled system of ODEs they satisfy. Two natural questions then arise: first, under which conditions such history dependence can be simplified; and second, how to tackle the issue of existence of LME when this simplification is not possible.

**Private values: one-dimensional shooting.** We say that an environment is one of

private values if the myopic player’s flow utility satisfies

\[
\hat{u}_\theta := \hat{U}_{\theta \theta} = 0,
\]

i.e., the myopic player’s best-reply does not directly depend on his belief about $\theta$, but only indirectly via the long-run player’s action. Otherwise, we say that the environment is one of

common values (despite the long-run player always knowing $\theta$).

In a private-value setting, the myopic player’s coefficient on $\dot{M}$ is $\delta_1 = \hat{u}_a \alpha_3$. In this case, there is a one-to-one mapping between $\gamma$ and $\chi$:

**Lemma 5.** Set $\sigma_X \in (0, \infty)$. Suppose that $\beta_1$ and $\beta_3$ are continuous and that $\delta_1 = \hat{u}_a \alpha_3$. If $\hat{u}_a \neq 0$, there are positive constants $c_1, c_2$ and $d$ independent of $\gamma^o$ such that

\[
\chi_t = \frac{c_1 c_2(1 - \gamma_t/\gamma^o)^d}{c_1 + c_2 [\gamma_t/\gamma^o]^d}.
\]

Moreover, (i) $0 \leq \chi_t < c_2 < 1$ for all $t \in [0, T]$ and (ii) $c_2 \to 0$ as $\sigma_X \to 0$ and $c_2 \to 1$ as $\sigma_X \to \infty$. If instead $\hat{u}_a = 0$, $\chi_t = 1 - \gamma_t/\gamma^o$ and $c_2 = 1$. 

27
It is easy to see that the right-hand side of the expression for \( \chi \) in the previous lemma is strictly decreasing in \( \gamma_t \). Consequently, when the ratio of the signaling coefficients is constant, the dimensionality of the (backward) shooting problem is reduced to the single variable. The lemma also states that, as long as \( \sigma_X < \infty \), \( \chi \) is always strictly below 1, reflecting that the scope for the history-inference effect is diminished relative to the no-feedback case. Further, the characterization of \( \chi \) obtained in the latter case (5) is recovered when \( \dot{u}_a = 0 \), as the public signal is then uninformative.

Thanks to the previous lemma, the standard shooting method based on the continuity of the solutions is applicable. We state below the BVP for the leading-by-example application of Section 2 when \( \sigma_X \in (0, \infty) \) in its undiscounted version: recall that in that setting, the follower wants to match the leader’s action, and so

\[
\dot{a}_t = \hat{E}_t[a_t] \Rightarrow \delta_1 = 0 \Longleftrightarrow \hat{u}_a = 1.
\]

(Since scaling \( U \) and \( \hat{U} \) each by a constant does not alter incentives, the ODEs below are obtained under \( U(a, \dot{a}, \theta) = -(\theta - a)^2 - (a - \dot{a})^2 \) and \( \hat{U}(a, \dot{a}, \theta) = -(\dot{a} - a)^2 \) as opposed to \( -(\theta - a)^2 - (a - \dot{a})^2 \)/4 and \( \hat{U}(a, \dot{a}, \theta) = -(\dot{a} - a)^2 \)/2, which would yield (i) in Assumption 1). We omit the \( \beta_0 \)-ODE due to being uncoupled from the rest and linear in itself:

\[
\begin{align*}
\dot{v}_{6t} &= \beta_{2t}^2 + 2\beta_{1t}\beta_{2t}(1 - \chi_t) - \beta_{1t}^2(1 - \chi_t)^2 + \frac{2v_{6t}\alpha_{32t}\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)} \\
\dot{v}_{8t} &= -2\beta_{2t} - 2(1 - 2\alpha_{3t})\beta_{1t}(1 - \chi_t) - 4\beta_{1t}\chi_t(1 - \chi_t) + \frac{v_{8t}\alpha_{32t}\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)} \\
\dot{\beta}_{1t} &= \frac{\alpha_{32t}\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2(\alpha_{3t} - \beta_{1t})\beta_{1t}(1 - \chi_t) - \alpha_{3t}\beta_{1t}\chi_t v_{8t} \\
&\quad - 2\sigma_Y^2\alpha_{3t}\chi_t(\beta_{2t} - \beta_{1t}[1 - \chi_t - 2\beta_{2t}\chi_t]) \right\} \\
\dot{\beta}_{2t} &= \frac{\alpha_{32t}\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2\beta_{1t}^2(1 - \chi_t)^2 + 2\sigma_Y^2\alpha_{3t}\beta_{2t}\chi_t^2(1 - 2\beta_{2t}) - \alpha_{3t}\gamma_t\chi_t(2v_{6t} + \beta_{2t}\chi_t) \right\} \\
\dot{\beta}_{3t} &= \frac{\alpha_{32t}\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ -2\sigma_X^2\beta_{1t}(1 - \chi_t)\beta_{3t} + 2\sigma_Y^2\alpha_{3t}\beta_{2t}\chi_t^2(1 - 2\beta_{3t}) - \alpha_{3t}\beta_{3t}\gamma_t v_{8t} \right\} \\
\dot{\gamma}_t &= -\frac{\gamma_t^2\alpha_{3t}}{\sigma_Y^2}
\end{align*}
\]

with boundary conditions \( v_{6T} = v_{8T} = 0 \), \( \beta_{1T} = \frac{1}{2(2 - \chi_T)} \), \( \beta_{2T} = \frac{1 - \chi_T}{2(2 - \chi_T)} \), \( \beta_{3T} = \frac{1}{2} \) and \( \gamma_0 = \gamma_o \), and where \( \alpha_3 := \beta_3 + \beta_{1X} \) and \( \chi_t \) is as in the previous lemma. We have the following:

**Theorem 1.** Let \( \sigma_X \in (0, \infty) \) and \( r = 0 \). Then, there exists a strictly positive function \( T(\gamma_o) \in O(1/\gamma_o) \) such that, for all \( T < T(\gamma_o) \), there exists a LME based on the solution to the previous BVP. In that equilibrium, \( \beta_{0t} = 0 \), \( \beta_{1t} + \beta_{2t} + \beta_{3t} = 1 \) and \( \alpha_{3t} > 0 \), \( t \in [0, T] \).
The key step behind the proof is to show that $(\beta_1, \beta_2, \beta_3, v_6, v_8, \gamma)$ can be bounded uniformly over $[0, T(\gamma^o))$, some $T(\gamma^o) > 0$, when $\gamma_t \in [0, \gamma^o]$ at all times. For a given $T < T(\gamma^o)$, therefore, this implies that tracing the (parametrized) initial condition of $\gamma$ in the (backward) IVP from 0 upwards as schematically in Figure 6 will lead to at least one $\gamma$-path landing exactly at $\gamma^o$ (while the rest of the ODEs still admitting solutions), due to the continuity of the solutions with respect to the initial conditions.\footnote{See Bonatti et al. (2017) for an application of this method to a symmetric oligopoly model featuring dispersed fixed private information, imperfect public monitoring, and multiple long-run players.}

![Figure 6: The one-dimensional shooting method.](image)

As expected, the signaling coefficient in the interior cases lies “in between” those found in the extreme cases of Section 2. Graphically for the $r = 0$ case:\footnote{In the discounted case, one can instead work with the ‘forward-looking’ component of $(\beta_1, \beta_2, \beta_3, v_6, v_8)$, which is defined as the latter dynamic coefficients net of their myopic counterpart (given the learning induced by the dynamic strategy). Such forward-looking system eliminates a component linear in $r$ present in the system that $(\beta_1, \beta_2, \beta_3, v_6, v_8)$ satisfies, and that is absent in the undiscounted version.}

**Common-value settings: fixed-point methods.** When $\alpha$ and $\delta$ cease to be proportional, $\chi$ can depend on both current and past values of $\gamma$ at all points in time (as it is the case in most coupled-ODE systems). The multi-dimensionality problem reappears.
Figure 7: As $\sigma_X$ ranges from 0 to $+\infty$ the signaling coefficient starts close to the public benchmark, and gradually becomes closer to the no-feedback case counterpart.

Observe that finding a solution to any given instance of the BVP (23) is, mathematically, a fixed-point problem. Specifically, notice that the static Nash equilibrium at time $T$ depends on the value that $\chi$ takes at that point. The latter value, however, depends on how much signaling has taken place along the way, i.e., on values of the coefficients $\bar{\beta}$ at times prior to $T$. Those values, in turn, depend on the value of the equilibrium coefficients at $T$ by backward induction, and we are back to the same point where we started.

Our approach therefore applies a fixed-point argument adapted from the literature on BVPs with intertemporal linear constraints (Keller, 1968) to our problem with intratemporal nonlinear constraints. Because the method is novel and has the generality required to become useful in other asymmetric settings, we briefly elaborate on how it works.\(^{20}\)

Let $t \mapsto b_t(s, \gamma^0, 0)$ denote the solution to the forward IVP version of (23) when the initial condition is $(s, \gamma^0, 0)$, $s \in \mathbb{R}^6$, provided a solution exists. From Lemma 3, the last two components of $b$, $\gamma$ and $\chi$, always admit solutions as long as the others do; moreover, there are no constraints on their terminal values. Thus, for the fixed-point argument, we can focus on the first six components in $b := (\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8, \gamma, \chi)$ by defining the gap function

$$g(s) = B(\chi_T(s, \gamma^0, 0)) - D_T \int_0^T f(b_t(s, \gamma^0, 0))dt.$$ 

This function measures the distance between the total growth of $(\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8)$ (last term in the display), and its target value, $B(\chi_T(s, \gamma^0, 0))$. By (24), $B(\chi)$ is nonlinear: the static Nash equilibrium imposes nonlinear relationships across variables at time $T$.\(^{21}\)

\(^{20}\)Our approach is inspired by Theorem 1.2.7 in (Keller, 1968), the proof of which is not provided.

\(^{21}\)The function $g$ takes only the first six components of $b$ because there are no “shooting” constraints on $\gamma$ and $\chi$. Yet, one is not really dispensing with $(\gamma, \chi)$, as this pair does affect $(\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8)$. 

30
Critically, using that, by definition, \( b_0(s, \gamma^0, 0) = s \), it follows that

\[
g(s) = s \iff \mathbf{B}(\chi_T(s, \gamma^0, 0)) = s + \mathbf{D}_T \int_0^T f(\mathbf{b}_t(s, \gamma^0, 0)) dt = \mathbf{D}_T \mathbf{b}_T(s, \gamma^0, 0),
\]

where the last equality follows from the definition of the ODE-system that \( \mathbf{D}_T \mathbf{b} \) satisfies. Thus, the shooting problem (i.e., find \( s, \gamma \) s.t. \( \mathbf{B}(\chi_T(s, \gamma^0, 0)) = \mathbf{D}_T \mathbf{b}_T(s, \gamma^0, 0) \)) can be transformed to one of finding a fixed point of the function \( g \).22

The bulk of the method then consists of finding a time \( T(\gamma^0) \) and a compact set \( \mathcal{S} \) of values for \( s \) such that (i) for all \( s \in \mathcal{S} \), a unique solution \( (\mathbf{b}_t(s, \gamma^0, 0))_{t \in T(\gamma^0)} \) exists for the IVP with initial condition \( (s, \gamma^0, 0) \), and (ii) \( g \) is continuous map from \( \mathcal{S} \) to itself. The natural choice for \( \mathcal{S} \) is a ball centered around \( s_0 := \mathbf{B}(0) \), the terminal condition of the trivial game with \( T = 0 \). With this in hand, part (i) can be accomplished by bounding the solutions uniformly as in the one-dimensional shooting method, but now over \( [0, T(\gamma^0)] \times \mathcal{S} \). In turn, the continuity requirement of (ii) is guaranteed if the system of ODEs has enough regularity, while the self-map condition can be ensured due to the system scaling with \( \gamma^0 \) and \( T \).23

We can now establish our main existence result for a variation of the leading-by-example application in which the follower’s best-reply is given by

\[
\dot{a}_t = \hat{u}_\theta \mathbb{E}_t[\theta] + \hat{\mathbb{E}}_t[a_t] \Rightarrow \delta_{1t} = \dot{u}_\theta + \alpha_3 t, \text{ where } \dot{u}_\theta > 0.
\]

(The positivity constraint ensures that (ii) in Assumption 1 is satisfied.24) The associated BVP is given by (B.32)-(B.38) in the Appendix.

**Theorem 2.** Set \( \sigma_X \in (0, \infty) \), \( \dot{u} > 0 \) and \( r = 0 \) in the leadership model. Then, there is a strictly positive function \( T(\gamma^0) \in O(1/\gamma^0) \) such that if \( T < T(\gamma^0) \), there exists a LME based on the BVP (B.32)-(B.38). In such an equilibrium, \( \alpha_3 > 0 \).

---

22 A BVP with intertemporal linear constraints differs from ours in that \( \mathbf{D}_0 \mathbf{b}_0 + \mathbf{D}_T \mathbf{b}_T = (\mathbf{B}(\chi_T)', \gamma^0, 0)' \) becomes \( A \mathbf{b}_0 + B \mathbf{b}_T = \alpha \), where \( A \) and \( B \) are not necessarily diagonal matrices and \( \alpha \) is a constant vector. Thus, unlike in our analysis, one may not be able to dispense with a subset of the system (even if the associated ODEs can be shown to exist independently from the rest): when \( A \) and \( B \) are not diagonal, the analog of \( g(\cdot) \) in that case may carry constraints on all coordinates. A complication that arises in our setting, however, is that our version of \( \alpha \) is a nonlinear function of a subset of components of \( \mathbf{b}_T \). This requires estimating \( \mathbf{B}(\chi_T(s, \gamma^0, 0)) \) for all values of \( s \) over which \( g(\cdot) \) must be shown to be a self-map.

23 In general, it is more useful to work with a change of variables that eliminates \( 1 - \chi_t \) from denominator in the system, and which reflects play when the state variable \( L \) is replaced by \( (1 - \chi)L \). Having shown existence of the associated BVP in this case, we can then recover a solution to our original BVP by reversing the change of variables and applying Lemma 3 (which ensures that \( 1 - \chi_t > 0 \) for all \( t \in [0, T] \), and hence that the right-hand side of our system of interest is well-defined). This approach avoids the unnecessary task of finding a uniform upper bound for \( \chi \) that is strictly less than 1, and that would be required at the moment of bounding the system uniformly. In all cases, \( \gamma_t \in [0, \gamma^0] \) due the IVP under consideration being in its forward version (Lemma 3).

24 Since \( \dot{U}_{a\theta} = \dot{u}_\theta > 0, \dot{U}_{a\dot{a}} = \dot{u}_a = 1 \) and \( U_{a\theta} = 1/2 > 0 \), it follows that \( U_{a\theta}(U_{a\theta} + U_{a\dot{a}}\dot{U}_{a\theta}) > 0 \).
We conclude with three observations that distill from this theorem. First, the self-map condition, while not affecting the order of \( T(\gamma^o) \) relative to a traditional one-dimensional shooting case, is not vacuous either. In fact, since \( s_0 = B(0) \) is the center of \( S \), we have that

\[
g(s) - s_0 = B(\chi_T(s, \gamma^o, 0)) - B(0) - D_T \int_0^T f(b_t(s, \gamma^o, 0)) dt.
\]

Thus, bounding \( B(\chi_T(s, \gamma^o, 0)) - B(0) \) imposes an additional constraint relative to those that ensure that the system is uniformly bounded (and which guarantee that the last term in the previous expression is bounded too). In other words, the self-map condition reduces the constant of proportionality in \( T(\gamma^o) \in O(1/\gamma^o) \).

Second, the set of times for which a LME is guaranteed to exist increases without bound as \( \gamma^o \downarrow 0 \): this is because the rate of growth of the system of ODEs scale with this parameter, and so its solution converges to the full-information \((v_6, v_8, \beta_1, \beta_2, \beta_3, \chi, \gamma) = (0, 0, 0, 0, 1, 0, 0)\), which is defined for all \( T > 0 \).

Finally, the bound \( T(\gamma^o) \) is obtained under minimal knowledge of the system: it imposes crude bounds that only use the degree of the polynomial vector \( f(b) \), and that do not exploit any relationship between the coefficients. Thus, the proof technique is both general and improvable, provided more is known about the system in specific settings.

### 5 Extensions

As noted in Remark 1, our model can be generalized to accommodate a quadratic terminal payoff or to allow the long-run player to affect the public signal. To demonstrate, we first explore a political setting a politician’s payoff depends on her terminal reputation, and then trading model a la Kyle (1985) exhibiting private monitoring of an insider’s trades.

#### 5.1 Reputation for Neutrality

We consider an application in which the long-run player is an expert or politician with career concerns. The politician has a hidden ideological bias \( \theta \) and takes repeated actions.

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25Inspection of the \( \vec{\beta} \)-ODEs in the previous BVP indicates that \( v_6 \) and \( v_8 \) are always appear multiplied by \( \gamma \). Thus, we can instead look at the system with \( \tilde{u}_i = \gamma v_i, i = 6, 8, \) and these ODEs scale with \( \gamma \). Since the system is uniformly bounded, \( \gamma \) never vanishes, and we can recover \( v_i, i = 6, 8 \).
— for example, adopting positions on critical issues\textsuperscript{26} or making campaign promises.\textsuperscript{27} She receives utility from taking actions that conform to her bias but also from attaining a neutral reputation at the end of the horizon; hence, she must trade off her ideological desires with her career concerns.

We model this specification with

$$-\int_0^T e^{-rt}(\hat{a}_t - \theta)^2 dt - e^{-rT}\psi\hat{a}_T^2$$

as the payoff for the long-run player, where $\psi > 0$ is common knowledge and governs the intensity of career concerns, and a flow payoff of $\hat{U}(a_t, \hat{a}_t, \theta) = - (\hat{a}_t - \theta)^2$ for the myopic player. Since the myopic player optimally chooses $\hat{a}_t = \hat{M}_t$ at each $t \in [0, T]$, the long-run player’s termination payoff is effectively $-e^{-rT}\psi\hat{M}_T^2$. The myopic player can be interpreted as a decision-maker (or in reduced form, an electorate) whose actions are direct communication, journalism, or opinion polls which convey his belief about the long-run player.

As in the leading by example application, we study the role of public feedback in determining learning and payoffs for the long-run player in equilibrium. Note that the direct effect on payoffs of removing public feedback is negative: due to the concavity of the termination payoff, greater uncertainty about the myopic player’s belief hurts the long-run player. However, an indirect effect runs the opposite direction. All else equal, the long-run player prefers higher actions when her type is higher, and hence her equilibrium strategy attaches positive weight to her type. But the concavity of the termination payoff implies that the greater the perceived value of $\hat{M}$, the greater the incentive the long-run player has to manipulate it downward. Higher types therefore must offset higher beliefs from their perspectives, leading to a negative history-inference effect, which dampens the signaling coefficient $\alpha$. With reduced signaling, the belief is less volatile from an ex ante perspective, which improves payoffs due to the concavity of the objective function.\textsuperscript{28} Indeed, provided the objective is not too concave, the indirect effect dominates, and the politician is better off:

\textsuperscript{26}Mayhew (1974) in a classic political science text outlines three kinds of activities congresspeople engage in for electoral reasons: advertising, credit claiming, and (as in the current model) position taking. He describes the dynamic nature of position taking:

...it might be rational for members in electoral danger to resort to innovation. The form of innovation available is entrepreneurial position taking, its logic being that for a member facing defeat with his old array of positions, it makes good sense to gamble on some new ones.

\textsuperscript{27}Campaign promises may be costly either due to a politician’s honesty (Callander and Wilkie, 2007) or because the electorate might not reelect politicians who renege on promises (Aragonès et al., 2007).

\textsuperscript{28}It is easy to show that the ex ante expectation of $\hat{M}_T^2$ is $\gamma_0 - \gamma_T$, so that greater learning by the myopic player results in larger terminal losses for the long-run player.
Proposition 5. Suppose that $\psi < \sigma_Y^2/\gamma^o$ and $r = 0$. Then, for all $T > 0$: (i) there are unique LME in the public and no feedback cases, and (ii) learning is lower and ex ante payoffs higher in the no feedback case.

Proposition 5 highlights one mechanism through which an expert might benefit from committing to not following polls or journalism that publicly convey her reputation for bias. The present environment is one of common values. Hence, one can establish the existence of a LME in the interior version of this problem with analogous methods to those in Section 4.3. The only difference is that our baseline model had terminal conditions that were a function of $\chi$ exclusively, whereas the presence of a terminal payoff delivers a terminal condition for $\beta_1$ that also depends on $\gamma$ according to

$$\beta_{1T} = -\frac{\psi\gamma_T}{\sigma_Y^2 + \psi\gamma_T\chi_T},$$

reflecting the fact that the incentive to manipulate the myopic player’s belief in the final moment is decreasing in the precision of that belief. To the extent that $B$ depending on $\gamma$ and $\chi$ is of class $C^1$, our fixed point method goes through.\(^{29,30}\)

5.2 Insider Trading

An asset with fixed fundamental value $\theta$, is traded in continuous time until date $T$ when its fundamental value is revealed, ending the game. The long-run player, or insider, privately observes $\theta$ prior to the start of the game. The myopic player has a technology which allows him to obtain private, noisy signals of the insider’s trades, as in Yang and Zhu (2018). Both players and a flow of noise traders submit continuous orders to a third party, the market maker, who executes those trades at a price $L_t$, which is public information.

We depart from the baseline model along three dimensions. First, the myopic player’s flow payoff depends on $L$ according to $\xi(\theta - L)\hat{a} - \frac{\hat{a}^2}{2}$, where $\xi \geq 0$, the interpretation being that $L$ is the action of the market maker.\(^{31}\) Second, the long-run player’s flow payoff is

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\(^{29}\)The only adjustment needed is in proving the self-map condition for $g$, where $B_\gamma$ appears.

\(^{30}\)We can also study the case $\psi < 0$, where the long-run player wants to appear as extreme as possible at the end of the horizon. In that case, the history-inference effect becomes positive; higher types forecast higher beliefs by the myopic player and have greater incentive to further manipulate those beliefs. The history-inference effect thus amplifies the signaling coefficient, which benefits the long-run player by increasing learning and hence the terminal reward. This effect reinforces the positive direct effect of greater uncertainty in the presence of a convex terminal payoff, so the long-run player again prefers the environment with no feedback. This result is valid for mildly convex rewards, as $\beta_{1T}$ is not well-defined if $\psi$ is too negative.

\(^{31}\)The quadratic loss term strengthens our non-existence result, as it limits the myopic player’s ability to exploit the private information he acquires. The parameter $\xi$ can then be interpreted as the size of the myopic player or (inverse of) his transaction costs.
simply \((\theta - L_t)a_t\), i.e., it is linear in \(a_t\). Finally, the public signal now includes the long-run player’s action: \(dX_t = (a_t + \hat{a}_t)dt + \sigma_X dZ^X_t\). Hence, the myopic player learns from both the private monitoring channel and the public price.

Following the literature, we seek an equilibrium in which the informed trader reveals her private information gradually over time through a linear strategy of the form (12). Hence, we require that the coefficients of the insider’s strategy be \(C^1\) functions over strict compact subsets of \([0, T)\).\(^{32}\) We can then apply Lemmas 2 and 3 to such sets.\(^{33}\)

Clearly, when \(\xi = 0\), the model reduces to the classic model of Kyle (1985) (see also Back (1992)), and hence a LME with trading strategy of the form \(\beta_3(\theta - L)\) always exists. This is not the case when \(\xi > 0\).

**Proposition 6.** Fix \(\xi > 0\). Then for all \(\sigma_Y > 0\), there does not exist a linear Markov equilibrium of the insider trading game.

The intuition for this result is as follows. As the myopic player privately observes a signal of the insider’s trades, he acquires private information about \(\theta\) over time. The myopic player’s own trades then carry further information to the market maker, beyond that which the market maker learns from the insider alone. This introduces momentum into the law of motion for the price from the insider’s perspective, measured by a term \(\xi(m - l)\) in the price drift; the insider’s trading at any time not only causes an immediate price impact but also sets forth continued future price impacts as the myopic player’s trades continue to inform the market maker. These repeated price impacts via the myopic player make future trades less attractive to the insider, thereby putting the insider in a “race against herself” and inducing her to trade away all information in the first instant.

This result is intimately related to a non-existence result in Yang and Zhu (2018). In a two-period model, they show that a linear equilibrium ceases to exist if the private signal of a back-runner—a trader who only participates in the last round after receiving noisy information of the informed player’s first-period trade—is sufficiently precise, situation in which a mixed strategy equilibrium emerges. More generally, the existence problem relates to how, with pure strategies, an informed player’s rush to trade depends on the number of trading opportunities in certain settings. In this line, Foster and Viswanathan (1994) show, in an asymmetric environment where one long-run trader’s information nests another’s, that the better informed trader quickly trades the commonly known piece of information to exploit

\(^{32}\)By not imposing this requirement over \([0, T]\), we maintain the possibility of full revelation of the insider’s information through an explosion of trades near the end of the game, as is standard in insider trading models. In addition, this requirement ensures that the total order can be “inverted” from the price, and hence it is without loss to make \(X\) public to all players.

\(^{33}\)Specifically, the proof of Lemma 2 provides the learning ODEs for the case \(\nu > 0\), and it is easy to see that the steps of Lemma 3 (with \(\hat{u}_\theta = \xi, \hat{u}_a = 0\)) go through for this case.
her superior information only later on. While there are important differences between our settings (the belief of the lesser informed player is, in their model, always known to the first, and their common information exogenous) there is a common theme: once common information is created (either exogenously or endogenously), there is a pressure to trade quickly on it. Such pressure is increasing in the number of trading opportunities.\footnote{In symmetric settings, \cite{Holden1992} show that intense trading occurs in early periods between two identically informed traders, and \cite{Back2000} obtain the corresponding nonexistence result directly in continuous time.}

## 6 Conclusion

We have examined the implications of a minimal—yet natural—departure from an extensive literature on signaling games: namely, that the signal observed by a receiver is both noisy and private. We showed that, unlike in settings where such a signal is public, the sender’s history of play affects the informativeness of her actions at all points in time, and we explored the learning and payoff implications of such history-inference effect in applications. In the process, we have introduced an approach for establishing the existence of LME in dynamic games of asymmetric learning. Let us now discuss three assumptions of the model: its asymmetry, the presence of a myopic player, and the linear-quadratic-Gaussian structure.

The asymmetry of the environment studied indeed provides us with enough tractability, in the sense that it allows us to “close” the set of states at the second order. If instead the long-run player had a stochastic type, or access to an imperfect private signal, even higher-order beliefs would become payoff-relevant states. While some economic environments may feature some of these assumptions, a natural question that arises is whether we believe economic behavior in such settings is effectively driven by such higher-order inferences.

Second, the presence of a myopic player is not a major technical limitation. In fact, most of the results are derived for, or can be generalized to, continuous coefficients $\delta$. With a long-run “receiver” such coefficients solve ODEs capturing optimality and correct beliefs but (i) no additional states are needed, and (ii) the fixed-point argument is applicable (to an enlarged boundary value problem).

Finally, the linear-quadratic-Gaussian class examined is clearly a limited class. Yet, its advantage lies in that it is a powerful framework for uncovering economic effects that are likely to be key in other, more nonlinear, environments. From that perspective, the way in which the inference of others’ private histories interacts with payoffs in shaping signaling, along with the time-effects that learning has on incentives, seem to exhaust the set of effects that we would expect to be of first order in other settings.
Appendix A: Proofs for Section 2

Proofs for Section 2.1

Since player 2 attempts to match player 1’s action, we have

\[ \hat{a}_t = \mathbb{E}_t[\beta_0 + \beta_1 M_t + \beta_3 \theta] \]
\[ = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_3 \end{bmatrix}_t M_t. \]

The HJB equation for player 1 is

\[ rV(\theta, m, t) = \sup_a \left\{ -\left( a - \theta \right)^2 - \left( a - \hat{a}_t \right)^2 + \Lambda_t \mu_t(a) V_M(\theta, m, t) + \frac{\Lambda_t^2 \sigma_Y^2}{2} V_{MM}(\theta, m, t) + V_t(\theta, m, t) \right\}, \]

(A.1)

where

\[ \Lambda_t := \frac{\beta_3 \gamma_t}{\sigma_Y^2} \]
\[ \mu_t(a) := a - \beta_0 - (\beta_1 + \beta_3)m. \]

To obtain the maximizer of the RHS of (A.1), we impose the first order condition

\[ 0 = -2(a - \theta) - 2(a - \beta_0 - (\beta_1 + \beta_3)m) + \frac{\beta_3 \gamma_t[v_{2t} + 2m v_{4t} + \theta v_{5t}]}{\sigma_Y^2} \]
\[ \implies 0 = -2(\beta_0 + \beta_1 m + \beta_3 \theta - \theta) - 2\beta_3(\theta - m) + \frac{\beta_3 \gamma_t[v_{2t} + 2m v_{4t} + \theta v_{5t}]}{\sigma_Y^2}, \]

(A.2)

where in the second line we have used that the maximizer must be \( a^* := \beta_0 + \beta_1 m + \beta_3 \theta \). Since A.2 must hold for all \((\theta, m, t) \in \mathbb{R}^2 \times [0, T]\), the coefficients on \( \theta, m \) and \( t \) must vanish, and we obtain

\[ v_{2t} := \frac{2\sigma_Y^2 \beta_0}{\beta_3 \gamma_t}, \]
\[ v_{4t} := \frac{\sigma_Y^2 (\beta_1 - \beta_3)}{\beta_3 \gamma_t}, \]
\[ v_{5t} := \frac{2\sigma_Y^2 (2\beta_3 - 1)}{\beta_3 \gamma_t}. \]

(A.3) (A.4) (A.5)
Since $v_{iT} = 0$ for all $i \in \{0, 1, \ldots, 5\}$, (A.3)-(A.5) imply the terminal values of the coefficients

\begin{align*}
\beta_{0T} &:= 0 \quad \text{(A.6)} \\
\beta_{1T} &:= \frac{1}{2} \quad \text{(A.7)} \\
\beta_{3T} &:= \frac{1}{2} \quad \text{(A.8)}
\end{align*}

which are also the myopic equilibrium coefficients, that is, the coefficients for the game in which both players act myopically at each instant (while still updating their beliefs).

Substituting $a^*$ into (A.1) yields

\begin{align*}
0 = &-r[v_0t + v_1t\theta + v_2tm + v_3t\theta^2 + v_4tm^2 + v_5t\theta m] \\
&- [\beta_{0t} + m\beta_{1t} + \theta(\beta_{3t} - 1)]^2 - (m - \theta)^2\beta_{3t}^2 - \frac{(m - \theta)[v_2t + 2mv_4t + \theta v_5t]\beta_{3t}^2\gamma_t}{\sigma_Y^2} \\
&+ \frac{v_4t\beta_{3t}^2\gamma_t^2}{\sigma_Y^2} + \dot{v}_0t + \dot{v}_1t\theta + \dot{v}_2tm + \dot{v}_3t\theta^2 + \dot{v}_4tm^2 + \dot{v}_5t\theta m, \quad \text{(A.9)}
\end{align*}

which again must hold for all $(\theta, m, t) \in \mathbb{R}^2 \times [0, T]$. Using (A.3)-(A.5) and their derivatives, we eliminate $(v_2t, v_4t, v_5t, \dot{v}_2t, \dot{v}_4t, \dot{v}_5t)$ from (A.9) to obtain a new equation. As the constant term and the coefficients on $\theta, m, \theta^2, m^2$ and $\theta m$ in this new equation must vanish, we obtain along with (A.13) a system of ODEs for $(v_0, v_1, v_3, \beta_0, \beta_1, \beta_3, \gamma)$. This system contains a subsystem for $(\beta_0, \beta_1, \beta_3, \gamma)$:

\begin{align*}
\dot{\beta}_{0t} &= 2r\beta_{0t}\beta_{3t} \quad \text{(A.10)} \\
\dot{\beta}_{1t} &= \beta_{3t} \left[ r(2\beta_{1t} - 1) + \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2} \right] \quad \text{(A.11)} \\
\dot{\beta}_{3t} &= \beta_{3t} \left[ r(2\beta_{3t} - 1) - \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2} \right] \quad \text{(A.12)} \\
\dot{\gamma}_t &= -\frac{(\beta_{3t}\gamma_t)^2}{\sigma_Y^2}. \quad \text{(A.13)}
\end{align*}

Hence, a linear Markov equilibrium must solve the following boundary value problem: (A.10)-(A.13) with boundary conditions $\gamma_0 = \gamma^o$ and (A.6)-(A.8). We show that a solution to this boundary value problem always exists.

It is useful to transform this boundary value problem into an initial value problem by reversing the direction of time and making a guess for the initial value of $\gamma$. Hence, we define the backward system
\[ \dot{\beta}_0 = -2r \beta_0 \beta_3 t \]  
(A.14)

\[ \dot{\beta}_1 = \beta_3 \left[ r(1 - 2\beta_1) - \frac{\beta_1 \beta_3 \gamma t}{\sigma_y^2} \right] \]  
(A.15)

\[ \dot{\beta}_3 = \beta_3 \left[ r(1 - 2\beta_3) + \frac{\beta_1 \beta_3 \gamma t}{\sigma_y^2} \right] \]  
(A.16)

\[ \dot{\gamma}_t = \frac{\beta_3^2 \gamma_t^2}{\sigma_y^2}, \]  
(A.17)

with initial conditions \( \beta_{00} = 0, \beta_{10} = \beta_{30} = \frac{1}{2} \) and \( \gamma_0 = \gamma^F \geq 0 \).

Define \( B_t^{pub} := \beta_{1t} + \beta_{3t} \).

**Lemma A.1.** In any solution to the backward system over an interval \([0, T]\), if \( \gamma^F > 0 \), then (i) \( B_t^{pub} = 1 \) for all \( t \in [0, T] \), (ii) \( \beta_{3t} \in (1/2, 1) \) and \( \beta_{1t} \in (0, 1/2) \) for all \( t \in (0, T] \), (iii) \( \beta_3 \) is monotonically increasing while \( \beta_1 \) is monotonically decreasing, and (iv) \( \gamma \) is strictly increasing. If \( \gamma^F = 0 \), then \( \beta_{1t} = \beta_{3t} = \frac{1}{2} \) and \( \gamma_t = 0 \) for all \( t \in [0, T] \). For any \( \gamma^F, \beta_0 = 0 \).

**Proof of Lemma A.1.** We first claim that if a solution exists over some interval \([0, T]\), then \( \beta_{3t} > 0 \) for all \( t \in [0, T] \). To see this let \( f^{3t}(t, \beta_{3t}) \) denote the RHS of (A.16). Letting \( x_t := 0 \) for all \( t \in [0, T] \), we have \( \beta_{30} = 1/2 > x_0 \) and \( \dot{\beta}_{3t} - f^{3t}(t, \beta_{3t}) = 0 = \dot{x}_t - f^{3t}(t, x_t) \). By the comparison theorem in Teschl (2012, Theorem 1.3), the claim holds.

Next, we define \( B_t^{pub} := \beta_{1t} + \beta_{3t} \) and show that \( B_t^{pub} = 1 \). Adding (A.15) and (A.16) yields

\[ \dot{B}_t^{pub} = 2r \beta_3 (1 - B_t^{pub}), \]

which, given the initial condition and any \( \beta_{3t} \), has solution of the form \( B_t^{pub} = 1 - \tilde{C} e^{-\int_0^t 2r \beta_3 \, ds} \).

Since \( B_0^{pub} = 1 = 1 - \tilde{C} \), we have \( \tilde{C} = 0 \) and \( B_t^{pub} = 1 \).

Hence we can rewrite the \( \beta_{3t} \) ODE as

\[ \dot{\beta}_3 = \beta_3 \left[ r(1 - 2\beta_3) + \frac{\beta_3(1 - \beta_3) \gamma t}{\sigma_y^2} \right]. \]  
(A.18)

We now show that \( \beta_3 < 1 \). Let \( f^{3t}(t, \beta_{3t}) \) now denote the RHS of (A.18), and define \( x_t := 1 \) for all \( t \in [0, T] \). Then \( x_0 = 1 > \beta_{30} = \frac{1}{2} \), and \( \dot{\beta}_{3t} - f^{3t}(t, \beta_{3t}) = 0 \leq r = \dot{x}_t - f^{3t}(t, x_t) \), so by the comparison theorem, the claim holds. Since \( \beta_{1t} = 1 - \beta_{3t} \), we have \( \beta_1 > 0 \).

Consider the case \( \gamma^F = 0 \). Since \( \beta_3 > 0, \gamma_t = 0 \) is the unique solution to (A.17). Letting \( z_t := \beta_{3t} - \beta_{1t} \), we have \( \dot{z}_t = -2r \beta_3 \gamma t \). As this is a linear ODE with initial condition \( z_0 = 0 \),
the unique solution is $z_t = 0$ for all $t \in [0, T]$. Since $\beta_{1t} + \beta_{3t} = 1$, we have $\beta_{1t} = \beta_{3t} = 1/2$ for all $t \in [0, T]$, proving the claim in the proposition statement.

Next consider the case $\gamma^F > 0$. Since $\beta_3 > 0$, (A.17) implies $\gamma$ is strictly increasing, and hence $\gamma_t > 0$ for all $t \in [0, T]$. Now whenever $\beta_{3t} = \frac{1}{2}$, we have $\dot{\beta}_{3t} = \frac{1}{2} \left[ 0 + \frac{\eta}{4\sigma^2_Y} \right] > 0$, and thus $\beta_{3t} > 1/2$ for all $t \in (0, T]$. Since $\beta_{1t} = 1 - \beta_{3t}$, we have $\beta_{1t} < 1/2$ for all such $t$.

We now turn to (iii). Since $\dot{\beta}_{1t} + \dot{\beta}_{3t} = 0$ for all $t \in [0, T]$, it suffices to show that $\dot{\beta}_{3t} > 0$ for all $t \in [0, T]$; in turn, it suffices to show that $H_t := r(1 - 2\beta_{3t}) + \frac{\beta_{3t}(1-2\beta_{3t})}{\sigma^2_Y} > 0$ for all $t \in [0, T]$, where $\dot{H}_t = \frac{(1-\beta_{3t})\beta_{3t}^2}{\sigma^2_Y} > 0$. It follows that $H_t > 0$ for all $t \in [0, T]$, as desired.

Finally, note that in all cases, we have $\beta_3 > 0$, so the unique solution to (A.14) consistent with the initial condition $\beta_0 = 0$ is $\beta_0 = 0$.

**Proof of Proposition 1.** Existence of LME reduces to existence of a solution to the boundary value problem defined by (A.10)- (A.13) with associated boundary conditions. Using the uniform bounds in Lemma A.1, we can obtain existence using an analogous backward shooting argument to the one used in Theorem 1. It is then straightforward to verify that the remaining value function coefficients are well-defined and that the HJB equation is satisfied. The remaining claims have been established in Lemma A.1.

We conclude with a derivation of the closed form solution for the undiscounted case, to which we refer in Propositions 3 and 4.

**Lemma A.2.** For $r = 0$, the leading by example game has a unique LME for the public case, and $(\beta_0, \beta_1, \beta_3, \gamma)$ can be expressed in closed form.

**Proof.** We derive the following closed form solution:

$$\gamma_t = \frac{\gamma T}{2} + \frac{1}{2} \frac{1}{\gamma T} \frac{T-t}{\sigma^2_Y} \quad \text{(A.19)}$$

$$\beta_{3t} = \frac{1}{2} - \frac{\gamma T}{2\sigma^2_Y}, \quad \text{where} \quad \text{(A.20)}$$

$$\gamma_T = \frac{\gamma^o T + 2\sigma^2_Y - \sqrt{(\gamma T)^2 + 4\sigma^4_Y}}{T} \quad \text{(A.21)}$$

and where $\beta_0 \equiv 0$ and $\beta_1 \equiv 1 - \beta_3$.

Observe that $\dot{\beta}_{3t} \gamma_t + \dot{\beta}_{3t} \gamma_t = \frac{\beta_{3t}^2 \gamma^2_t}{\sigma^2_Y}$. Hence, define $\Pi_t := \beta_{3t} \gamma_t$, which has ODE $\dot{\Pi}_t = \frac{\Pi_t^2}{\sigma^2_Y}$
with initial condition \( \Pi_0 = \beta_{30}\gamma^F = \gamma^F/2 \); its solution is

\[
\Pi_t = \frac{1}{2 \gamma_{\text{pub}} - \frac{t}{\sigma^2}}.
\]

Substitute \( \Pi \) into (A.17) to obtain

\[
\dot{\gamma}_t = \frac{1}{\sigma^2} \left( 2 \gamma_{\text{pub}} - \frac{t}{\sigma^2} \right)^2 \Rightarrow \gamma_t = C_\gamma + \frac{1}{2 \gamma_{\text{pub}} - \frac{t}{\sigma^2}}.
\]

As \( \gamma_0 = \gamma_{\text{pub}}^F \), we have \( C_\gamma = \gamma_{\text{pub}}^F/2 \) and thus

\[
\gamma_t = \frac{\gamma_{\text{pub}}^F}{2} + \frac{1}{2 \gamma_{\text{pub}} - \frac{t}{\sigma^2}}. \tag{A.22}
\]

Moreover,

\[
\gamma_T = \gamma^o = \frac{\gamma_{\text{pub}}^F}{2} + \frac{1}{2 \gamma_{\text{pub}} - \frac{T}{\sigma^2}},
\]

which is equivalent to the quadratic

\[
\frac{T}{2} \left( \gamma_{\text{pub}}^F \right)^2 - \left( \gamma^o T + 2\sigma^2 \right) \gamma_{\text{pub}}^F + 2\sigma^2 \gamma^o = 0.
\]

The quadratic on the LHS is convex and evaluates to \( 2\sigma^2 \gamma^o > 0 \) at \( \gamma_{\text{pub}}^F = 0 \) and evaluates to \( -\left( \gamma^o \right)^2 T/2 < 0 \) at \( \gamma_{\text{pub}}^F = \gamma^o \), so there is a unique solution in \( (0, \gamma^o) \) which in the forward system is \( \gamma_T \) as in the proposition statement. Substituting this into (A.22) and returning to the forward system by replacing \( t \) with \( T - t \) yields \( \gamma_t \) in the forward system. It is easy to verify that \( \gamma_t > 0 \) for all \( t \).

Lastly, we have (in the forward system) \( \beta_{3t} = \Pi_t/\gamma_t = \frac{1}{2 \gamma_{\text{pub}}(T-t) - 2\sigma^2} \) and \( \beta_{1t} = 1 - \beta_{3t} \).

Proofs for Section 2.2

Proof of Lemma 1. Let \( \beta_{0t} + \beta_{1t} \hat{M}_t + \beta_{3t}\theta \) denote the long-run player’s strategy. Thus, \( \hat{E}_t[a_t] = \alpha_{0t} + \alpha_{3t} \hat{M}_t \), where \( \alpha_0 = \beta_0 + \beta_1(1 - \chi)\bar{m}_0 \) and \( \alpha_3 = \beta_3 + \chi\beta_1 \). Then, \( d\hat{M}_t = \)
\[ \frac{\alpha_3 \gamma_{11}}{\sigma_Y^2} [dY_t - (\alpha_{0t} + \alpha_{1t} \dot{M}_t) dt], \]
so letting \( R(t, s) = \exp(-\int_s^t \frac{\alpha_3 \gamma_{11}}{\sigma_Y^2} du) \)

\[
\dot{M}_t = \dot{m}_0 R(t, 0) + \int_0^t R(t, s) \frac{\alpha_3 \gamma_{11}}{\sigma_Y^2} [(a_{2t} - \alpha_{0t}) dt + \sigma_Y dZ_t^Y] \\
\Rightarrow M_t = \dot{m}_0 R(t, 0) + \int_0^t R(t, s) \frac{\alpha_3 \gamma_{11}}{\sigma_Y^2} (a_{2t} - \alpha_{0t}) dt \\
\gamma_{1t} = -\frac{\gamma_1^2 \alpha_3^2}{\sigma_Y^2}. \quad (A.23)
\]

On the path of play, however, \( a_{2t} = \beta_{0t} + \beta_{11} M_t + \beta_{3t} \theta \), so we obtain

\[
M_t = \dot{m}_0 R(t, 0) + \theta \int_0^t R(t, s) \frac{\alpha_3 \gamma_{11}}{\sigma_Y^2} [- (1 - \chi_s) \beta_{1s} \dot{m}_0 + \beta_{1s} M_{2s} + \beta_{3s} \theta] ds
\]
where we used that \( \beta_{0s} - \alpha_{0s} = -(1 - \chi_s) \beta_{1s} \dot{m}_0 \). In particular,

\[
dM_t = \left( -M_{2t} \left[ \frac{\alpha_3 \gamma_s}{\sigma_Y^2} (\alpha_{3s} + \beta_{1s}) \right] + \frac{\alpha_3 \gamma_s}{\sigma_Y^2} [-\beta_{1s}(1 - \chi_s) \dot{m}_0 + \beta_{3s} \theta] \right) dt,
\]
and so, letting \( \tilde{R}(t, s) = \exp(-\int_s^t \frac{\alpha_3 \gamma_{11}}{\sigma_Y^2} (\alpha_{3u} - \beta_{1u}) du) \),

\[
M_t = \dot{m}_0 \left( \tilde{R}(t, 0) - \int_0^t \tilde{R}(t, s) \frac{\alpha_3 \gamma_s}{\sigma_Y^2} \beta_{1s}(1 - \chi_s) ds \right) + \theta \int_0^t \tilde{R}(t, s) \frac{\alpha_3 \gamma_s}{\sigma_Y^2} \beta_{3s} ds.
\]

From here

\[
\chi_t = \int_0^t \tilde{R}(t, s) \frac{\alpha_3 \gamma_s}{\sigma_Y^2} \beta_{3s} ds \quad \text{and} \quad 1 - \chi_t = \tilde{R}(t, 0) - \int_0^t \tilde{R}(t, s) \frac{\alpha_3 \gamma_s}{\sigma_Y^2} \beta_{1s}(1 - \chi_s) ds.
\]

Critically, observe that the second constraint is a direct consequence of the first. Specifically, adding and subtracting \( \beta_{3s} \) and noticing that \( \alpha_{3s} - \beta_{1s} = \beta_{3s} - \beta_{1s}(1 - \chi_s) \) we can write

\[
- \int_0^t \tilde{R}(t, s) \frac{\alpha_3 \gamma_s}{\sigma_Y^2} \beta_{1s}(1 - \chi_s) ds = \int_0^t \tilde{R}(t, s) \frac{\alpha_3 \gamma_s}{\sigma_Y^2} [\alpha_{3s} - \beta_{1s}] ds - \chi_t
\]

\[
= \int_0^t \frac{d \tilde{R}(t, s)}{ds} ds - \chi_t
\]

\[
= 1 - \tilde{R}(t, 0) - \chi_t \quad (A.24)
\]

where in the last equality we used that \( \tilde{R}(t, t) = 1 \).

With this in hand, the relevant constraint is the first. In differential form, and using that
\[ \alpha_{3t} = \beta_3 + \beta_1 \chi, \]

\[ \dot{\chi}_t = -\frac{\alpha_{3t}}{\sigma_Y^2}[\alpha_{3t} - \beta_{1t}] \chi_t + \frac{\alpha_{3t} \beta_{3t} \gamma_t}{\sigma_Y^2} = \frac{(\beta_{3t} + \beta_{1t} \chi_t)^2 \gamma_t}{\sigma_Y^2}(1 - \chi_t) = \frac{\alpha_{3t}^2 \gamma_t}{\sigma_Y^2}(1 - \chi_t). \]

Using the exact same arguments in the proof of Lemma 3, we conclude that the resulting \((\chi, \gamma)\) system admits a unique solution with the same properties as in that lemma. Furthermore, it is easy to check that \(1 - \gamma_t/\gamma^o\) satisfies the \(\chi\)-ODE, concluding the proof.

We now turn to characterizing equilibrium for the no feedback case. From Lemma 1, given a conjecture by the follower about \((\alpha, \gamma)\), the variance of the follower's belief evolves deterministically as \(\dot{\chi}_t = -\frac{\alpha_2}{\sigma_Y^2} \chi_t^2\) and \(\chi \equiv 1 - \gamma/\gamma^o\).

The follower matches the expectation of the leader's action by playing

\[ \hat{\mu}_t = \hat{\beta}_t^2[\beta_0 + \beta_{1t} \tilde{M}_t + \beta_3 \theta] + \hat{\beta}_t[\beta_0 + \beta_{1t} (\tilde{m}_0[1 - \chi_t] + \chi_t \theta) + \beta_3 \theta] \]

Define \(\mu_{1t} := \alpha_{1t} \gamma_t / \sigma_Y^2\), \(\nu_{0t} = -\mu_{1t}[\beta_0 + \beta_{1t} \tilde{m}_0(1 - \chi_t)]\) and \(\nu_{2t} = -\alpha_{1t} \nu_{1t}\), where \(\alpha_{1t} = \beta_{1t} \chi_t + \beta_3 \).

The HJB equation is

\[ rV(\theta, m, t) = \sup_a \left\{ -\left( a^2 - 2a[\delta_{0t} + \delta_{1t} m] + \delta_{0t}^2 + 2\delta_{0t}\delta_{1t} m + \delta_{1t}^2[\gamma_t \chi_t + m^2] \right) \right\} = \mathbb{E}_t[(a - \hat{a}_t)^2] \]

\[ -(a - \theta)^2 + (\nu_{0t} + m \nu_{1t} + m^2 \nu_{2t})V_m(\theta, m, t) + V_t(\theta, m, t), \]

where we have used that \(\mathbb{E}_t[\tilde{M}_t] = M_t = m\) and \(\mathbb{E}_t[\tilde{M}_t^2] = \left( \mathbb{E}_t[\tilde{M}_t] \right)^2 + \text{Var}[\tilde{M}_t] = m^2 + \gamma_t \chi_t.\)

Imposing the first order condition on the RHS of (A.25) and evaluating at \(a = \beta_0 + \beta_{1t} m + \beta_3 \theta\), yields

\[ 0 = -4[\beta_0 + \beta_{1t} m + \beta_3 \theta] + 2\theta + \frac{\alpha_t \gamma_t [v_{2t} + 2m v_{4t} + \theta v_{5t}]}{\sigma_Y^2} + 2[\beta_0 + \tilde{m}_0 \beta_{1t} (1 - \chi_t) + m \alpha_t]. \]

(A.26)
Matching coefficients, we obtain
\[ v_{2t} = \frac{2\sigma_Y^2[\beta_{0t} - \hat{m}_0\beta_{1t}(1 - \chi_t)]}{\alpha_t\gamma_t} \] (A.27)
\[ v_{4t} = \frac{-\sigma_Y^2[\beta_{3t} - \beta_{1t}(2 - \chi_t)]}{\alpha_t\gamma_t} \] (A.28)
\[ v_{5t} = \frac{2\sigma_Y^2[2\beta_{3t} - 1]}{\alpha_t\gamma_t} \] (A.29)
provided that \( \alpha_t, \gamma_t > 0 \), which we will confirm below. Setting \( v_{iT} = 0 \) for \( i = 2, 4, 5 \) yields the terminal conditions stated below. As in the public benchmark, these conditions also yield the equilibrium coefficients for the case where both players behave myopically.

We now evaluate (A.25) at \( a = \beta_{0t} + \beta_{1t}m + \beta_{3t}\theta \), using (A.27)-(A.29) and their derivatives to eliminate \( (v_{2t}, v_{4t}, v_{5t}, \dot{v}_{2t}, \dot{v}_{4t}, \dot{v}_{5t}) \) and in turn using (A.33) and (A.34) to eliminate \( \dot{\gamma}_t \) and \( \dot{\chi}_t \). The resulting subsystem gives a system of ODEs for \((v_0, v_1, v_3, \beta_0, \beta_1, \beta_3, \gamma)\), which contains the following subsystem for \((\beta_0, \beta_1, \beta_3, \gamma)\):

\[ \dot{\beta}_{0t} = \frac{\alpha_t}{2\sigma_Y^2} \{ r\sigma_Y^2\beta_{0t}(2 - \chi_t) - r\sigma_Y^2(1 - \chi_t) + 2\gamma_t\beta_{1t}^2(1 - \chi_t) \} \] (A.30)
\[ \dot{\beta}_{1t} = \frac{\alpha_t}{2\sigma_Y^2} \{ -r\sigma_Y^2 + 2\beta_{1t}[\beta_{3t}\gamma_t + r\sigma_Y^2(2 - \chi_t)] - 2\beta_{1t}^2\gamma_t(1 - \chi_t) \} \] (A.31)
\[ \dot{\beta}_{3t} = \frac{\alpha_t}{2\sigma_Y^2} \{ -r\sigma_Y^2(2 - \chi_t) - 2\beta_{3t}[\beta_{1t}\gamma_t - r\sigma_Y^2(2 - \chi_t)] \} \] (A.32)
\[ \dot{\gamma}_t = -\frac{\alpha_t^2\gamma_t^2}{\sigma_Y^2} \] (A.33)
\[ \dot{\chi}_t = \gamma_t\frac{\alpha_t^2(1 - \chi_t)}{\sigma_Y^2} \] (A.34)

with boundary conditions \( \gamma_0 = \gamma^0, \beta_{0T} = \frac{1 - x_T}{2(2 - x_T)}, \beta_{1T} = \frac{1}{2(2 - x_T)}, \beta_{3T} = \frac{1}{2}, \gamma_0 = \gamma^0 \) and \( \chi_0 = 0 \). The ODE for \( \alpha \) is

\[ \dot{\alpha}_t = -r\alpha_t[1 - \alpha_t(2 - \chi_t)], \] (A.35)

and its terminal value is \( \alpha_T = \frac{1}{2 - \chi_T} \).

Let \((\beta_{0m}^m, \beta_{1m}^m, \beta_{3m}^m, \alpha_{m}^m) = \left( \frac{1 - x_T}{2(2 - x_T)}, \frac{1}{2(2 - x_T)}, \frac{1}{2}, \frac{1}{2 - \chi_T} \right)\) denote the myopic coefficients, that is the myopic equilibrium given the variance \( \gamma \) induced by the dynamic candidate equilibrium strategy.

**Proof of Proposition 2.** We work with the backward system, using an initial guess \( \gamma_0 = \gamma^F \) to formulate an initial value problem parameterized by \( \gamma^F \).
The backward system for the coefficients is

\[
\begin{align*}
\dot{\beta}_0 &= \frac{\alpha_t}{2\sigma_Y^2} \{ -r\sigma_Y^2 \beta_0(2 - \chi_t) + r\sigma_Y^2 (1 - \chi_t) - 2\gamma_t \beta_t^2 (1 - \chi_t) \} \tag{A.36} \\
\dot{\beta}_1 &= \frac{\alpha_t}{2\sigma_Y^2} \{ r\sigma_Y^2 - 2\beta_1 [\beta_3 \gamma_t + r\sigma_Y^2 (2 - \chi_t)] + 2\beta_1^2 \gamma_t (1 - \chi_t) \} \tag{A.37} \\
\dot{\beta}_3 &= \frac{\alpha_t}{2\sigma_Y^2} \{ r\sigma_Y^2 (2 - \chi_t) + 2\beta_3 [\beta_1 \gamma_t - r\sigma_Y^2 (2 - \chi_t)] \} \tag{A.38} \\
\dot{\alpha}_t &= r\alpha_t [1 - \alpha_t (2 - \chi_t)] \tag{A.39} \\
\dot{\gamma}_t &= \frac{\alpha_t^2 \gamma_t^2}{\sigma_Y^2} \tag{A.40}
\end{align*}
\]

with boundary conditions \(\beta_{00} = \frac{1 - \chi_0}{2(2 - \chi_0)}, \beta_{10} = \frac{1}{2(2 - \chi_0)}, \beta_{30} = \frac{1}{2}, \) and \(\alpha_0 = \frac{1}{2 - \chi_0} > 0.\)

By Teschl’s comparison theorem, \(\alpha > 0\) in any solution to the IVP. It follows that for \(\gamma^F = 0,\) the IVP has a unique solution \((\beta_0, \beta_1, \beta_3, \gamma, \chi) = (0, 1/2, 1/2, 0, 1).\) By continuity, suppose now that \(\gamma^F > 0\) is sufficiently small that there exists a solution to the IVP. By the same argument as in the proof of Lemma A.1, \(\gamma\) is increasing.

Given such \((\gamma_t)_{t \in [0, T]},\) we can write the right-hand side of the \(\alpha\)-ODE as a function of the form \(f^\alpha(t, \alpha)\) that is of class \(C^1.\) Using that \(\chi_t = 1 - \gamma_t / \gamma^o,\) observe that, in the backward system,

\[
\frac{d}{dt} \left( \frac{1}{2 - \chi_t} \right) - f^\alpha(t, 1/(2 - \chi_t)) = - \frac{\dot{\gamma}_t}{\gamma^o (1 + \gamma_t/\gamma^o)^2} < 0 = \dot{\alpha}_t - f^\alpha(t, \alpha_t)
\]

where the first inequality follows from \(\gamma_t\) being increasing in the backward system. Moreover, \(\alpha_0 = \frac{1}{2 - \chi_0}.\) The comparison theorem allows us to conclude that \(\alpha_t \geq 1/(2 - \chi_t),\) and in turn \(\dot{\alpha}_t \leq 0\) (and hence \(\dot{\gamma}_t \geq 0\) in the forward system), for all \(t \in [0, T]\) with both inequalities strict for \(t \in (0, T)\) (\(t \in [0, T]\) in the forward system) if and only if \(r > 0.\) It follows that for all \(t \in (0, T],\) \(\alpha_t < \alpha_0 = \frac{1}{2 - \chi_0} < 1.\)

Now by simple addition of the ODEs, we obtain that \(B^\infty_t := \beta_0 + \beta_1 + \beta_3\) satisfies

\[
\dot{B}^\infty_t = \frac{\alpha_t}{2\sigma_Y^2} \{ 2r\sigma_Y^2 (2 - \chi_t) [1 - B^\infty_t] \}, \text{ with } B^\infty_0 = 1.
\]

It is easy to see that the unique solution is \(B^\infty_t = 1\) (one can integrate the previous ODE between 0 and \(t\) small, and argue by contradiction using the continuity of \(B^\infty).\)

Next, we establish uniform bounds on \(\beta_1\) and \(\beta_3\) (and hence \(\beta_0\)). Observe that \(\beta_{30} = \beta_{30}^m = 1/2\) and \(\beta_{10} = \beta_{10}^m > 0;\) we first argue that \(\beta_{3t} > \frac{1}{2}\) and \(\beta_{1t} > 0\) for all \(t \in (0, T].\)

By way of contradiction, suppose that \(\tau \in (0, T]\) is the first strictly positive time either of these properties fails. If \(\beta_{1\tau} = 0,\) then \(\dot{\beta}_{1\tau} = r\beta_{1\tau}^3 > 0,\) a contradiction; hence \(\beta_{1\tau} > 0.\) But
if $\beta_{3r} = 1/2$, then $\dot{\beta}_{3r} = \frac{\beta_{1r} \gamma(1+2\beta_{1r} \gamma r)}{4\sigma_Y^3} > 0$, also a contradiction. We conclude that $\beta_{3t} > 1/2$ and $\beta_{1t} > 0$ for all $t \in (0, T]$.

By definition, $\alpha_t = \beta_{1t} \chi_t + \beta_{3t}$ and thus $\beta_{3t} \leq \alpha_t < 1$. Finally, note that $\beta_{10} = \beta_{10}^m < 1$ and the RHS of the $\beta_1$ ODE can be written as $f^{\beta_1}(t, \beta_1)$ of class $C^1$, and with algebra it can be shown that

$$\dot{\beta}_{1t}^m - f^{\beta_1}(t, \beta_{1t}^m) = \frac{\gamma t}{4\sigma_Y^2(2-\chi t)^4} (\beta_{3t}[2-\chi t] - [1-\chi t]) (2\beta_{3t}[2-\chi t] + \chi t)$$

$$> 0 = \dot{\beta}_{1t} - f^{\beta_1}(t, \beta_{1t}),$$

so by the comparison theorem, we have $\beta_{1t} < \beta_{1t}^m < 1$ for all $t \in (0, T]$.

By an analogous one-dimensional shooting argument as in the public case, we obtain existence of a solution to the BVP and hence existence of an LME.

The final claim to prove is that as $T \to \infty$, $\alpha_T \to 1$, for which we use the forward system. Recalling that $\alpha > 1/2$, we have $\gamma_T \to 0$ as $T \to \infty$, and thus $\chi_T \to 1$ and $\alpha_T = 1/(2-\chi_T) \to 1$.

We refer to the following result in Propositions 3 and 4.

**Lemma A.3.** For $r = 0$, the leading by example game has a unique LME for the no feedback case, expressed in closed form as follows:

$$\beta_{1t} = \frac{\gamma o[(\gamma o + \gamma_T)^2\sigma_Y^2 - (T-t)(\gamma o)^2\gamma_T]}{(\gamma o + \gamma_T)[2\sigma_Y^2(\gamma o + \gamma_T)^2 - (T-t)(\gamma o)^2\gamma_T]}$$

(A.41)

$$\beta_{3t} = \frac{\sigma_Y^2(\gamma o + \gamma_T)^2}{2\sigma_Y^2(\gamma o + \gamma_T)^2 - (T-t)(\gamma o)^2\gamma_T}$$

(A.42)

$$\alpha_t = \frac{\gamma o}{\gamma o + \gamma_T}$$

(A.43)

$$\gamma_T = \frac{\gamma_T\sigma_Y^2(\gamma o + \gamma_T)^2}{\sigma_Y^2(\gamma o + \gamma_T)^2 - (T-t)(\gamma o)^2\gamma_T},$$

(A.44)

for all $t \in [0, T]$, where $\chi_t = 1 - \gamma_{1t}/\gamma o$ and $\gamma_T \in (0, \gamma o)$ is the unique solution in $(0, \gamma o)$ to the cubic $\gamma_T T (\gamma o)^3 + (\gamma_T - \gamma o) (\gamma_T + \gamma o)^2 \sigma_Y^2 = 0$, and $\beta_0 \equiv 1 - \beta_1 - \beta_3$.

**Proof.** We work with the backward system. First note that by setting $r = 0$ in (A.39), $\alpha$ must be constant and equal to its initial value $\alpha_0 = \frac{1}{2-\chi o}$. Next, recall that by Lemma 1, $\chi_t = 1 - \frac{2t}{\gamma o}$, so $\chi_0 = 1 - \frac{\gamma Y F}{\gamma o}$ and thus $\alpha_t = \alpha = \frac{\gamma o}{\gamma Y F + \gamma o}$ for all $t \in [0, T]$. Next, note that the ODE $\dot{\gamma}_t = \frac{\chi_t^2}{\sigma_Y^2}$ given an initial value $\gamma_{NF}^F$ has solution $\gamma_t = \frac{\gamma_{NF}^F \sigma_Y^2}{(\gamma_{NF}^F + \gamma o)^2} t$; switching back to the forward system by replacing $t$ with $T-t$ yields the expression in the original
statement. Now the terminal condition $\gamma_T = \gamma^o$ is equivalent to the following cubic equation for $\gamma_{NF}^F$:

$$q(\gamma_{NF}^F) := \gamma_{NF}^F T (\gamma^o)^3 + (\gamma_{NF}^F - \gamma^o) (\gamma_{NF}^F + \gamma^o)^2 \sigma_Y^2 = 0. \quad (A.45)$$

Note $q(\gamma_{NF}^F) > 0$ for $\gamma_{NF}^F \geq \gamma^o$ and $q(\gamma_{NF}^F) \leq 0$ for $\gamma_{NF}^F \leq 0$, so all real roots must lie in $(0, \gamma^o)$. Now any root to the cubic must satisfy

$$\frac{T(\gamma^o)^3}{\gamma^o - \gamma_{NF}^F} = \frac{\sigma_Y^2 (\gamma_{NF}^F + \gamma^o)^2}{\gamma_{NF}^F}. \quad (A.46)$$

The LHS of (A.46) is strictly increasing for $\gamma_{NF}^F \in (0, \gamma^o)$ while the RHS is strictly decreasing in this interval, so $q$ has a unique real root. Returning to the $\beta_1$ ODE, using $\alpha = \beta_1 \chi + \beta_3$, we have $\dot{\beta}_1 = \frac{\alpha \gamma_t}{\sigma_Y^2} (\alpha - \beta_1 t)$. This ODE can be solved by integration after moving $\beta_1 (\alpha - \beta_1)$ to the LHS, and with algebra, one obtains (in the forward system) the expression in the proposition statement. One then obtains $\beta_3t$ from these known quantities using $\beta_3t = \alpha - \beta_1 t \chi_t$.

**Proofs for Section 2.3**

Here we prove Propositions 3 and 4. Since these results require some preliminary lemmas, we organize them into separate sections.

**Proof of Proposition 3**

We treat the patient and myopic cases one at a time. The following lemma compares signaling and learning between the public and no feedback cases for $r = 0$.

**Lemma A.4.** For $r = 0$ and all values of $T, \gamma^o$ and $\sigma_Y$, more information is revealed in the no feedback case than in the public benchmark case: $\gamma_{pub} > \gamma_{NF}^F$. In the public benchmark, there is more aggressive signaling early in the game and less aggressive signaling later in the game, relative to the no feedback case; i.e., there exists $T^* \in (0, T)$ such that $\beta_{pub}^3 > \alpha^N F$ if and only if $t < T^*$.

**Proof.** For the first claim, recall that $\gamma_{NF}^F$ is the unique positive root of the cubic equation $q(\gamma^F) = 0$ defined in (A.45), where for $\gamma > 0$, $q(\gamma) > 0$ iff $\gamma > \gamma_{NF}^F$. Hence, to prove the
claim, it suffices to show that \( q(\gamma_{\text{pub}}^F) > 0 \). By direct calculation, we have

\[
q(\gamma_{\text{pub}}^F) = (\gamma^o)^3 \left(T\gamma^o + 2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right)
\]

\[
+ \frac{\sigma_Y^2}{T^3} \left(2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right) \left(2T\gamma^o + 2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right)^2
\]

\[
= (\gamma^o)^4 T q_2(S),
\]

where

\[
q_2(S) := 1 + 2S - \sqrt{1 + 4S^2} + S \left(2S - \sqrt{1 + 4S^2}\right) \left(2 + 2S - \sqrt{1 + 4S^2}\right)^2
\]

and

\[
S := \frac{\sigma_Y^2}{T\gamma^o}.
\]

We now show that \( q_2(S) > 0 \) for all \( S > 0 \) (observe that \( q_2(0) = 0 \)). Let \( R(S) = 1 + 2S - \sqrt{1 + 4S^2} \); it is straightforward to verify that \( R(0) = 0 \) and that for all \( S \geq 0, R'(S) > 0 \) and \( R(S) < 1 \). Moreover, the inverse of \( R \) is the function \( S : [0, 1) \to [0, \infty) \) characterized by \( S(R) := \frac{R(2 - R)}{4(1 - R)} \). Hence, by change of variables, \( q_2(S) > 0 \) for all \( S > 0 \) iff \( q_3(R) > 0 \), where

\[
q_3(R) := R - S(R)(1 - R)(R + 1)^2.
\]

Now for \( R \in [0, 1) \),

\[
q_3(R) > 0 \iff S(R) = \frac{R(2 - R)}{4(1 - R)} < \frac{R}{(1 - R)(R + 1)^2}
\]

\[
\iff q_4(R) := (2 - R)(R + 1)^2 < 4.
\]

It is straightforward to verify that over the interval \([0, 1]\), \( q_4(R) \) attains its maximum value of 4 at \( R = 1 \), and tracing our steps backwards this implies that \( q(\gamma_{\text{pub}}^F) > 0 \), so \( \gamma_{\text{pub}}^F > \gamma_{NF}^F \), proving the first claim.

For the second claim, using the forward system, since \( \beta_{3T}^\text{pub} = \frac{1}{2} < \alpha_{NF}^F \) and \( \beta_t^\text{pub} \) is monotonically decreasing, it suffices to show that \( \beta_{30}^\text{pub} > \alpha_{NF}^F \). Using the associated expressions from Lemmas A.2 and A.3, this is equivalent to

\[
\frac{1}{2 - \frac{\gamma_{\text{pub}}^F T}{2\sigma_Y^2}} > \frac{\gamma^o}{\gamma^o + \gamma_{NF}^F}
\]

\[
\iff \hat{\gamma} := \gamma^o \left(1 - \frac{\gamma_{\text{pub}}^F T}{2\sigma_Y^2}\right) < \gamma_{NF}^F.
\]
It suffices to show that $q(\hat{\gamma}) := T\hat{\gamma}(\gamma^o)^3 + (\hat{\gamma} - \gamma^o)(\hat{\gamma} + \gamma^o)^2 \sigma_Y^2 < 0$. Recalling that

$$\gamma_F^{\text{pub}} = \frac{\gamma^o T + 2\sigma_Y^2 - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}}{T}$$

one can show that

$$q(\hat{\gamma}) = \frac{(\gamma^o)^4 T[-\gamma^o T + \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}]}{2\sigma_Y^2} \left[ 1 - \frac{2\sigma_Y^2 - (\gamma^o T - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4})}{2\sigma_Y^2} \right]$$

$$= -\frac{T(\gamma^o)^4}{2\sigma_Y^4} \left[ (T\gamma^o)^2 + 2\sigma_Y^4 - T\gamma^o \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4} \right].$$

The expression in square brackets can be written as $x^2 + y^2 - \sqrt{xy} > 0$ where $x = (T\gamma^o)^2 > 0$ and $y = (T\gamma^o)^2 + 4\sigma_Y^4 > 0$, and thus $q(\hat{\gamma}) < 0$, concluding the proof.

Continuing toward the proof of Proposition 3, we now handle the myopic case. We begin by deriving the solution for the public and no feedback cases for a myopic leader.

**Lemma A.5.** Suppose the leader is myopic. In the LME for the public case, $\beta^3_t = 1/2$ and $\gamma_t^{\text{pub}} = \frac{4\sigma_Y^2 \gamma^o}{4\sigma_Y^2 + \gamma^o\chi_t}$. In the LME for the no feedback case, $\alpha_t = \frac{\gamma^o}{\gamma^o + \gamma_t^{\text{NF}}}$, where $\gamma_t^{\text{NF}}$ is defined implicitly as the unique solution in $(0, \gamma^o)$ of the equation $2\ln(\gamma_t^{\text{NF}}/\gamma^o) - \gamma^o/\gamma_t^{\text{NF}} + \gamma_t^{\text{NF}}/\gamma^o = -\frac{\gamma^o_t}{\sigma_Y^2}$.

**Proof.** We first consider the public benchmark case, where in the myopic solution, $\beta^3_t = 1/2$, and thus (in the forward system)

$$z_t^{\text{pub}} = -\beta^2_3(\gamma_t^{\text{pub}})^2 = -\frac{(\gamma_t^{\text{pub}})^2}{4\sigma_Y^2}$$

$$\implies \gamma_t^{\text{pub}} = \frac{4\sigma_Y^2 \gamma^o}{4\sigma_Y^2 + \gamma^o\chi_t},$$

and thus $u_t^{\text{pub}} = \frac{\gamma_t^{\text{pub}}}{2} = \frac{2\sigma_Y^2 \gamma^o}{4\sigma_Y^2 + \gamma^o\chi_t}$.

In the myopic solution to the no feedback case, $\alpha_t = \frac{1}{2 - \chi_t} = \frac{1}{1 + \gamma_t^{\text{NF}}/\gamma^o}$, where $\gamma_t^{\text{NF}}$ solves the ODE

$$z_t^{\text{NF}} = -\alpha_t^2 \left( \frac{\gamma_t^{\text{NF}}}{\sigma_Y^2} \right)^2 = -\frac{1}{\sigma_Y^2} \left( \frac{\gamma^o \gamma_t^{\text{NF}}}{\gamma^o + \gamma_t^{\text{NF}}} \right)^2$$

$$\implies 2\frac{z_t^{\text{NF}}}{\gamma_t^{\text{NF}}} + \gamma^o \frac{z_t^{\text{NF}}}{(\gamma_t^{\text{NF}})^2} = -\gamma^o \frac{1}{\sigma_Y^2}. \tag{A.47}$$

By integrating both sides of (A.48) and using that $\gamma_0^{\text{NF}} = \gamma^o$ to pin down the constant of
integration, we obtain $(\gamma_t^{NF})_{t \in [0, T]}$ to \eqref{eq:gamma_NF} solves
\begin{equation}
2 \ln(\gamma_t^{NF} / \gamma^0) - \gamma^0 / \gamma_t^{NF} + \gamma_t^{NF} / \gamma^0 = -\frac{\gamma_t^{NF}}{\sigma^2_Y}.
\end{equation}
\label{eq:gamma_NF}

To verify that $\gamma_t^{NF} \in (0, \gamma^0]$ is well-defined as such, define $f : (0, 1] \to \mathbb{R}$ by
\[ f(y) := 2 \ln(y) - 1/y + y, \]
and note that $f(y)$ is strictly increasing as $f'(y) = (1 + 1/y)^2 > 0$, and moreover, $f(1) = 0 \geq -\frac{\gamma_t^{NF}}{\sigma^2_Y}$ while $\lim_{y \to 0} f(y) = -\infty < -\frac{\gamma_t^{NF}}{\sigma^2_Y}$. It follows that for all $t \in [0, T]$, $\gamma_t^{NF} \in (0, \gamma^0]$ is uniquely determined by \eqref{eq:gamma_NF}.\hfill \tag*{□}

\begin{lemma}
For the myopic case, $\gamma_t^{Pub} > \gamma_t^{NF}$ for all $t \in (0, T]$.
\end{lemma}
\begin{proof}
Observe that $\gamma_0^{NF} = \gamma_0^{Pub} = \gamma^0$. Solving the ODEs for $\gamma^{Pub}$ and $\gamma^{NF}$ by integration, and using that $\alpha_t \geq \beta_3 t = 1/2$ with strict inequality for all $t > 0$ then yields the result.\hfill \tag*{□}
\end{proof}

The last step toward proving Proposition \ref{prop:uniform_convergence} is to establish uniform convergence of solutions to the myopic solutions as $r \to \infty$. For arbitrary $T > 0$ and $r \geq 0$, let $BVP^{Pub}(r)$ denote the boundary value problem for $(\beta_1, \beta_3, \gamma)$ defined by \eqref{eq:beta1}-\eqref{eq:gamma} and the associated boundary conditions, parameterized by $r$, and likewise let $BVP^{NF}(r)$ denote the boundary value problem for $(\alpha, \gamma, \chi)$ defined by \eqref{eq:alpha}-\eqref{eq:chi} and the associated boundary conditions.

\[ \Xi^{pub} := \{(\beta_1, \beta_3, \gamma) \in C^1([0, T])^3 \text{ such that } (\beta_1, \beta_3, \gamma) \text{ solves } BVP^{Pub}(r) \text{ for some } r \geq 0\} \]
\[ \Xi^{NF} := \{(\alpha, \gamma, \chi) \in C^1([0, T])^3 \text{ such that } (\alpha, \gamma, \chi) \text{ solves } BVP^{NF}(r) \text{ for some } r \geq 0\}. \]

\begin{lemma}
The families $\{ (\hat{\beta}_1, \hat{\beta}_3, \hat{\gamma}) : (\beta_1, \beta_3, \gamma) \in \Xi^{pub} \}$ and $\{ (\hat{\alpha}, \hat{\gamma}, \hat{\chi}) : (\alpha, \gamma, \chi) \in \Xi^{NF} \}$ are uniformly bounded, and hence $\Xi^{pub}$ and $\Xi^{NF}$ are equicontinuous.
\end{lemma}
\begin{proof}
We begin with the public case. Recall that $\Xi^{pub}$ is uniformly bounded, and in particular, we have $(\beta_1, \beta_3, \gamma_t) \in [0, 1/2] \times [1/2, 1] \times [0, \gamma^0]$ for all $(\beta_1, \beta_3, \gamma) \in \Xi^{pub}$ and all $t \in [0, T]$. It follows that $|\dot{\gamma}_t| = \beta_3 \dot{\gamma}_t^2 \leq \frac{(\gamma^2 + \gamma^0)^2}{\sigma_Y^2}$. We now establish a uniform bound on $\dot{\beta}_3$. If we define $\beta_3^m := 1/2$ and $\beta_3^f = \beta_3 - \beta_3^m$, we have from the (backward system) $\beta_3$ ODE
\[ \dot{\beta}_3^m = \dot{\beta}_3 = \beta_3 [ -2 r \beta_3^f + \beta_3 (1 - \beta_3) \gamma_t / \sigma_Y^2], \]
which is linear in $\beta_3^f$. Solving this ODE and multiplying through by $r$, we obtain
\[ r \beta_3^f = \int_0^t r e^{-r t} \beta_3^m (1 - \beta_3) \gamma_s / \sigma_Y^2 ds. \]
Now \( |\beta_{3\alpha}^2(1 - \beta_{3\alpha})\gamma_s/\sigma_Y^2| \leq \bar{g}_{\text{pub}} := \frac{\gamma}{\sigma_Y}. \) Moreover, \( 2\beta_{3\alpha} \geq 1/2, \) and thus

\[
|r\beta_{3\alpha}'| \leq \bar{g}_{\text{pub}} \int_0^t re^{-r} f_s' 2\beta_{3\alpha} du \, ds \leq \bar{g}_{\text{pub}} \int_0^t re^{-r(t-s)} \, ds = \bar{g}_{\text{pub}}(1 - e^{-rt}) < \bar{g}_{\text{pub}}.
\]

It follows that

\[
|\beta_{3\alpha}'| = |\beta_{3\alpha}''| = |\beta_{3\alpha}[-2r\beta_{3\alpha}' + \beta_{3\alpha}(1 - \beta_{3\alpha})\gamma_t/\sigma_Y^2]| \\
\leq | - 2\beta_{3\alpha} | \cdot | r\beta_{3\alpha}' | + | \beta_{3\alpha}^2(1 - \beta_{3\alpha})\gamma_t/\sigma_Y^2 | \\
\leq 1 \cdot \bar{g}_{\text{pub}} + \bar{g}_{\text{pub}},
\]

which is the desired uniform bound as \( \bar{g}_{\text{pub}} \) is independent of \( r. \) Now since \( \beta_1 + \beta_3 \equiv 1, |\beta_{3\alpha}'| \)
is also uniformly bounded above by \( 2\bar{g}_{\text{pub}}. \) Hence we have established uniform bounds on the derivatives \((\beta_1, \beta_3, \gamma)\) for \((\beta_1, \beta_3, \gamma) \in \Xi_{\text{pub}}, \) and thus \( \Xi_{\text{pub}} \) is equicontinuous.

Next, we turn to the no feedback case, where we recall the uniform bounds \( \alpha_t \in [1/(2 - \chi_t), 1] \subset [1/2, 1], \gamma_t \in [0, \gamma^0] \) and \( \chi_t \in [0, 1]. \) Immediately, we have \( |\gamma_t| = |\frac{\alpha_t^2 \gamma^2}{\sigma_Y^2}| \leq \frac{(\gamma^0)^2}{\sigma_Y^2}, \) and since \( \chi \equiv 1 - \gamma/\gamma^0, |\chi_t| = | - \gamma_t/\gamma^0| \leq \frac{\gamma}{\sigma_Y} =: \bar{g}_{\text{NF}}. \) We now uniformly bound \( \alpha_t. \)

Set \( \alpha_t^m := 1/(2 - \chi_t) \) and \( \alpha_t^f := \alpha_t - \alpha_t^m, \) and note that \( \dot{\alpha}_t^m = \dot{\chi}_t/(2 - \chi_t)^2. \) We

\[
\dot{\alpha}_t^f := \dot{\alpha}_t - \dot{\alpha}_t^m = - r\alpha_t (2 - \chi_t) \alpha_t^f - \dot{\chi}_t/(2 - \chi_t)^2,
\]

which is linear in \( \alpha_t^f. \) As in the public case, solving this ODE and multiplying through by \( r \) yields

\[
r \alpha_t^f = \int_0^t re^{-r} f_s' \alpha_t (2 - \chi_u) du [-\dot{\chi}_s/(2 - \chi_s)^2] ds \\
\implies |r \alpha_t^f| \leq \int_0^t re^{-r} f_s' \alpha_t (2 - \chi_u) du |\dot{\chi}_s/(2 - \chi_s)^2| ds.
\]

Now \( |\dot{\chi}_s/(2 - \chi_s)^2| \leq |\dot{\chi}_s| \leq \bar{g}_{\text{NF}} \) as noted above, so

\[
|r \alpha_t^f| \leq \bar{g}_{\text{NF}} \int_0^t re^{-r} f_s' \alpha_t (2 - \chi_u) du ds \\
\leq \bar{g}_{\text{NF}} \int_0^t r e^{-r(t-s)} ds = \bar{g}_{\text{NF}}(1 - e^{-rt}) < \bar{g}_{\text{NF}},
\]

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where we have used that \( \alpha_u \geq 1/(2 - \chi_u) \implies \int_s^t \alpha_u(2 - \chi_u) du \geq (t - s) \). We now have

\[
|\dot{\alpha}_t| = | - r\alpha_t(2 - \chi_t)\alpha_t^\ell | = |r\alpha_t^\ell| \cdot |\alpha_t| \cdot |2 - \chi_t| \\
\leq \bar{g}^{NF} \cdot 1 \cdot 2.
\]

We have shown that \((\dot{\alpha}, \dot{\gamma}, \dot{\chi})\) are uniformly bounded for \((\alpha, \gamma, \chi) \in \Xi^{NF}\), and thus \(\Xi^{NF}\) is equicontinuous. \(\square\)

Let \((\beta_1^{pub, \infty}, \beta_3^{pub, \infty}, \gamma^{pub, \infty})\) and \((\alpha^{NF, \infty}, \gamma^{NF, \infty}, \chi^{NF, \infty})\) denote the myopic equilibrium coefficients for the public and no feedback cases, respectively.

**Proposition A.1.** Fix any \(T > 0\) and let \(\{r_n\}_{n=1}^\infty\) be a sequence with \(\lim_{n \to \infty} r_n = \infty\). Let \(\{(\beta_1^{pub, n}, \beta_3^{pub, n}, \gamma^{pub, n})\}_{n=1}^\infty\) and \(\{(\alpha^{NF, n}, \gamma^{NF, n}, \chi^{NF, n})\}_{n=1}^\infty\) be sequences of solutions to \(\text{BVP}^{Pub}(r_n)\) and \(\text{BVP}^{NF}(r_n)\), respectively. Then uniformly, \((\beta_1^{pub, n}, \beta_3^{pub, n}, \gamma^{pub, n}) \to (\beta_1^{pub, \infty}, \beta_3^{pub, \infty})\) and \((\alpha^{NF, n}, \gamma^{NF, n}, \chi^{NF, n}) \to (\alpha^{NF, \infty}, \gamma^{NF, \infty}, \chi^{NF, \infty})\).

**Proof.** First, note that \(r_n \to \infty\) implies that \(r_n = 0\) for at most finitely many \(n\), so there exists \(N\) such that \(r_n > 0\) for all \(n \geq N\); it suffices to consider only such \(n\). In the public benchmark, recall that \(\beta_1^{pub, \infty} = \beta_3^{pub, \infty} = 1/2\). We have \(\beta_3 \geq 1/2 > 0\), so the \(\beta_3\) ODE can be rearranged to obtain

\[
\beta_3^{pub, n} = \frac{1}{2} + \frac{1}{r_n} \left[ \frac{(\beta_3^{pub, n})^2 (1 - \beta_3^{pub, n}) \gamma^{pub, n}}{2 \beta_3^{pub, n} \sigma_2^2 Y} - \frac{\dot{\beta}_3^{pub, n}}{2 \beta_3^{pub, n}} \right] .
\]

Since \(\beta_3^{pub, n} \geq 1/2\), the expression in brackets is uniformly bounded in absolute value by some constant \(K^{pub}\) (independent of \(r_n\) and \(t\)). Hence \(\beta_3^{pub, n}\) converges pointwise to \(1/2 = \beta_3^{pub, \infty}\). Since \(\beta_1^{pub, n} \equiv 1 - \beta_3^{pub, n}\), \(\beta_1^{pub, n}\) converges pointwise to \(1/2 = \beta_1^{pub, \infty}\). As \([0, T]\) is compact and the sequence \(\{(\beta_1^{pub, n}, \beta_3^{pub, n})\}_{n=1}^\infty\) is equicontinuous by Lemma A.7, we apply Lemma 39 in Royden (1988, p. 168) and obtain uniform convergence for \(\beta_1^{pub}\) and \(\beta_3^{pub}\). Finally, we prove uniform convergence for \(\gamma^{pub}\), by proving pointwise convergence and invoking the same result to obtain uniform convergence. Note that for \(i \in \mathbb{N} \cup \{\infty\}\)

\[
\gamma_t^{pub, i} = \frac{\gamma^o \sigma_2^2 Y}{\sigma_2^2 + \gamma^o \int_t^T (\beta_3^{pub, i})^2 ds}.
\]

Since for \(n \in \mathbb{N}\), the functions \((\beta_3^{pub, n})^2\) are bounded uniformly by a constant and converge pointwise to \((\beta_3^{pub, \infty})^2\), the dominated convergence theorem (Royden, 1988, p. 267, Theorem
16) implies \( \int_t^T (\beta^{\text{pub},n})^2 ds \to \int_t^T (\beta^{\text{pub},\infty})^2 ds \) and thus for all \( t \in [0, T] \) pointwise,

\[
\gamma^{\text{pub},n}_t \to \frac{\gamma^0 \sigma_Y^2}{\sigma_Y^2 + \gamma^0 \int_t^T (\beta^{\text{pub},\infty})^2 ds} = \gamma^{\text{pub},\infty}_t.
\]

By Lemma A.7, the sequence of \( \gamma^{\text{pub},n} \) is equicontinuous, and thus Royden (1988, p. 168, Lemma 39) gives uniform convergence to \( \gamma^{\text{pub},\infty} \).

Next, consider the no feedback case. By Royden (1988, p. 169, Theorem 40), there exists a subsequence of \( \{(\alpha^{NF,n}, \gamma^{NF,n}, \chi^{NF,n})\}_{n=1}^\infty \) indexed by \( \{k(n)\}_{n=1}^\infty \) which converges uniformly on \([0, T]\) to some limit \((\alpha^*, \gamma^*, \chi^*)\). We argue that \((\alpha^*, \gamma^*, \chi^*) = (\alpha^{NF,\infty}, \gamma^{NF,\infty}, \chi^{NF,\infty})\), and thus the original sequence converges uniformly to \((\alpha^{NF,\infty}, \gamma^{NF,\infty}, \chi^{NF,\infty})\).

Using a similar dominated convergence argument to before, we have that pointwise \( \gamma^{NF,k(n)}_t \to \frac{\gamma^0 \sigma_Y^2}{\sigma_Y^2 + \gamma^0 \int_t^T (\alpha^*)^2 ds} \), so \( \gamma^*_t = \frac{\gamma^0 \sigma_Y^2}{\sigma_Y^2 + \gamma^0 \int_t^T (\alpha^*)^2 ds} \) and \( \chi^{NF,k(n)}_t \to 1 - \gamma^*_t / \gamma^0 \). Since \( \chi^{NF,k(n)}_t \to \chi^* \) uniformly, we have \( \chi^* = 1 - \gamma^* / \gamma^0 \).

Let \( \alpha^{m,n}_t := 1/(2 - \chi^{NF,n}_t) \). Since \( \alpha^{NF,n}_t > 0 \) for all \( n \), the ODE can be rearranged to obtain

\[
\alpha^{NF,n}_t - \alpha^{m,n}_t = \frac{1}{r_n} \left[ - \frac{\dot{\alpha}^{NF,n}_t}{\alpha^{NF,n}_t (2 - \chi^{NF,n}_t)} \right].
\]

Since \( 2 - \chi^{NF,n}_t > 1 \) and \( \alpha^{NF,n}_t > 1/2 \), the expression in brackets is uniformly bounded in absolute value by some constant \( K^{NF} \) (independent of \( r_n \) and \( t \)), and thus \( |\alpha^{NF,n}_t - \alpha^{m,n}_t| \to 0 \) pointwise. It follows that pointwise, and by familiar arguments uniformly, \( \alpha^{m,k(n)} \to \alpha^* \). But \( \alpha^{m,k(n)} = 1/(2 - \chi^{NF,n}_t) \to 1/(2 - \chi^*) \), so \( \alpha^* = 1/(2 - \chi^*) = 1/(1 + \gamma^*/\gamma^0) \).

By differentiating the equation \( \gamma^*_t = \frac{\gamma^0 \sigma_Y^2}{\sigma_Y^2 + \gamma^0 \int_t^T (\alpha^*)^2 ds} \), we obtain \( \dot{\gamma}^*_t = \frac{(\alpha^* \gamma^*_t)^2}{\sigma_Y^2 + \gamma^0 \int_t^T (\alpha^*)^2 ds} = \frac{(\gamma^*_t)^2}{\sigma_Y^2 (1 + \gamma^*/\gamma^0)^2} \), subject to initial condition \( \gamma^*_0 = \gamma^0 \). This equation has a unique solution which is \( \gamma^{NF,\infty} \) as obtained in the proof of Lemma A.5, from which \( \alpha^* = \alpha^{NF,\infty} \) and \( \chi^* = \chi^{NF,\infty} \). This establishes that all convergent subsequences have the same limit, and hence the original sequences converges to that limit.

**Proof of Proposition 3.** Part (i) is a restatement of results from Lemma A.4. For part (ii), choose any \( \delta \in (0, T) \). Define \( \tilde{\gamma} := \min_{t \in [T - \delta, T]} (\gamma^{\text{pub},\infty}_t - \gamma^{NF,\infty}_t) \), where we use \( \gamma^{\tau,r} \) to denote the solution for the case \( x \in \{\text{Pub, NF}\} \) and \( r \) is the discount rate. By Lemma A.6, \( \tilde{\gamma} > 0 \). By Proposition A.1, \( \gamma^{\text{pub},r} - \gamma^{NF,r} \) converges uniformly to \( \gamma^{\text{pub},\infty} - \gamma^{NF,\infty} \) as \( r \to \infty \), and thus for any \( \varepsilon \in (0, \tilde{\gamma}) \), there exists \( \bar{r} > 0 \) such that for all \( r > \bar{r} \) and all \( t \in [T - \delta, T] \), we have \( \gamma^{\text{pub},r}_t - \gamma^{NF,r}_t > \gamma^{\text{pub},\infty}_t - \gamma^{NF,\infty}_t - \varepsilon \geq \tilde{\gamma} - \varepsilon > 0 \), as desired. \( \square \)
Proof of Proposition 4

We begin by calculating expected flow losses in the public and no feedback cases.

Lemma A.8. Then the expected flow payoffs to player 1 in the public benchmark and no feedback case have magnitudes

\[ u_{t}^{\text{pub}} = \gamma_{t}^{\text{pub}}[\beta_{3t}^2 + (1 - \beta_{3t})^2] \]

\[ u_{t}^{\text{NF}} = (1 - \alpha_{3t})^2 \gamma^{\circ} + \alpha_{3t}^2 \gamma_{t}^{\text{NF}}. \]

Proof. Let \( E_{\theta} \) denote ex ante expectations over realizations of \( \theta \sim N(\hat{m}_0, \gamma^{\circ}) \). In the public benchmark, using \( \beta_{1t} = 1 - \beta_{3t} \), we have

\[ u_{t}^{\text{pub}} = E_{\theta} (E_0 [(\theta - a_t)^2 + (a_t - \hat{a}_t)^2]) \]

\[ = E_{\theta} (E_0 [(\theta - \beta_{1t} M_t - \beta_{3t} \theta)^2 + (M_t - \beta_{1t} M_t - \beta_{3t} \theta)^2]) \]

\[ = E_{\theta} (E_0 [(\theta - M_t)^2 (1 - \beta_{3t})] + \beta_{3t}^2]) \]

\[ = \gamma_{t}^{\text{pub}}[\beta_{3t}^2 + (1 - \beta_{3t})^2] \]

where in the second to last step we have used that the long-run player and myopic player have the same information prior to \( \theta \) being realized, along with the law of iterated expectations. In the no feedback case, we have

\[ u_{t}^{\text{NF}} = E_{\theta} (E_0 [(\theta - a_t)^2 + (a_t - \hat{a}_t)^2]) \]

\[ = E_{\theta} (E_0 [(\theta - (1 - \alpha_t) \hat{m}_0 - \alpha_t \theta)^2 + (1 - \alpha_t) \hat{m}_0 + \alpha_t \theta - (1 - \alpha_t) \hat{m}_0 - \alpha_t \hat{M}_t)^2]) \]

\[ = \gamma_{t}^{\circ}(1 - \alpha_{t})^2 + \alpha_{t}^2 E_{\theta} (E_0 \{E_t [(M_t - \theta)^2]\}) \]

where we have again used the law of iterated expectations. In the second term, we expand

\[ E_t [(\hat{M}_t - \theta)^2] = E_t [\hat{M}_t^2 - 2 \hat{M}_t \theta + \theta^2] \]

\[ = E_t [\hat{M}_t^2] - 2 \theta M_t + \theta^2 \]

\[ = \gamma_{t}^{\text{NF}} M_t^2 - 2 \theta M_t + \theta^2 \]

\[ = \gamma_{t}^{\text{NF}} \chi_t + (M_t - \theta)^2 \]

\[ = \gamma_{t}^{\text{NF}} \chi_t + (1 - \chi_t)^2 (\theta - \hat{m}_0)^2. \]
This term is unchanged under the expectation $E_0$. Under the outer expectation $E_\theta$, it reduces to $\gamma_t^{NF} \chi_t + (1 - \chi_t)^2 \gamma^o$, and using $\chi_t = 1 - \gamma_t^{NF} / \gamma^o$, the result follows. \hfill $\square$

Let $V^{Pub}$ and $V^{NF}$ denote Player 1’s ex ante expected payoff, i.e., at time 0 taking expectations over $\theta \sim N(\hat{m}_0, \gamma^o)$.

**Lemma A.9.** Assume $r = 0$. For all $T, \gamma^o, \sigma_Y^2 > 0$, $V^{Pub} > V^{NF}$.

**Proof.** For $i \in \{pub, NF\}$, define $\tilde{T}^i := \frac{T\gamma^{F}_i}{\sigma_Y^2}$ and define $\rho := \gamma^{F}_{NF} / \gamma^o$. We claim that the quantities under comparison are

\[
V^{Pub} = -\sigma_Y^2 \left\{ \frac{\tilde{T}^{pub}}{2} - \ln \left[ \frac{16 - 8\tilde{T}^{pub}}{(4 - \tilde{T}^{pub})^2} \right] \right\},
\]

\[
V^{NF} = -\sigma_Y^2 \left\{ \rho(1 - \rho) - \ln \rho \right\}.
\]

First consider the public benchmark, where from Lemma A.8,

\[
V^{pub} = E_0 \left( \int_0^T \left[ -(\theta - a_t)^2 - (a_t - \hat{a}_t)^2 \right] dt \right)
= -\int_0^T u_t^{pub} dt
= -\int_0^T \left[ \gamma_t \left( [1 - \beta_{3t}]^2 + \beta_{3t}^2 \right) \right] dt
\]

Using the closed-form expressions for $\gamma_t$ and $\beta_{3t}$, the integrand simplifies to

\[
\gamma_t \left( [1 - \beta_{3t}]^2 + \beta_{3t}^2 \right) = \frac{\gamma^{F}_{pub}}{2} \left[ 1 + \frac{2t\gamma^{F}_{pub} \sigma_Y^2}{(2\sigma_Y^2 - t\gamma^{F}_{pub})(4\sigma_Y^2 - t\gamma^{F}_{pub})} \right]
= \frac{\gamma^{F}_{pub}}{2} \left[ 1 + \frac{2\tilde{t}^{pub}}{(2 - \tilde{t}^{pub})(4 - \tilde{t}^{pub})} \right],
\]

where $\tilde{t}^{pub} := t\gamma^{F}_{pub} / \sigma_Y^2$. Using that the function $g : x \mapsto \frac{x}{(2-x)(4-x)}$ has antiderivative

\[
\ln \left[ \frac{(4-x)^2}{2-x} \right]
\]

and integrating the second term w.r.t. $\tilde{t}^{pub}$ over $[0, \tilde{T}^{pub}]$ and rescaling by $\sigma_Y^2 / \gamma^{F}_{pub}$, we obtain

\[
V^{pub} = -\frac{T\gamma^{F}_{pub}}{2} - \sigma_Y^2 \left\{ \tilde{T}^{pub} / 2 - \ln \left[ \frac{16 - 8\tilde{T}^{pub}}{(4 - \tilde{T}^{pub})^2} \right] \right\}.
\]
Next, consider the no feedback case, where by Lemma A.8,

\[ V^{NF} = E_0 \left( \int_0^T \left[ -(\theta - a_t)^2 - (a_t - \hat{a}_t)^2 \right] dt \right) \]

\[ = - \int_0^T u_t^{NF} dt \]

\[ = - \int_0^T \left[ (1 - \alpha)^2 \gamma_o' + \alpha^2 \left( [1 - \chi_t] \sigma^2 + \gamma_t \chi_t \right) \right] dt. \quad (A.50) \]

By plugging in the analytic solutions for \( \alpha, \chi_t \) and \( \gamma_t \) in terms of \( \gamma_{NF}^F \), the integrand in (A.50) is

\[ \frac{\gamma_o \gamma_{NF}^F}{(\gamma_o + \gamma_{NF}^F)^2} \frac{t \left( \gamma_o \gamma_{NF}^F \right)^2 - (\gamma_o + \gamma_{NF}^F)^3 \sigma_Y^2}{t \left( \gamma_o - \gamma_{NF}^F \right) - T \gamma_o} \quad (A.51) \]

From the equation defining \( \gamma_{NF}^F \), we have \( (\gamma_o + \gamma_{NF}^F)^2 \sigma_Y^2 = T \gamma_{NF}^F (\gamma_o)^3 / (\gamma_o - \gamma_{NF}^F) \), and thus (A.51) reduces to

\[ \frac{\gamma_o \gamma_{NF}^F}{(\gamma_o + \gamma_{NF}^F)^2} \frac{t \gamma_{NF}^F (\gamma_o - \gamma_{NF}^F) - T \gamma_o (\gamma_o + \gamma_{NF}^F)}{t \left( \gamma_o - \gamma_{NF}^F \right) - T \gamma_o} \quad (A.52) \]

Integrating (A.52) yields

\[ V^{NF} = - \int_0^T \left[ \frac{\gamma_o \gamma_{NF}^F}{(\gamma_o + \gamma_{NF}^F)^2} \frac{t \gamma_{NF}^F (\gamma_o - \gamma_{NF}^F) - T \gamma_o (\gamma_o + \gamma_{NF}^F)}{t \left( \gamma_o - \gamma_{NF}^F \right) - T \gamma_o} \right] dt \]

\[ = - \frac{T \gamma_o \gamma_{NF}^F}{(\gamma_o + \gamma_{NF}^F)^2} \left[ \frac{\gamma_{NF}^F}{\gamma_o + \gamma_{NF}^F} + \frac{(\gamma_o)^2 \ln \frac{\gamma_o}{\gamma_{NF}^F}}{\gamma_o - \gamma_{NF}^F} \right] \]

\[ = - \sigma_Y^2 \left( \gamma_o - \gamma_{NF}^F \right) \left[ \frac{\gamma_{NF}^F}{\gamma_o + \gamma_{NF}^F} + \frac{(\gamma_o)^2 \ln \frac{\gamma_o}{\gamma_{NF}^F}}{\gamma_o - \gamma_{NF}^F} \right] \]

\[ = - \sigma_Y^2 \left\{ \rho(1 - \rho) - \ln \rho \right\}. \]

Next, we prove that \( V^{NF} < V^{pub} \). Substituting in the expressions for \( \gamma_{NF}^F \), and simplify-
ing, we obtain
\[ V^{\text{NF}} - V^{\text{pub}} = \sigma_Y^2 \left\{ \rho(\rho - 1) + \ln \rho + 1 + \tilde{T}/2 - \frac{\sqrt{4 + \tilde{T}^2}}{2} \right. \]
\[ \left. - \ln \left[ 8 \left( -\tilde{T} + \sqrt{4 + (\tilde{T})^2} \right) \right] + 2 \ln \left[ 2 - \tilde{T} + \sqrt{4 + (\tilde{T})^2} \right] \right\} \]
\[ = \frac{\sigma_Y^2}{2\rho} f(\rho), \]
where again \( \tilde{T} := \frac{T_\gamma^o}{\sigma_Y} \) and \( \rho := \frac{\gamma^{\text{MF}}}{\gamma^{\text{NF}}} \), where in the second line we have used that the cubic equation defining \( \gamma^{\text{NF}} \) rearranges to \( \tilde{T} = \frac{(1 - \rho)(1 + \rho)^2}{\rho} \), and where

\[
\begin{align*}
  f(x) &:= A_1(x) + 2x \ln \left( \frac{A_2(x)^2}{A_3(x)} \right) \\
  A_1(x) &:= x^3 - 3x^2 + 3x + 1 - z(x) \\
  A_2(x) &:= x^3 + x^2 + x - 1 + z(x) \\
  A_3(x) &:= 8[x^3 + x^2 - x - 1 + z(x)] \\
  z(x) &:= \sqrt{4x^2 + (1 - x)^2(1 + x)^4}.
\end{align*}
\]

We now show that \( f(x) < 0 \) for all \( x \in (0, 1) \), so that in particular \( f(\rho) < 0 \), from which the desired result follows. To that end, we first show that \( A_2(x) > 0 \) and \( A_3(x) > 0 \) for all \( x > 0 \). By inspection, for all \( x > 0 \), we have \( A_2(x) > A_3(x)/8 \), and

\[
\begin{align*}
  A_3(x)/8 &= (x - 1)(x + 1)^2 + \sqrt{4x^2 + (1 - x)^2(1 + x)^4} \\
  &> (x + 1)^2 \left[ x - 1 + \sqrt{(1 - x)^2} \right] \\
  &\geq 0.
\end{align*}
\]

Next, we apply the inequality \( \ln(y) \leq 2 \left( y^{\frac{1}{2}} - 1 \right) \) for \( y > 0 \) using \( y = \frac{A_2(x)^2}{A_3(x)} \) in (A.53) to obtain

\[
\begin{align*}
f(x) &\leq A_1(x) + 4x \left( \frac{A_2(x)}{\sqrt{A_3(x)}} - 1 \right). \tag{A.54}
\end{align*}
\]

For \( x > 0 \), the RHS of (A.54) is negative if and only if

\[
\frac{A_2(x)}{\sqrt{A_3(x)}} < -\frac{A_1}{4x} + 1. \tag{A.55}
\]
For \( x \in (0, 1) \), the RHS of (A.55) is strictly positive:

\[
\frac{-A_1}{4x} + 1 = \frac{1}{4x} \left[ \sqrt{4x^2 + (1-x)^2(1+x)^4} - x^3 + 3x^2 + x - 1 \right]
\]

\[
> \frac{1}{4x} \left[ (1-x)(1+x)^2 - x^3 + 3x^2 + x - 1 \right]
\]

\[
= \frac{1}{2} [1 + x(1-x)] > 0.
\]

Hence, for \( x \in (0, 1) \), (A.55) is equivalent to

\[
0 > A_2(x)^2 - A_3(x) \left( -\frac{A_1}{4x} + 1 \right)^2 = \frac{2}{x^2} \left[ (1-x)^2 A_4(x) + A_5(x) z(x) \right]
\]

where

\[
A_4(x) = x^6 - 4x^4 - x^3 + 4x^2 + 3x + 1
\]

\[
A_5(x) = x^5 - 3x^4 + x^3 + 2x^2 - 1.
\]

Now by Descartes’ rule of signs, \( A_5 \) has 3 sign changes and at most 3 positive real roots, counting multiplicity. It is easy to verify that there is a double root at \( x = 1 \), that \( A_5(2) = -1 < 0 \), and that \( \lim_{x\to+\infty} A_5(x) = +\infty \), so there is a positive root at some \( x > 2 \). This implies there are no roots in \( (0, 1) \). Since \( A_5(0) = -1 < 0 \), it follows that \( A_5(x) < 0 \) for all \( x \in (0, 1) \). Thus, without signing \( A_4(x) \), it suffices to show that \((1-x)^2 |A_4(x)| < -A_5(x)z(x)\), or equivalently (since \( A_5(x) < 0 \))

\[
0 > (1-x)^4 A_4(x)^2 - z(x)^2 A_5(x)^2 = 4(1-x)^4 x^5 (x^4 - 2x^2 - 3x - 1).
\]

Now for \( x \in (0, 1) \), we have in (A.57) \( x^4 - 2x^2 - 3x - 1 < -2x^2 - 3x < 0 \), and the outside factor is clearly positive. Hence we have shown that \( f(x) < 0 \) for \( x \in (0, 1) \), concluding the proof.

\[\square\]

**Lemma A.10.** Suppose the leader is myopic. There exists \( \bar{T} > 0 \) such that for all \( T > \bar{T} \) the team’s ex ante flow payoff is strictly larger in the no feedback case over \([\bar{T}, T]\).

**Proof.** Plugging the solutions from Lemma A.5 into the expressions for the expected flow
losses in Lemma A.8, we have

\[ u_{t}^{\text{pub}} = \gamma_{t}^{\text{pub}} [(1 - \beta_{3t})^2 + \beta_{3t}^2] = \frac{\gamma_{t}^{\text{pub}}}{2} = \frac{2\sigma_{Y}^2 \gamma \circ t}{4\sigma_{Y}^2 + \gamma \circ t} \]

\[ u_{t}^{NF} = (1 - \alpha_{3t})^2 \gamma \circ t + 2\alpha_{3t} \gamma_{t}^{NF} = \frac{\gamma \circ t^{NF}}{\gamma \circ t + \gamma_{t}^{NF}} \]  

Let \( \bar{t} := \frac{\tau \circ}{\sigma_{Y}} \). Flow losses are worse in the public benchmark case at time \( t \) if and only if

\[ u_{t}^{\text{pub}} > u_{t}^{NF} \]

\[ \iff \frac{2\sigma_{Y}^2 \gamma \circ}{4\sigma_{Y}^2 + \tau \circ} > \frac{\gamma \circ \gamma_{t}^{NF}}{\gamma \circ t + \gamma_{t}^{NF}} \]

\[ \iff \gamma_{t}^{NF} < \frac{2\sigma_{Y}^2 \gamma \circ}{2\sigma_{Y}^2 + \gamma \circ t} = \frac{2\gamma \circ \gamma_{t}^{NF}}{2 + \gamma \circ t} \]  

Since \( f(\gamma / \gamma \circ) \) is strictly increasing in \( \gamma \), where \( f(\gamma \circ / \gamma \circ) \) is the LHS of (A.49), it suffices to show that there exists \( \bar{T} \) such that \( f(\gamma / \gamma \circ)_{\gamma = \frac{\gamma \circ}{2+\gamma \circ}} > -\frac{2\gamma \circ}{\sigma_{Y}^2} = -\bar{t} \) if \( t > \bar{T} \). By direct calculation,

\[ f(\gamma / \gamma \circ)_{\gamma = \frac{\gamma \circ}{2+\gamma \circ}} + \bar{t} = \frac{f_{2}(\bar{t})}{2(\bar{t} + 2)} \]

where

\[ f_{2}(\bar{t}) := \bar{t}^2 + 4(\bar{t} + 2) \ln \left( \frac{2}{\bar{t} + 2} \right) \].

Now the denominator of the RHS of (A.58) is strictly positive, and it is easy to verify that in the numerator, \( f_{2}(0) = 0 \) and \( f_{2}'(0) = -4 \). Moreover, \( f_{2}'(\bar{t}) = 2\bar{t} - 4 + 4 \ln \left( \frac{2}{\bar{t} + 2} \right) \) with \( \lim_{\bar{t} \to +\infty} f_{2}'(\bar{t}) = +\infty \) as the linear term dominates the log term, and \( f_{2}''(\bar{t}) = 2 - \frac{4}{(\bar{t} + 2)^2} \geq 0 \) with strict inequality for all \( \bar{t} > 0 \). It follows that there is a unique positive threshold \( \bar{t}^{*} \) defined by \( f_{2}(\bar{t}^{*}) = 0 \) such that the RHS of (A.58) is strictly positive, and \( u_{t}^{\text{pub}} > u_{t}^{NF} \), if and only if \( \bar{t} > \bar{t}^{*} \). Letting \( \bar{T} = \bar{t}^{*} \sigma_{Y}^2 / \gamma \circ \) yields the result. \( \square \)

**Proof of Proposition 4.** Part (i) of the proposition is simply a restatement of Lemma A.9. For part (ii), use superscript \( r \) for the discount rate as in the proof of Proposition 3. Let \( \bar{T} \) be as in Lemma A.10 and define \( \bar{u} := \min_{t \in [\bar{T}, T]} (u_{t}^{\text{pub, } \infty} - u_{t}^{NF, \infty}) \), which is strictly positive. Now the mappings \( \gamma \mapsto \gamma / 2 \) and \( \gamma \mapsto \gamma / \gamma \circ + \gamma \) underlying the expressions for expected flow losses in Lemma A.8 are continuous for \( \gamma \in [0, \gamma \circ] \), a compact set, and hence uniformly continuous; together with Proposition A.1, this implies that \( u_{t}^{\text{pub, } r} \) and \( u_{t}^{NF, r} \) converge uniformly to \( u_{t}^{\text{pub, } \infty} \) and \( u_{t}^{NF, \infty} \), respectively. Hence for any \( \epsilon \in (0, \bar{u}) \), there exists \( \bar{r} \) such that for all \( r > \bar{r} \) and all \( t \in [\bar{T}, T] \), \( u_{t}^{\text{pub, } r} - u_{t}^{NF, r} > u_{t}^{\text{pub, } \infty} - u_{t}^{NF, \infty} - \epsilon \geq -\bar{u} - \epsilon > 0 \). \( \square \)
Appendix B: Proofs for Section 4

In the proofs for this section, we denote the prior by \( \hat{m}_0 := \mu \).

**Proof of Lemma 2.** We establish a more general version of the lemma for a drift of the form \( \hat{a}_t + \nu a_t, \nu \in [0,1] \), in \( X \). We use “p1” and “p2” to refer to the long-run player and the myopic player, respectively. Without fear of confusion, we also use \( \gamma_{1t} \) for the posterior variance of p2 (\( \gamma_t \) in the main body), as this variance appears in the first filtering step of the proof. Likewise, p1’s posterior variance will be denoted by \( \gamma_2 \), as it is obtained from a second filtering step.

Inserting (11) in (12), we can write

\[
a_t = \alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\theta,
\]

with

\[
\alpha_{0t} = \beta_{0t}, \quad \alpha_{2t} = \beta_{2t} + \beta_{1t}(1 - \chi_t), \quad \text{and} \quad \alpha_{3t} = \beta_{3t} + \beta_{1t}\chi_t.
\]

that p2 conjectures drives the public signal \( X \). \(^{35}\) With this in hand, p2’s filtering problem can be obtained using the Kalman filter (Chapters 11 and 12 in Liptser and Shiryaev, 1977).

Specifically, define

\[
\begin{align*}
dX_t^2 &:= dX_t - [\hat{a}_t + \nu(\alpha_{0t} + \alpha_{2t}L_t)]dt = \nu\alpha_{3t}\theta dt + \sigma_X dZ_X^t \\
dY_t^2 &:= dY_t - [\alpha_{0t} + \alpha_{2t}L_t]dt = \alpha_{3t}\theta dt + \sigma_Y dZ_Y^t
\end{align*}
\]

which are in p2’s information set. Then, by Theorems 12.6 and 12.7 in Liptser and Shiryaev (1977), \( \theta|\mathcal{F}_t \sim \mathcal{N}(\hat{M}_t, \gamma_{1t}) \), where,

\[
\begin{align*}
d\hat{M}_t &= \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X^2}[dX_t^2 - \nu\alpha_{3t}\hat{M}_t dt] + \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2}[dY_t^2 - \alpha_{3t}\hat{M}_t dt] \\
\gamma_{1t} &= -\gamma_{1t}^2\alpha_{3t}^2 \Sigma
\end{align*}
\]

and \( \Sigma := \left( \frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right) \). (These expressions hold for any (admissible) strategies of the players, as deviations go undetected.)

P1 can affect \( \hat{M}_t \) via her choice \( (a_t)_{t\in[0,T]} \). It is easy to see that, from her perspective,

\[
\begin{align*}
d\hat{M}_t &= \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X^2}[(\nu a_t - \nu(\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\hat{M}_t))dt + \sigma_X dZ_X^t] \\
&\quad + \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2}[(a_t - \{\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\hat{M}_t\})dt + \sigma_Y dZ_Y^t]
\end{align*}
\]

\(^{35}\)Since \( \alpha_{3t} \) plays a key role in the economic analysis and appears frequently throughout the paper, we sometimes abbreviate it to \( \alpha_t \).
under any admissible strategy \((a_t)_{t \in [0,T]}\). Rearranging terms we can write

\[
d\hat{M}_t = (\mu_0 t + \mu_1 a_t + \mu_2 \hat{M}_t)dt + B_t^X dZ_t^X + B_t^Y dZ_t^Y, \tag{B.1}
\]

where

\[
\begin{align*}
\mu_1 t &= \alpha_3 \gamma_1 t \Sigma, \\
\mu_0 t &= -\mu_1 t [\alpha_0 + \alpha_2 L_t], \\
\mu_2 t &= -\alpha_3 \mu_1 t, \\
B_t^X &= \frac{\nu \alpha_3 \gamma_1 t}{\sigma_X}, \\
B_t^Y &= \frac{\alpha_3 \gamma_1 t}{\sigma_Y}. \tag{B.2}
\end{align*}
\]

This dynamic is linear in \(\hat{M}\). Also, since \(L_t\) depends only on the paths of \(X\), \(\mu_0 t\) is in \(p1\)'s information set. Similarly with \((a_t)_{t \in [0,T]}\), which is measurable with respect to \((\theta, X)\).

On the other hand, because \(p1\) always thinks that \(p2\) is on path, the public signal follows

\[
dX_t = (\nu a_t + \delta_0 t + \delta_1 \hat{M}_t dt + \delta_2 L_t)dt + \sigma_X dZ_t^X,
\]

from her perspective. This dynamic is also affine in the unobserved state \(\hat{M}\), with an intercept that is again measurable with respect to \((\theta, X)\). The hidden state \(\hat{M}\) along with the observation state \((\theta, X)\) is thus conditionally Gaussian. In particular, applying the filtering equations in Theorem 12.7 in Liptser and Shiryaev (1977) yields that \(M_t := \mathbb{E}_t[\hat{M}_t]\) and \(\gamma_{2t} := \mathbb{E}_t[(M_t - \hat{M}_t)^2]\) satisfy

\[
\begin{align*}
dM_t &= \underbrace{(\mu_0 t + \mu_1 a_t + \mu_2 \hat{M}_t)dt}_{=\mathbb{E}_t[(\mu_0 t + \mu_1 a_t + \mu_2 \hat{M}_t)dt]} + \frac{\sigma_X B_t^X + \gamma_2 t \delta_1 t}{\sigma_X^2} [dX_t - (\nu a_t + \delta_0 t + \delta_1 \hat{M}_t + \delta_2 L_t)dt], \tag{B.3}
\end{align*}
\]

\[
\dot{\gamma}_{2t} = 2\mu_2 t \gamma_{2t} + (B_t^X)^2 + (B_t^Y)^2 - \left(\frac{\sigma_X B_t^X + \gamma_2 t \delta_1 t}{\sigma_X}\right)^2, \tag{B.4}
\]

and where \(dZ_t := [dX_t - (\nu a_t + \delta_0 t + \delta_1 \hat{M}_t + \delta_2 L_t)dt]/\sigma_X\) is a Brownian motion with respect to the long-run player’s standpoint. (For notational simplicity, we have omitted the dependence of \(M\) and \(Z\) on the strategy \((a_t)_{t \in [0,T]}\) that is being followed.)

Inserting \(a_t = \beta_0 t + \beta_1 M_t + \beta_2 L_t + \beta_3 \theta\) into the right-hand side of \(dM_t\), and solving for \(dM_t\) as a function of the increment \(dX_t\), we obtain

\[
dM_t = [\mu_0 t + \mu_1 t M_t + \mu_2 t L_t + \mu_3 t \theta]dt + \hat{B}_t dX_t
\]
where

\[
\begin{align*}
\hat{\mu}_{0t} &= -\alpha_3 \gamma_{1t} \alpha_{0t} \Sigma + \alpha_3 \gamma_{1t} \beta_{0t} \Sigma + \frac{\nu \alpha_3 \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma_X^2} [-\nu \beta_{0t} - \delta_{0t}] \\
\hat{\mu}_{1t} &= \alpha_3 \gamma_{1t} \beta_{1t} \Sigma + \frac{\nu \alpha_3 \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma_X^2} [-\nu \beta_{1t} - \delta_{1t}] \\
\hat{\mu}_{2t} &= \frac{\nu \alpha_3 \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma_X^2} \beta_{3t} \\
\hat{\beta}_t &= \frac{\nu \alpha_3 \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma_X^2}.
\end{align*}
\]

Let \( R(t, s) = \exp(\int_t^s \hat{\mu}_{1u} du) \), and suppose and denote the prior distribution of \( \theta \) by \( \mathcal{N}(\hat{m}_0, \gamma_{10}) \); in particular, \( M_0 = \hat{m}_0 \). Path-by-path of \( X \), therefore,

\[
M_t = R(t, 0) \hat{m}_0 + \theta \int_0^t R(t, s) \hat{\mu}_{3s} ds + \int_0^t R(t, s)[\hat{\mu}_{0s} + \hat{\mu}_{2s} L_s] ds + \int_0^t R(t, s) \hat{B}_s dX_s,
\]

which, after recalling that we started from (11), yields a system of two equations:

\[
\begin{align*}
\chi_t &= \int_0^t R(t, s) \hat{\mu}_{3s} ds, \\
L_t &= \frac{R(t, 0) \hat{m}_0 + \int_0^t R(t, s)[\hat{\mu}_{0s} + \hat{\mu}_{2s} L_s] ds + \int_0^t R(t, s) \hat{B}_s dX_s}{1 - \chi_t}.
\end{align*}
\]

The validity of the construction boils down to finding a solution to the previously stated equation for \( \chi \) that takes values in \([0, 1]\). In fact, when this is the case,

\[
(1 - \chi_t) dL_t - L_t d\chi_t = [\hat{\mu}_{0t} + (\hat{\mu}_{1t}(1 - \chi_t) + \hat{\mu}_{2t}) L_t] dt + \hat{B}_t dX_t \\
\implies dL_t = \frac{L_t[\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}] dt + \hat{\mu}_{0t} dt + \hat{B}_t dX_t}{1 - \chi_t}.
\]

Thus, letting \( R_2(t, s) := \exp\left(\int_s^t \frac{\hat{\mu}_{1u} + \hat{\mu}_{2u} + \hat{\mu}_{3u}}{1 - \chi_u} du\right) \), we obtain that

\[
L_t = R_2(t, 0) \hat{m}_0 + \int_0^t R_2(t, s) \frac{\hat{\mu}_{0s}}{1 - \chi_s} ds + \int_0^t R_2(t, s) \frac{\hat{B}_s}{1 - \chi_s} dX_s,
\]

(B.5)
i.e., $L$ is a (linear) function of the paths of $X$ as conjectured. Moreover, in the particular case of $\nu = 0$, it is easy to verify that $dL_t = (l_{0t} + l_{1t}L_t)dt + B_tdX_t$, where

$$l_{0t} = -\frac{\gamma_t \chi_t \delta_{0t} \delta_{1t}}{\sigma^2_X (1 - \chi_t)} \quad \text{(B.6)}$$

$$l_{1t} = -\frac{\gamma_t \chi_t \delta_{1t} (\delta_{1t} + \delta_{2t})}{\sigma^2_X} \quad \text{(B.7)}$$

$$B_t = \frac{\gamma_t \chi_t \delta_{1t}}{\sigma^2_X (1 - \chi_t)}. \quad \text{(B.8)}$$

We will ultimately find a solution to the equation for $\chi$ that is of class $C^1$ and that takes values in $[0, 1)$. In particular, if $\chi$ is differentiable,

$$\dot{\chi}_t = \dot{\mu}_t \chi_t + \dot{\mu}_t$$

$$= \alpha_{3t} \gamma_{1t} \sum \left[ \chi_t \beta_{1t} - \alpha_{3t} \chi_t \right] + \chi_t \frac{\nu \alpha_{3t} \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma^2_X} [-\nu \beta_{1t} - \delta_{1t}]$$

$$e^{\alpha_{3t} (1 - \chi_t) - \beta_{3t}} + \alpha_{3t} \gamma_{1t} \beta_{3t} \Sigma + \frac{\nu \alpha_{3t} \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma^2_X} [-\nu \beta_{3t}]. \quad \text{(B.9)}$$

Using that $\alpha_{3t} = \beta_{3t} + \beta_{1t} \chi_t$, we obtain the following system of ODEs

$$\dot{\gamma}_{1t} = -\gamma_{1t}^2 (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma$$

$$\dot{\gamma}_{2t} = -2 \gamma_2 \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma + \gamma_{2t}^2 (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma - \left( \frac{\nu \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t) + \gamma_{2t} \delta_{1t}}{\sigma_X} \right)^2$$

$$\dot{\chi}_t = \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma (1 - \chi_t) - (\nu [\beta_{3t} + \beta_{1t} \chi_t] + \delta_{1t} \chi_t) \left( \frac{\nu \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t) + \gamma_{2t} \delta_{1t}}{\sigma^2_X} \right).$$

In the proof of the next lemma we establish that $\chi = \gamma_2 / \gamma_1$. After replacing $\nu = 0$ and $\gamma_2 = \chi \gamma$ in the third ODE, and using $\gamma$ for $\gamma_1$, the first and third equations of the previous system correspond to (15)–(16) as desired. The representation $L_t = \mathbb{E}[\theta | F_t^X]$ is proved in Lemma B.1 at the end of this subsection. \hfill \Box

**Proof of Lemma 3.** Consider the system $(\gamma_1, \gamma_2, \chi)$ from the proof of the previous lemma when $\nu = 0$ (in particular, $\Sigma$ becomes $1/\sigma^2_X$). Also, let $\delta_{1t} := \dot{u}_t + \dot{u}_t \alpha_{3t}.^{36}$ The local existence of a solution follows from continuity of the associated operator. Suppose that the maximal interval of existence is $[0, \tilde{T})$, with $\tilde{T} \leq T$.

Since the system is locally Lipschitz continuous in $(\gamma_1, \gamma_2, \chi)$ uniformly in $t \in [0, T]$ for

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36 All the results in this proof extend to a generic continuous function $\delta_1$ over $[0, T]$ in which the explicit dependence on $\beta$ and $\chi$ is not recognized, which happens when the myopic player becomes forward looking.
given continuous coefficients, it solution is unique over the same interval (Picard-Lindelof).
In particular, observe that \((\gamma_1 t, \gamma_2 t, \chi) = (\gamma^o, 0, 0)\) solves the system as long as \(\beta_3 = 0\). Without loss of generality then, assume \(\beta_{30} \neq 0\).

Observe that \(\gamma_1\) is (weakly) decreasing over \([0, \tilde{T}]\), so \(\gamma_{1t} \leq \gamma^o\). Suppose there is a time at which \(\gamma_1\) is strictly negative. Let \(s < t\) be the first time \(\gamma_1\) crosses zero, and notice that for \(t > s\) close to \(s\),

\[
0 > \gamma_{1t} = \int_s^t \dot{\gamma}_{1u} du = -\int_s^t \gamma_{1u}^2 [\beta_{3u} + \beta_{1u} \chi_u]^2 \Sigma ds \geq 0,
\]

which is a contradiction. Thus, \(\gamma_{1t} \in [0, \gamma^o]\) for all \(t \in [0, \tilde{T}]\). Moreover, if \(\gamma_t > 0\), straightforward integration shows that

\[
\gamma_{1t} = \frac{\gamma^o}{1 + \int_0^t [\beta_{3u} + \beta_{1u} \chi_u]^2 \Sigma ds}.
\]

Since \(\tilde{\beta}\) is continuous over \([0, T]\), if \(\gamma\) ever vanishes in \([0, \tilde{T}]\) we must have that \(\chi\) diverges at such point; by definition of \(\tilde{T}\), however, that point must be \(\tilde{T}\). Thus, \(\gamma_{1t} > 0\) in \([0, \tilde{T}]\) (regardless of whether \(\chi\) diverges at \(\tilde{T}\) or not).

We now show that \(0 < \gamma_{2t} < \gamma_{1t}\) for \(t > 0\). In fact, since \(\gamma_{20} = 0\), \(\gamma_{10} > 0\) and \(\beta_{30} > 0\), we have \(\gamma_{2t} > 0\) for \(\epsilon\) small. Consider now \([\epsilon, \tilde{t}]\) with \(\tilde{t} \in (\epsilon, \tilde{T})\). Then,

\[
f^{\gamma_2}(t, x) := -2x \frac{\gamma_{1t}(\beta_{3t} + \beta_{1t} \chi_{1t})^2}{\sigma_Y^2} + \frac{\gamma_{1t}^2 (\beta_{3t} + \beta_{1t} \chi_{1t})^2}{\sigma_Y^2} - \left( \frac{x \delta_{1t}}{\sigma_X} \right)^2,
\]

is locally Lipschitz continuous with respect to \(x\) uniformly in \(t \in [\epsilon, \tilde{t}]\). Since \(0 = f_{\gamma_2}(t, 0) \leq 0 = \dot{\gamma}_{2t} - f^{\gamma_2}(t, \gamma_{2t})\) and \(0 < \gamma_{2t}\), we obtain that \(\gamma_{2t} > 0\) for all \(t \in [0, \tilde{t}]\) by means of standard comparison theorems (e.g., Theorem 1.3 in Teschl), and hence over \((0, \tilde{T})\) as well.

Now, let \(z_t := \gamma_{2t} - \gamma_{1t}, t < \tilde{T}\). Using the ODEs for \(\gamma_1\) and \(\gamma_2\) we deduce that

\[
\dot{z}_t < -\frac{2 \gamma_{1t}(\beta_{3t} + \beta_{1t} \chi_{1t}) z_t}{\sigma_Y^2}, \quad z_0 = \gamma_{20} - \gamma_{10} = -\gamma^o < 0.
\]

It is then easy to conclude that (Gronwall’s inequality),

\[
z_t < z_0 \exp \left( -\int_0^t \frac{2 \gamma_{1s}(\beta_{3s} + \beta_{1s} \chi_{1s})}{\sigma_Y^2} ds \right) < 0, \quad t < \tilde{T},
\]

as \(\gamma_{1t}(\beta_{3t} + \beta_{1t} \chi_{1t})\) is continuous over \([0, t], t < \tilde{T}\). Thus, \(\gamma_{2t} < \gamma_{1t}\) for all \(t \in [0, \tilde{T}]\).

With this in hand, \(\gamma_{2t}/\gamma_{1t} \in (0, 1)\) for all \(t \in (0, \tilde{T})\), and \(\gamma_{20}/\gamma_{10} = 0\). Moreover, it is easy
to verify that the previous ratio solves the χ–ODE. By uniqueness, χ = γ/γ. Replacing
γ = χγ and ν = 0 in the χ–ODE above yields (16), i.e.,

\[ \dot{x}_t = \gamma_{1t}(\beta_{3t} + \beta_{1t}x_t)^2 \frac{(1 - \chi_t)}{\sigma_Y^2} - \gamma_{1t} \frac{(\delta_{1t}x_t)^2}{\sigma_X^2}, \quad t \in [0, \tilde{T}). \]

By the previous analysis, (γ1, γ2, χ) is bounded over [0, \tilde{T}). If \( \tilde{T} < T \), the solution can be extended strictly beyond \( \tilde{T} \) thanks to the continuity of the associated operator (Peano’s theorem), contradicting the definition of \( \tilde{T} \). Thus, the only option is that \( \tilde{T} = T \), in which case the system admits a continuous extension to \( T \).\(^{37}\) By continuity, such an extension is unique, the desired properties (\( \chi = \gamma_2/\gamma_1 \) stated in Lemma 2; \( \chi \) solves (16) and \( \chi \in (0, 1) \); and \( \gamma^o \in (0, \gamma^o) \)) hold up to \( T \) by the exact same arguments previously applied over strict compact subsets of \( [0, \tilde{T}) \) now over \( [0, T) \).\(^{38}\)

**Proof of Lemma 4.** The long-run player’s problem is to choose an admissible \( a := (a_t)_{t \in [0, T]} \) that maximizes

\[ U(a) := \mathbb{E}_0 \left[ \int_0^T e^{-rt}U(a_t, \delta_{0t} + \delta_{1t} \dot{M}_t + \delta_{2t} L_t, \theta)dt \right] \]

where \( (\dot{M}_t)_{t \geq 0} \) is given by (B.1) and \( (L_t)_{t \geq 0} \) by (B.5). Using that the flow is quadratic, we obtain that

\[ U(a) = \mathbb{E}_0 \left[ \int_0^T e^{-rt}U(a_t, \delta_{0t} + \delta_{1t} \dot{M}_t + \delta_{2t} L_t, \theta)dt \right] + \frac{U_\delta}{2} \mathbb{E}_0 \left[ \int_0^T e^{-rt}\mathbb{E}_t[(M^a_t - \dot{M}^a_t)^2]dt \right] \]

with \( M^a_t := \mathbb{E}_t[\dot{M}^a_t] \), where we have made explicit the dependence of both processes on the strategy followed. By the proof of Lemma 2, \( (M^a_t)_{t \in [0, T]} \) evolves as in (B.3), i.e.,

\[ dM^a_t = (\mu_{0t} + \mu_{1t}a_t + \mu_{2t}M^a_t)dt \]

\[ + \frac{\sigma_X B^X_t + \gamma_{2t} \delta_{1t} \sigma_X \dot{Z}^a_t}{\sigma_X} \]

where \( \dot{Z}^a_t := [dX_t - (\nu a_t + \delta_{0t} + \delta_{1t} M^a_t + \delta_{2t} L_t)dt]/\sigma_X \) is a Brownian motion from the long-run player’s standpoint, \( (\mu_0, \mu_1, \mu_2, B^X_t) \) are given by (B.2), and where \( \gamma_{2t} \) evolves as in (B.4). Moreover, from the same filtering equations (B.3)–(B.4) we know that \( \mathbb{E}_t[(M^a_t - \dot{M}^a_t)^2] \) is independent of the strategy followed, and that it coincides with \( \gamma_{2t}, \ t \in [0, T] \). Thus, the

\(^{37}\)For a generic system \( \dot{z}_t = f(t, z_t) \), if \( z \) is bounded over \( [0, T] \) and \( f \) continuous, there exists \( K \) s.t. \( |x_t - x_s| < M|t - s| \); but this implies that \( (x_s)_{s,T} \) is Cauchy, and hence the limit exists.

\(^{38}\)An alternative way of seeing that \( \chi < 1 \) is that \( \dot{\chi} \leq \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t)^2 (1 - \chi_t)/\sigma_Y^2 \), and so \( \chi_t \leq 1 - \gamma_t/\gamma^o \) by standard comparison theorems, as the latter function satisfies \( \dot{z}_t = \gamma_{1t} (\beta_{3t} + \beta_{1t} z_t)^2 (1 - z_t)/\sigma_Y^2, \ z_0 = 0. \)
long-run player’s problem reduces to

\[
\max_{(a_t)_{t \geq 0} \text{ admissible}} \mathbb{E}_0 \left[ \int_0^T e^{-rt} U(a_t, \delta_{0t} + \delta_{1t}M^a_t + \delta_{2t}L_t, \theta)dt \right]
\]

where \((M^a_t)_{t \in [0,T]}\) is as above, and \((L_t)_{t \geq 0}\) is linear in the paths of \(X\) according to (B.5). In differential form, the latter process can be written as

\[
dL_t = \frac{1}{1 - \chi_1} \left\{ L_t[\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}] + \hat{\mu}_{0t} + \hat{B}_t[\nu a_t + \delta_{0t} + \delta_{1t}M^a_t + \delta_{2t}L_t] \right\} dt + \frac{\sigma_X \hat{B}_t}{1 - \chi_t} dZ^a_t.
\]

where we used that \(dX_t = (\nu a_t + \delta_{0t} + \delta_{1t}M^a_t + \delta_{2t}L_t)dt + \sigma_X dZ^a_t\) from the long-run player’s standpoint. (Refer to the proof of Lemma 2 for the expressions for \((\hat{\mu}_{0t}, \hat{\mu}_{1t}, \hat{\mu}_{2t}, \hat{\mu}_{3t}, \hat{B}^X_t)\).)

So far, we have fixed an admissible strategy \((a_t)_{t \in [0,T]}\) (in the sense of Section 3) for the long-run player, and then obtained processes \(M^a\) and \(Z^a\) that potentially depend on that choice. The above problem thus differs from traditional control problems with perfectly observed states in that the Brownian motion is, in principle, affected by the choice of strategy.

With linear dynamics, however, the separation principle (e.g., Liptser and Shiryaev, 1977, Chapter 16), applies. In fact, the solution to the long-run player’s problem can be found by first fixing a Brownian motion, say, \(Z_t := Z^0_t\) (i.e., \(Z^a_t\) when \(a \equiv 0\)), and then solving the optimization problem that replaces \(Z^a\) by \(Z\) in the laws of motion of \(\hat{M}^a\) and \(L\). The method works to the extent that \(Z^a \equiv Z\) for all \((a_t)_{t \geq 0}\): it is easy to conclude from (B.1) and (B.3) that the process \(\hat{M}^a_t - M^a_t\) is independent of the strategy followed, and hence so is \(Z^a_t\), given that \(\sigma_X dZ^a_t = dX_t - (\nu a_t + \delta_{0t} + \delta_{1t}M^a_t + \delta_{2t}L_t)dt = \delta_{1t}(\hat{M}^a_t - M^a_t)dt + \sigma_X dZ^X_t\) under the true data-generating process, thanks to the linearity of the dynamics. In this procedure, therefore, one filters as a first step, and then optimizes afterwards using the posterior mean as a controlled state.\(^{39}\)

Returning to the \(\nu = 0\) case, we can then insert \(Z_t\) in the dynamic of \(M^a_t\). Omitting the dependence of the resulting process on \(a\) (as any control problem does), it is easy to see that

\[
dM_t = \frac{\gamma_t \alpha_{3t}}{\sigma^2_X} (a_t - [\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}M_t])dt + \frac{\chi_t \gamma_t \delta_{1t}}{\sigma_X} dZ_t.
\]

As for the expression for \(L\) (display (19)), this one follows from (17) using that \(dX_t =

\(^{39}\)Relative to Chapter 16 in Liptser and Shiryaev (1977), our problem is more general in that it allows for a linear component in the flow, and the public signal can be controlled (when \(\nu \neq 0\)). The first generalization is clearly innocuous. As for the second, the key behind the separation principle is that the innovations \(dX_t - \mathbb{E}_t[dX_t]\) are independent of the strategy followed, which also happens when \(\nu \neq 0\). Given any admissible strategy \((a_t)_{t \geq 0}\), therefore, the fact that the filtrations of \(Z, Z^a\) and \(X^a\) satisfy \(\mathcal{F}^Z_t = \mathcal{F}^{Z^a}_t \subseteq \mathcal{F}^{X^a}_t, t \geq 0\), means the optimal control found by using \(Z\) is weakly better than any such \((a_t)_{t \geq 0}\). See p.183 in section 16.1.4 in Liptser and Shiryaev (1977) for more details in a context of a quadratic regulator problem.
(δ₀ + δ₂L_t + δ₁1M_t)dt + σ_XdZ_t from the long-run player’s perspective. In fact, it is easy to see from (B.6)–(B.8) that

\[ l_{0t} + B_t(δ_{0t} + (l_{1t} + B_tδ_{1t}))L_t + B_tδ_{1t}M_t = \frac{γ_t(1 - χ_t)}{σ^2_X}(M_t - L_t). \]

This concludes the proof. □

**Lemma B.1.** The process \( L \) is the belief about \( θ \) held by an outsider who observes only \( X \).
Moreover, \( \left( \theta \middle| M_{1t} \right) \mathcal{F}_t^X \sim \mathcal{N}(M_{t}^{\text{out}}, \gamma_{t}^{\text{out}}) \) where \( M_{t}^{\text{out}} = \begin{pmatrix} L_t \\ L_t \end{pmatrix} \) and \( \gamma_{t}^{\text{out}} = \begin{pmatrix} \frac{γ_t(1 - χ_t)}{σ^2_X} \\ \frac{γ_t(1 - χ_t)}{σ^2_X} \end{pmatrix}. \)

**Proof.** The outsider jointly filters the state \( v_t = (θ_t, M_{1t})' \). For the evolution of the state and the signal, we adopt notation from Liptser and Shiryaev (1977) (Section 12.3). From the outsider’s perspective, both players (and in particular player 2) are on the equilibrium path, and thus the outsider believes that \( v_t \) evolves as

\[ dv_t = a_1(t, X^{\text{out}})v_tdt + b_1(t, X)dW_1(t) + b_2(t, X)dW_2(t), \]

where \( a_1(t, X^{\text{out}}) := \begin{pmatrix} 0 & 0 \\ z_t & -z_t \end{pmatrix} \) for \( z_t := \frac{(να_{3t})^2γ_{1t}}{σ^2_X} + σ^2_{3t}γ_{1t}, b_1(t, X) := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), \( b_2(t, X) := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), and \( W_1(t) := \begin{pmatrix} W_{11}(t) \\ Z_t^{Y} \end{pmatrix} \) and \( W_2(t) := Z_t^{X} \), where \( W_{11}(t) \) is a standard Brownian motion and \( W_{11}(t), Z_t^{Y} \) and \( Z_t^{X} \) are mutually independent. The signal is

\[ dX_t^{\text{out}} := dX_t - [δ_{0t} + δ_{2t}L_t + ν(α_{0t} + α_{2t}L_t)]dt = A_1(t, X)v_t + B_1(t, X)W_1(t) + B_2(t, X)W_2(t), \]

where \( A_1(t, X) := (να_{3t} \ δ_{1t}), B_1(t, X) := \begin{pmatrix} 0 & 0 \end{pmatrix} \) and \( B_2(t, X) = σ_X. \)

Hence, denoting \( M_{t}^{\text{out}} = \begin{pmatrix} M_{t}^{\text{out}, 1} \\ M_{t}^{\text{out}, 2} \end{pmatrix} \) and \( \gamma_{t}^{\text{out}} = \begin{pmatrix} \gamma_{t}^{\text{out}, 11} & \gamma_{t}^{\text{out}, 12} \\ \gamma_{t}^{\text{out}, 21} & \gamma_{t}^{\text{out}, 22} \end{pmatrix} \) and imposing \( \gamma_{t}^{\text{out}, 21} = \gamma_{t}^{\text{out}, 12} \), we have from the standard filtering equations of Liptser and Shiryaev (1977) (Theorem 12.7)
that \( (\theta \mid M_t) \mid \mathcal{F}_t^X \sim \mathcal{N}(M_{t}^{\text{out}}, \gamma_{t}^{\text{out}}) \), where \( M^{\text{out}} \) and \( \gamma^{\text{out}} \) are the unique solutions to

\[
dM_t^{\text{out}} = a_1(t, X) M_t^{\text{out}} + \frac{1}{\sigma_X^2} \left[ \begin{pmatrix} 0 \\ \nu \alpha_3 \gamma_{1t} \end{pmatrix} + \gamma_t^{\text{out}} \begin{pmatrix} \nu \alpha_3 t \\ \delta_{1t} \end{pmatrix} \right] \times \cdots \times \left\{ dX_t^{\text{out}} - (\nu \alpha_3 t M_{t,1}^{\text{out}} + \delta_{1t} M_{t,2}^{\text{out}}) dt \right\} \tag{B.10}
\]

\[
\gamma_t^{\text{out}} = a_1(t, X) \gamma_t^{\text{out}} + \gamma_t^{\text{out}} a_1^{*} + \frac{1}{\sigma_X^2} \left[ \begin{pmatrix} 0 \\ \nu \alpha_3 \gamma_{1t} \end{pmatrix} + \gamma_t^{\text{out}} \begin{pmatrix} \nu \alpha_3 t \\ \delta_{1t} \end{pmatrix} \right] \times \cdots \times \left[ \begin{pmatrix} 0 \\ \nu \alpha_3 \gamma_{1t} \end{pmatrix} + \gamma_t^{\text{out}} \begin{pmatrix} \nu \alpha_3 t \\ \delta_{1t} \end{pmatrix} \right]^{*} \tag{B.11}
\]

with initial conditions \( M_0^{\text{out}} = \begin{pmatrix} \hat{m}_0 \\ \hat{m}_0 \end{pmatrix} \) and \( \gamma_0^{\text{out}} = \begin{pmatrix} \gamma^o \\ 0 \\ 0 \end{pmatrix} \).

Recall that

\[
\dot{\gamma}_{1t} = -\frac{\alpha_2^2}{\sigma_Y^2} \gamma_{1t}^2
\]

\[
\dot{\chi}_t = \frac{\gamma_{1t} \alpha_3^2 (1 - \chi_t)}{\sigma_Y^2} - \frac{\gamma_{1t} \delta_{1t}^2 \chi_t^2}{\sigma_X^2}
\]

with initial conditions \( \gamma_{10} = \gamma^o \) and \( \chi_0 = 0 \). It is then straightforward to verify that \( \gamma_t^{\text{out}} = \begin{pmatrix} \gamma_{1t} \\ \gamma_{1t} \chi_t \\ \gamma_{1t} \chi_t \end{pmatrix} \) satisfies the differential equation for \( \gamma^{\text{out}} \) above along with given initial condition. Moreover, \( \gamma_t^{\text{out}} = \begin{pmatrix} \gamma_{1t} \\ \gamma_{1t} \chi_t \\ \gamma_{1t} \chi_t \end{pmatrix} \) is positive semidefinite as its leading principal minors are positive multiples of 1 and \( \chi - \chi^2 > 0 \).

Next, substitute given the solution \( \gamma_t^{\text{out}} \) into (B.11) and subtract the equation for the second component from its first to obtain the following SDE for \( \bar{M} := M_{t}^{\text{out}} - M_{2}^{\text{out}} \)

\[
d\bar{M}_t = -\Sigma \bar{M}_t \alpha_3^2 \gamma_{1t}
\]

with initial condition \( \bar{M}_0 = 0 \). Now if \( \bar{M}_t > 0 \), then \( d\bar{M}_t < 0 \), giving us a contradiction; likewise for the case \( \bar{M}_t < 0 \). It follows that \( \bar{M}_t = 0 \), and thus \( M_{t,1}^{\text{out}} = M_{t,2}^{\text{out}} \), for all \( t \geq 0 \). Substituting this back into (B.11), we have
The goal is to find a function $f$ such that

$$
d M_{t,1}^{\text{out}} = \frac{\gamma_I(\nu \alpha_3 t + \delta_{1t} \chi_I)}{\sigma_X^2(1 - \chi_I)} (d X_t^{\text{out}} - (\nu \alpha_3 t + \delta_{1t}) M_{t,1}^{\text{out}}) dt
$$

$$
= \frac{\gamma_I(\nu \alpha_3 t + \delta_{1t} \chi_I)}{\sigma_X^2(1 - \chi_I)} [d X_t - (\nu \alpha_0 t + \delta_{0t} + M_{t,1}^{\text{out}}(\nu \alpha_3 t + \delta_{1t} t) + L_t(\nu \alpha_2 t + \delta_{2t})) dt] \quad \text{(B.13)}
$$

On the other hand, we have

$$
d L_t = \frac{L_t [\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}]}{1 - \chi_I} dt + \hat{\mu}_{0t} dt + \hat{B}_t dX_t
$$

$$
= \frac{\gamma_I(\nu \alpha_3 t + \delta_{1t} \chi_I)}{\sigma_X^2(1 - \chi_I)} [d X_t - (\delta_{0t} + \nu \alpha_0 t + L_t[(\nu(\alpha_2 t + \alpha_3 t) + \delta_{1t} + \delta_{2t})) dt] \quad \text{(B.14)}
$$

Defining $\bar{L}_t := M_{t,1}^{\text{out}} - L_t$ and subtracting (B.14) from (B.13), we obtain

$$
d \bar{L}_t = -\frac{\gamma_I(\nu \alpha_3 t + \delta_{1t} \chi_I)}{\sigma_X^2(1 - \chi_I)} \bar{L}_t (\nu \alpha_3 t + \delta_{1t})
$$

with initial condition $\bar{L}_0 = \hat{m}_0 - \hat{m}_0 = 0$. We conclude that $\bar{L}_t = 0$, and thus $L_t = M_{t,1}^{\text{out}} = M_{t,2}^{\text{out}}$, for all $t \geq 0$.

**Proof of Lemma 5.** Suppose $\delta_1 = \hat{u}_a \alpha_3, \hat{u}_a > 0$. The $\chi$-ODE for $\nu \in [0,1]$ boils down to

$$
\dot{\chi}_t = \gamma_I \alpha_3^2 \left( \frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right) (1 - \chi_t) - \frac{(\nu + \hat{u}_a \chi_t)^2}{\sigma_X^2} := -\gamma_I \alpha_3^2 Q(\chi_t).
$$

The goal is to find a function $f : [0, \bar{\chi}) \to [0, \gamma^0]$, some $\bar{\chi} \in (0,1)$, such that $f(\chi_t) = \gamma_t$ for all $t \geq 0$. When this is the case, and such $f$ is differentiable, $f'(\chi_t) \chi_t = \gamma_t$. Thus, if $\alpha_3 t > 0$,

$$
\frac{f'(\chi_t)}{f(\chi_t)} = \frac{1}{\sigma_Y^2 Q(\chi_t)}.
$$

Thus, we aim to solve the ODE

$$
\frac{f'(\chi)}{f(\chi)} = \frac{1}{\sigma_Y^2 Q(\chi)}, \quad \chi \in (0, \bar{\chi}), \quad \text{and} \quad f(0) = \gamma^0,
$$

over some domain $[0, \bar{\chi})$, with the property that $f(\chi) > 0$ if $\chi > 0$.

To this end, let

$$
c_2 = \sqrt{\frac{b^2 + 4(\hat{u}_a)^2}{2(\hat{u}_a/\sigma_X)^2}} - b \quad \text{and} \quad c_1 = -\sqrt{\frac{b^2 + 4(\hat{u}_a)^2}{2(\hat{u}_a/\sigma_X)^2}} - b,
$$

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where \( b := [\nu^2/\sigma_X^2 + 1/\sigma_Y^2] + 2\nu \hat{u}_a/\sigma_X^2 \), be the roots of the quadratic

\[
Q(\chi) = \left( \frac{\hat{u}_a}{\sigma_X} \right)^2 \chi^2 + \chi \left( \frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right) + 2\nu \hat{u}_a^2/\sigma_X^2 - \frac{1}{\sigma_Y^2}.
\]

Clearly, \(-c_1 < 0 < c_2\). Also, it is easy to verify that \( c_2 < 1 \).\(^{40}\) Thus,

\[
\frac{1}{\sigma_Y^2}Q(\chi) = -\frac{\sigma_X^2}{(\sigma_Y^2 \hat{u}_a)^2(c_1 + c_2)} \left[ \frac{1}{\chi + c_1} - \frac{1}{\chi - c_1} \right]
\]

is well defined (and negative) over \([0, c_2]\) with \(1/(\chi + c_1) > 0\) and \(-1/(\chi - c_2) > 0\) over the same domain. We can then set \( \chi = c_2 \) and solve

\[
\int_0^\chi \frac{f'(s)}{f(s)} ds = -\frac{\sigma_X^2}{(\sigma_Y^2 \hat{u}_a)^2(c_1 + c_2)} \log \left( \frac{\chi + c_1}{c_2 - \chi c_1} \right) \Rightarrow f(\chi) = f(0) \left( \frac{c_1}{c_2} \right)^{1/d} \left( \frac{c_2 - \chi}{\chi + c_1} \right)^{1/d}
\]

where \(1/d = \sigma_X^2/[(\sigma_Y^2 \hat{u}_a)^2(c_1 + c_2)] > 0\). We then impose \( f(0) = \gamma^o \), thus obtaining a strictly positive and decreasing function that has the initial condition we look for. Moreover, letting \( \gamma = f(\chi) \), its inverse is decreasing and given by

\[
\chi(\gamma) = f^{-1}(\gamma) = c_1 c_2 \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d}.
\]

When \( \gamma_1 = \gamma^o \), we have that \( \chi = 0 \), whereas when \( \gamma = 0 \), it follows that \( \chi = c_2 \) as desired.

We verify that this candidate satisfies the \( \chi \)-ODE; we do this for the \( \nu = 0 \) case only. To this end, it is easy to verify that

\[
\frac{d(\chi(\gamma_t))}{dt} = \frac{\sigma_X^2 \gamma_t}{\sigma_Y^2[c_1 + c_2(\gamma/\gamma^o)^d]^2} c_1 c_2 d[c_1 + c_2] \left( \frac{\gamma_t}{\gamma^o} \right)^d.
\]

By construction, moreover,

\[
c_1 c_2 = c_1 - c_2 = \frac{\sigma_X^2}{\sigma_Y^2 \hat{u}_a^2}
\]

which follows from equating the first- and zero-order coefficients in \( Q(\chi) = \hat{u}_a^2 \chi^2/\sigma_X^2 + \chi/\sigma_Y^2 - 1/\sigma_Y^2 = \hat{u}_a^2(\chi - c_2)(\chi + c_1)/\sigma_X^2 \). Thus, \( dc_1 c_2 = c_1 + c_2 \).

On the other hand,

\[
\frac{[\hat{u}_a \chi(\gamma)]^2}{\sigma_X^2} = \frac{\hat{u}_a^2}{\sigma_X^2} \left[ c_1 c_2 \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d} \right]^2 = c_1^2(1 - c_2) \left[ \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d} \right]^2
\]

\(^{40}\)This follows from squaring both sides of \( \sqrt{b^2 + 4(\hat{u}_a)^2/|\sigma_X \sigma_Y|^2} < b + 2(\hat{u}_a)^2/\sigma_X^2 \) using that \( b + 2(\hat{u}_a)^2/\sigma_X^2 < 0 \) and \( b = [\nu^2/\sigma_X^2 + 1/\sigma_Y^2] + 2\nu \hat{u}_a/\sigma_X^2 \).
where we used that $c_1^2 c_2^2 / \sigma_X^2 = c_1^2 (1 - c_2) / \sigma_X^2$ follows from $\dot{u}_a^2 c_2^2 / \sigma_X^2 = (1 - c_2) / \sigma_X^2$, by definition of $c_2$. Thus, the right-hand side of the $\chi$-ODE evaluated at our candidate $\chi(\gamma)$ satisfies

$$
\gamma_1 \alpha_3^2 \left( \frac{1 - \chi}{\sigma_Y^2} \right) - \left( \frac{\dot{u}_a \chi}{\sigma_X^2} \right) \bigg|_{\chi=\chi(\gamma)} = \frac{\alpha_3^2 \gamma_1}{\sigma_Y^2} \left( 1 - \chi - c_1^2 (1 - c_2) \left[ \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2 (\gamma/\gamma^o)^d} \right]^2 \right).
$$

Thus, using that $c_1 c_2 d = c_1 + c_2$ in our expression for $d(\chi(\gamma))/dt$, it suffices to show that

$$
[c_1 + c_2]^2 \left( \frac{\gamma^a}{\gamma^o} \right)^d = (1 - \chi) [c_1 + c_2 (\gamma/\gamma^o)^d]^2 - c_1^2 (1 - c_2) [1 - (\gamma/\gamma^o)^d]^2.
$$

Using that $\chi[c_1 + c_2 (\gamma/\gamma^o)^d] = 1 - (\gamma/\gamma^o)$, it is easy to conclude that this equality reduces to three equations

$$
0 = c_1^2 - c_1^2 c_2 - c_1^2 c_1 = \frac{\dot{u}_a \chi}{\sigma_X^2}
$$

$$
(c_1 + c_2)^2 = 2c_1 c_2 - c_1 c_2 (c_2 - c_1) + 2c_1^2 (1 - c_2)
$$

$$
0 = c_2^2 + c_1 c_2 - c_1^2 (1 - c_2).
$$

capturing the conditions on the constant, $(\gamma/\gamma^o)^d$ and $(\gamma/\gamma^o)^{2d}$, respectively. The first condition is trivially satisfied. As for the third, by definition of $c_1$ and $c_2$ we have that $c_2^2 / (1 - c_2) = \sigma_X^2 / (\dot{u}_a \sigma_Y^2) = c_1^2 / (1 + c_1)$. Thus, $c_1^2 (1 - c_2) = c_2^2 (1 + c_1)$, and the result follows. For the second, use that $c_1 c_2 (c_2 - c_1) = -(c_1 - c_2)^2$ and that $c_1^2 (1 - c_2) = c_2^2 (1 + c_1)$ to conclude

$$
2c_1 c_2 - c_1 c_2 (c_2 - c_1) + 2c_1^2 (1 - c_2) = c_1^2 + c_2^2 + 2c_1^2 (1 + c_1) = c_1^2 + c_2^2 + 2c_2 + 2c_2 \underbrace{c_2 c_1}_{=c_1-c_2} = (c_1 + c_2)^2.
$$

Thus, $\chi(\gamma)$ as postulated satisfies the $\chi$-ODE. We then conclude by uniqueness of any such solution.

Finally, when $\dot{u}_a = 0$, we have that $\delta_1 \equiv 0$, and the $\chi$-ODE reduces to $\ddot{\chi} = \alpha_3^2 \gamma_1 (1 - \chi t) / \sigma_Y^2$, $\chi_0 = 0$. It is then easy to verify that $\chi(\gamma) = 1 - \gamma t / \gamma^o$ satisfy the ODE, and hence we conclude using the same uniqueness argument.

### Proof of Theorem 1

We begin by proving that a solution to the BVP exists, and any solution has the stated properties; from there we will establish the rest of the solution. Recall that $\alpha_t = \alpha_3 t = \beta_1 t \chi_t + \beta_3 t$. 

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We begin by reversing time and posing the associated IVP parameterized by an initial guess \( \gamma_0 = \gamma^F \in [0, \gamma^o] \):

\[
\begin{align*}
\dot{v}_{6t} &= -\beta_{2t}^2 - 2\beta_{1t}\beta_{2t}(1 - \chi_t) + \beta_{1t}^2(1 - \chi_t)^2 - \frac{2v_{6t} \alpha_2^2 \gamma \chi_t}{\sigma_X^2(1 - \chi_t)} \\
\dot{v}_{8t} &= 2\beta_{2t} + 2(1 - 2\alpha_t)\beta_{1t}(1 - \chi_t) + 4\beta_{1t}^2 \chi_t(1 - \chi_t) - \frac{v_{8t} \alpha_2^2 \gamma \chi_t}{\sigma_X^2(1 - \chi_t)} \\
\dot{\beta}_{1t} &= \frac{\alpha_t \gamma_t}{2\sigma_X^2(1 - \chi_t)} \left\{-2\sigma_X^2(\alpha_t - \beta_{1t})\beta_{1t}(1 - \chi_t) + 2\sigma_X^2 \alpha_t \chi_t(\beta_{2t}(1 + 2\beta_{1t}\chi_t) - \beta_{1t}[1 - \chi_t]) + \alpha_t^2 \beta_{1t} \gamma_t \chi_t v_{8t}\right\} \\
\dot{\beta}_{2t} &= \frac{\alpha_t \gamma_t}{2\sigma_X^2(1 - \chi_t)} \left\{-2\sigma_X^2 \beta_{1t}^2(1 - \chi_t)^2 - 2\sigma_X^2 \alpha_t \beta_{1t}^2 \chi_t^2(1 - 2\beta_{2t}) + \alpha_t^2 \beta_{3t} \gamma_t \chi_t v_{8t}\right\} \\
\dot{\beta}_{3t} &= \frac{\alpha_t \gamma_t}{2\sigma_X^2(1 - \chi_t)} \left\{2\sigma_X^2 \beta_{1t}(1 - \chi_t)\beta_{3t} - 2\sigma_X^2 \alpha_t \beta_{1t}^2 \chi_t^2(1 - 2\beta_{3t}) + \alpha_t^2 \beta_{3t} \gamma_t \chi_t v_{8t}\right\} \\
\dot{\alpha}_t &= \frac{\alpha_t^3 \gamma_t \chi_t}{2\sigma_X^2(1 - \chi_t)} \left\{4\sigma_X^2 \beta_{2t} \chi_t + \alpha_t \gamma_t v_{8t}\right\} \\
\dot{\gamma}_t &= \frac{\gamma_t^2 \alpha_t^2}{\sigma_Y^2} \left\{4\sigma_Y^2 \beta_{2t} \chi_t + \alpha_t \gamma_t v_{8t}\right\}
\end{align*}
\]

with initial conditions \( v_{60} = v_{80} = 0 \), \( \beta_{10} = \frac{1}{2(2 - \chi_0)} \), \( \beta_{20} = \frac{1 - \chi_0}{2(2 - \chi_0)} \), \( \beta_{30} = \frac{1}{2} \) and \( \gamma_0 = \gamma^F \), where \( \chi_t = \chi(\gamma_t) \) using the function \( \chi : \mathbb{R}_+ \to (-\infty, c_2) \) as defined in Lemma 5. Hereafter, we use the notation \( \bar{\chi} := c_2 \). Recall also that \( c_2 < 1 \), and note by inspection that \( \chi(\gamma) \downarrow -c_1 < 0 \) as \( \gamma \to +\infty \), and since \( \chi \) is decreasing, it has range \((-c_1, \bar{\chi})\). The right hand sides of the equations above are of class \( C^1 \) over the domain \( \{(v_6, v_8, \beta_1, \beta_2, \beta_3, \gamma) \in \mathbb{R}^5 \times \mathbb{R}_+\} \), and hence, subject to existence, they have unique solutions over [0, T] given the initial conditions. To solve the BVP, our goal is to show that there exists \( \gamma^F \in (0, \gamma^o) \) such that a (unique) solution to the IVP above exists and it satisfies \( \gamma_T(\gamma^F) = \gamma^o \).

Note that if \( \gamma^F = 0 \), then the IVP has the following (unique) solution: for all \( t \in [0, T] \), \( \chi_t = \bar{\chi} \), \( \beta_{1t} = \frac{1}{2(2 - \bar{\chi})} \), \( \beta_{2t} = \frac{1 - \bar{\chi}}{2(2 - \bar{\chi})} \), and \( \beta_{3t} = \frac{1}{2} \) (so that \( \alpha_t = \frac{1}{2(2 - \bar{\chi})} \)) and \( \gamma_t = 0 \), with \( v_6 \) and \( v_8 \) obtained from integration of their ODEs. (Clearly, this does not correspond to a solution to the BVP, since \( \gamma_T(0) = 0 < \gamma^o \).)

We now consider \( \gamma^F > 0 \). Given the \( C^1 \) property mentioned above, and the existence of a solution to the IVP for \( \gamma^F = 0 \), there exists \( \epsilon > 0 \) such that a (unique) solution to the IVP exists over [0, T] for each \( \gamma^F \in (0, \epsilon) \) (see Theorem on page 397 in Hirsch et al. (2004)).

Observe that for \( \gamma^F > 0 \), we can change variables using \( \bar{v}_i := \gamma v_i \) for \( i = 6, 8 \) and create a new IVP in \( (\bar{v}_6, \bar{v}_8, \beta_1, \beta_2, \beta_3, \gamma) \). We label this System 1, and it consists of (B.17)-(B.21)
γ solution on the initial conditions; in particular, the terminal value $C$ of class $\gamma$ with respect to set inclusion. Since the RHS of the equations that comprise System 2 are solving System 1 (and hence the BVP).

We can construct the solutions to (B.18) (which will be $1 - \gamma v$ since the RHS of the original ODEs. We will show that there exists a function $v$ of the other variables. In anticipation of this property (which we will soon verify), observe that in this system, we have substituted $1 - \gamma v$ when this system has a solution over $[0, T]$, $v$ and $\gamma v$ are well-defined, and hence a solution to the original IVP can be recovered.

System 1 has the property that in its solution, $v_6$ and $\beta_2$ can be expressed directly as functions of the other variables. In anticipation of this property (which we will soon verify), it is convenient to work with a reduced IVP in $(\tilde{v}_8, \beta_3, \gamma)$, which we call System 2, as follows:

$$\dot{v}_8 = \gamma_t \left\{ -\beta_2 + 2(1 - 2\alpha_t)\beta_1(1 - \chi_t) + 4\beta_1^2 \chi_t(1 - \chi_t) + \tilde{v}_8 \alpha_t^2 \left[ \frac{1}{\sigma_Y^2} - \frac{2\chi_t}{\sigma_X^2(1 - \chi_t)} \right] \right\} \tag{B.22}$$

$$\dot{v}_6 = \gamma_t \left\{ \frac{\beta_2 - 2\beta_1 \beta_2(1 - \chi_t) + \beta_1^2(1 - \chi_t)^2 + \tilde{v}_6 \alpha_t^2 \left[ \frac{1}{\sigma_Y^2} - \frac{2\chi_t}{\sigma_X^2(1 - \chi_t)} \right] \right\} \tag{B.23}$$

subject to $\tilde{v}_6(0) = \tilde{v}_8(0) = 0$ and the remaining initial conditions above. We will argue later that when this system has a solution over $[0, T]$, $v_6 = \tilde{v}_6/\gamma$ and $v_8 = \tilde{v}_8/\gamma$ are well-defined, and hence a solution to the original IVP can be recovered.

(Replacing all instances of $\gamma v_i$ with $\tilde{v}_i$, $i = 6, 8$) together with

$$\dot{v}_8 = \gamma_t \left\{ 2[1 - \beta_1 - \beta_3] + 2(1 - 2\alpha_t)\beta_1(1 - \chi_t) + 4\beta_1^2 \chi_t(1 - \chi_t) + \tilde{v}_8 \alpha_t^2 \left[ \frac{1}{\sigma_Y^2} - \frac{2\chi_t}{\sigma_X^2(1 - \chi_t)} \right] \right\} \tag{B.24}$$

$$\dot{\beta}_{1t} = \frac{\alpha_t \gamma_t}{2\sigma_Y^2 \sigma_X^2(1 - \chi_t)} \left\{ -2\sigma_X^2(\alpha_t - \beta_{1t})\beta_{1t}(1 - \chi_t) + 2\sigma_Y^2 \alpha_t \chi_t [1 - \beta_{1t} - \beta_3] - \beta_{1t} [1 - \chi_t] \right\} \tag{B.25}$$

$$\dot{\beta}_{3t} = \frac{\alpha_t \gamma_t}{2\sigma_X^2 \sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2 \beta_{1t}(1 - \chi_t) \beta_{3t} - 2\sigma_Y^2 \alpha_t [1 - \beta_{1t} - \beta_3 \chi_t(1 - \chi_t)^2(1 - 2\beta_3) + \alpha_t^2 \beta_{3t} \chi_t \tilde{v}_8 \right\} \tag{B.26}$$

and (B.21), subject to the initial conditions $\tilde{v}_8(0) = 0$, $\beta_{10} = \frac{1}{2(\gamma F)}$, $\beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma F$. Observe that in this system, we have substituted $1 - \beta_1 - \beta_3$ for all instances of $\beta_2$ in the original ODEs. We will show that there exists a $\gamma F \in (0, \gamma^*)$ such that a (unique) solution to System 2 exists and satisfies $\gamma_T(\gamma F) = \gamma^*$, and then we will show that given this solution, we can construct the solutions to (B.18) (which will be $1 - \beta_1 - \beta_3$) and (B.22) directly, solving System 1 (and hence the BVP).

Based on System 2, define

$$\bar{\gamma} := \sup \{ \gamma F > 0 \mid \text{a solution to System 2 with } \gamma_0 = \gamma F \text{ exists over } [0, T] \},$$

with respect to set inclusion. Since the RHS of the equations that comprise System 2 are of class $C^1$, the solution is unique when it exists, and there is continuous dependence of the solution on the initial conditions; in particular, the terminal value $\gamma_T$ is continuous in $\gamma F$.
(see Theorem on page 397 in Hirsch et al. (2004)). Hence if there exists \( \gamma^F \in (0, \bar{\gamma}) \) such that \( \gamma_T(\gamma^F) \geq \gamma^o \), by the intermediate value theorem there exists a \( \gamma^F \in (0, \bar{\gamma}) \) such that \( \gamma_T(\gamma^F) = \gamma^o \), allowing us to construct a solution to System 1. We rule out the alternative case by contradiction, and then we return to the case just described to complete the construction.

Suppose by way of contradiction that for all \( \gamma^F \in (0, \bar{\gamma}) \), \( \gamma_T(\gamma^F) < \gamma^o \). In particular, because \( \gamma_t \) is nondecreasing in the backward system for any initial condition, we have that \( \gamma_t \in (0, \gamma^o) \) and by Lemma 5, \( \chi_t \in (0, \bar{\chi}) \) for all \( t \in [0, T] \) when \( \gamma^F \in (0, \bar{\gamma}) \).

To reach a contradiction, it suffices to show that the solution to System 2 can be bounded uniformly over \( \gamma^F \in (0, \bar{\gamma}) \), as this would imply that the solution can be extended strictly to the right of \( \bar{\gamma} \) in this case, violating the definition of \( \bar{\gamma} \).

To establish uniform bounds, we decompose \( \beta_1 \) and \( \beta_3 \) as sums of forward-looking and myopic components and show that both of these components are uniformly bounded; for \( \tilde{v}_8 \), we establish the bound without any decomposition. Specifically, define \( \beta^m_{1i} := \frac{1}{2(2 + \chi_t)} \), \( \beta^m_{3i} = \frac{1}{2} \) and \( \beta^f_{it} := \beta_{it} - \beta^m_{it} \) for all \( t \in [0, T] \), \( i = 1, 3 \). Observe that for \( \gamma^F \in (0, \bar{\gamma}) \), \( \beta^m_{it} \) is uniformly bounded by \([1/4, 1/2] \subset [0, 1] \), as \( \chi_t = \chi(\gamma_t) \in [0, 1] \) for all \( t \in [0, T] \), and trivially \( \beta^m_{it} \in [0, 1] \). Hence, we define one final system, System 3, in \((\tilde{v}_8, \beta^f_1, \beta^f_3, \gamma_t)\) to be \((B.21), (B.24)\), and

\[
\dot{\beta}^f_{1i} = -\frac{\chi_t}{2(2 - \chi_t)^2} + \frac{\alpha_t \gamma_t}{2\sigma^2_X \sigma^2_Y (1 - \chi_t)} \left\{ -2\sigma^2_X (\alpha_t - [\beta^f_{1i} + \beta^m_{1i}]) [\beta^f_{1i} + \beta^m_{1i}] (1 - \chi_t) \\
+ 2\sigma^2_X \alpha_t \chi_t \left[ 1 - (\beta^f_{1i} + \beta^m_{1i}) - (\beta^f_{3i} + \beta^m_{3i}) \right][1 + 2(\beta^f_{1i} + \beta^m_{1i}) \chi_t] \right. \\
- 2\sigma^2_X \alpha_t \chi_t (\beta^f_{1i} + \beta^m_{1i}) (1 - \chi_t) + \alpha_t^2 (\beta^f_{1i} + \beta^m_{1i}) \chi_t \tilde{v}_{8t} \right\}, \tag{B.27}
\]

\[
\dot{\beta}^f_{3i} = \frac{\alpha_t \gamma_t}{2\sigma^2_X \sigma^2_Y (1 - \chi_t)} \left\{ 2\sigma^2_X (\beta^f_{1i} + \beta^m_{1i}) (1 - \chi_t) (\beta^f_{3i} + \beta^m_{3i}) + \alpha_t^2 (\beta^f_{3i} + \beta^m_{3i}) \chi_t \tilde{v}_{8t} \\
+ 4\sigma^2_X \alpha_t \beta^f_{3i} \chi_t^2 [1 - (\beta^f_{1i} + \beta^m_{1i}) - (\beta^f_{3i} + \beta^m_{3i})] \right\}, \tag{B.28}
\]

subject to initial conditions \( \tilde{v}_{80} = 0, \beta^f_{10} = 0, \beta^f_{30} = 0 \) and \( \gamma_0 = \gamma^F \in (0, \bar{\gamma}) \), where \( \alpha_t = [\beta^f_{3i} + \beta^m_{3i}] + [\beta^f_{1i} + \beta^m_{1i}] \chi_t \). Define \( h^{\tilde{v}}(\beta^f_{1i}, \beta^m_{1i}, \beta^f_{3i}, \beta^m_{3i}, \tilde{v}_{8t}, \gamma_t) \) as the RHS of \((B.24)\) with \( \beta^f_{it} + \beta^m_{it} \) substituted for \( \beta_{it} \), \( i = 1, 3 \).

Given that \( \beta^m_{1i} \) and \( \beta^m_{3i} \) are uniformly bounded, it suffices to show that the solutions \((\tilde{v}_8, \beta^f_1, \beta^f_3)\) are uniformly bounded by some \([-K, K]^3\). (Recall that \( \gamma \) is already bounded by \([0, \gamma^o] \).)

Define \( \bar{\alpha} = (K + 1) \bar{\chi} + (K + 1) \), where we suppress dependence on \( K \). Next, for \( x \in \)
\{\tilde{v}_8, \beta_{81}, \beta_8^f\} define \(\tilde{h}^x : \mathbb{R}_{++}^2 \to R_{++}\) as follows:

\[
\tilde{h}^{\tilde{v}}(K, \gamma^o) := \gamma^o \{2[1 + 2(K + 1)] + 2(1 + 2\tilde{\alpha})(K + 1) + 4(K + 1)^2\tilde{\chi} + K\tilde{\alpha}^2 \left[1/\sigma^2_Y + \tilde{\chi}/(\sigma^2_X[1 - \tilde{\chi}])\right]\} \\
\tilde{h}^{\beta_1}(K, \gamma^o) := \frac{\tilde{\alpha}\gamma^o[1/\sigma^2_Y + \tilde{\chi}/(\sigma^2_X[1 - \tilde{\chi}])]}{2(2 - \tilde{\chi})^2} + \frac{\tilde{\alpha}\gamma^o}{2\sigma^2_X\sigma^2_Y(1 - \tilde{\chi})} \left[2\sigma^2_X[\tilde{\alpha} + K + 1](K + 1) + 2\sigma^2_X\tilde{\alpha}\tilde{\chi}([1 + 2(K + 1)] [1 + 2(K + 1)] \tilde{\chi} + K + 1) + \tilde{\alpha}^2K(K + 1)\tilde{\chi}\right] \\
\tilde{h}^{\beta_3}(K, \gamma^o) := \frac{\tilde{\alpha}\gamma^o}{2\sigma^2_X\sigma^2_Y(1 - \tilde{\chi})} \left[2\sigma^2_X(K + 1)^2 + \tilde{\alpha}^2K(K + 1)\right] + 4\sigma^2_X\tilde{\alpha}\tilde{\chi}^2K[1 + 2(K + 1)] \right\},
\]

Define

\[
T(\gamma^o) := \max_{K' > 0} \min_{x \in \{\tilde{v}_8, \beta_{1,3}^f\}} \frac{K'}{\tilde{h}^x(K', \gamma^o)},
\]

and let \(K\) denote the arg max.\(^{41}\) We now show that given \(T < T(\gamma^o), (\tilde{v}_8, \beta_{1,3}^f)\) are uniformly bounded by \([-K, K]^3\). Suppose otherwise, and define \(\tau = \inf\{t > 0 : (\tilde{v}_8, \beta_{1,3}^f) \notin [-K, K]^3\}\); by supposition and continuity of the solutions, \(\tau \in (0, T)\) and \(|x_\tau| = K\), some \(x \in \{\tilde{v}_8, \beta_{1,3}^f\}\). Now by construction of the \(\tilde{h}^x(K, \gamma^o)\), for all \(t \in [0, \tau]\) and for each \(x \in \{\tilde{v}_8, \beta_{1,3}^f\}\) we have

\[
|\dot{x}_t| = |\tilde{h}^x(\beta_{1,3}^f, \beta_{1,3}^m, \beta_{1,3}^m, \tilde{v}_8, \gamma_t)| < \tilde{h}^x(K, \gamma^o)
\]

and thus by the triangle inequality,

\[
|x_\tau| < 0 + \tau \cdot \tilde{h}^x(K, \gamma^o) < T(\gamma^o)\tilde{h}^x(K, \gamma^o) \leq K,
\]

a contradiction. We conclude that the solutions \((\tilde{v}_8, \beta_{1,3}^f, \beta_{1,3}^m, \gamma)\) to System 3 are uniformly bounded by \([-K, K]^3 \times [0, \gamma^o]\), and hence the solutions \((\tilde{v}_8, \beta_1, \beta_3, \gamma)\) to System 2 are uniformly bounded by \([-K, K] \times [-K + 1, K + 1] \times [0, \gamma^o]\). This gives us the desired contradiction on the definition of \(\bar{\gamma}\) from before, so we conclude that there exists \(\gamma^F \in (0, \bar{\gamma})\) such that the solution to System 2 satisfies \(\gamma_T(\gamma^F) = \gamma^o\). (Note that any such \(\gamma^F\) lies in \((0, \gamma^o)\), as \(\gamma\) is nondecreasing in the backward system.)

Now consider any such \(\gamma^F\) and solution \((\tilde{v}_8, \beta_1, \beta_3, \gamma)\) to System 2 with \(\gamma_T(\gamma^F) = \gamma^o\). We claim using the comparison theorem that \(\alpha > 0\) over \([0, T]\). First, we have \(\alpha_0 > 0\), as

\(^{41}\)Note that \(T(\gamma^o), K < \infty\) as the \(\tilde{h}^x\) grow faster than linearly in \(K\).
\( \alpha_0 = \frac{1}{2 - \chi(\gamma^F)} \), and by Lemma 5, \( 0 \leq \chi(\gamma^F) < \bar{\chi} < 1 \). Now the RHS of (B.20) is locally Lipschitz continuous in \( \alpha \), uniformly in \( t \), as the remaining coefficients appearing in that ODE are bounded (being continuous functions of time over the compact set \([0, T]\)). By standard application of the comparison theorem in Teschl (2012, Theorem 1.3), we have \( \alpha_t > 0 \) for all \( t \in [0, T] \).

Hence, we can define

\[
\bar{v}^{\text{cand}}_{6t} := \frac{\sigma^2}{\alpha_t} \left[ -1 + 2 \beta_{1t} (1 - \chi_t) + \alpha_t \right] - \frac{\bar{v}_{8t}}{2},
\]

\[
\beta^{\text{cand}}_{2t} := 1 - \beta_{1t} - \beta_{3t}.
\]

Observe that in System 2, (B.24)-(B.26) was obtained by replacing \( \beta_{2t} \) with \( \beta_{2t}^{\text{cand}} \) in (B.23), (B.17) and (B.19), so \((\bar{v}_8, \beta_1, \beta_{2}^{\text{cand}}, \beta_3, \gamma)\) solves (B.23), (B.17), (B.19) and (B.21). It is tedious but straightforward to verify that \( \bar{v}^{\text{cand}}_{6t} \) and \( \beta^{\text{cand}}_{2t} \) solve (B.22) and (B.18), respectively. Now the RHS of (B.22) and (B.18), given the solutions \((\bar{v}_8, \beta_1, \beta_3, \gamma)\), are locally Lipschitz continuous in their respective variables, uniformly in \( t \), and thus \((\bar{v}^{\text{cand}}_{6t}, \beta^{\text{cand}}_{2t})\) are the unique solutions to (B.22) and (B.18) given the other variables.

We have \( \gamma_t \geq \gamma^F > 0 \) for all \( t \in [0, T] \). Thus, we can recover \( v_0 = \bar{v}_6/\gamma \) and \( v_8 = \bar{v}_8/\gamma \) as the solutions to (B.15) and (B.16). Hence, we have established the existence of \( \gamma^F \in (0, \gamma^o) \) and solution to the associated IVP posed at the beginning of the proof such that \( \gamma_T(\gamma^F) = \gamma^o \). By reversing the direction of time, this is a solution to the BVP in the theorem statement.

For the remainder of the proof, we refer to the forward system. Now in the full system of equations for the equilibrium, learning and value function coefficients, we have

\[
v_{2t} = \frac{2\sigma^2 \beta_{0t}}{\gamma_t \alpha_t},
\]

\[
v_{5t} = -\frac{\sigma^2 [\beta_{3t} - \beta_{1t}(2 - \chi_t)]}{\gamma_t \alpha_t},
\]

\[
v_{7t} = \frac{2\sigma^2 (2\beta_{3t} - 1)}{\gamma_t \alpha_t},
\]

\[
v_{9t} = \frac{2\sigma^2 [\beta_{2t} - \beta_{1t}(1 - \chi_t)]}{\gamma_t \alpha_t}.
\]

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and a system of ODEs for $v_0, v_1, v_3, v_4, \beta_0$:

$$
\begin{align*}
\dot{v}_0_t &:= \beta^2_0 + \frac{\alpha_t \gamma_t \chi_t}{\sigma^2_X(1 - \chi_t)^2} \left\{ \sigma^2_Y [-2\beta_2 \chi_t (1 - \chi_t) + \alpha_t \chi_t (1 - \chi_t)^2] + \alpha_t \sigma^2_X (1 - \chi_t)^2 - \alpha_t \gamma_t \chi_t v_6_t \right\} \\
\dot{v}_1_t &:= -2\beta_0 t \\
\dot{v}_3_t &:= \frac{\alpha_t^2 \gamma_t \chi_t v_3_t}{\sigma^2_X (1 - \chi_t)} + 2\beta_0 [\beta_1 (1 - \chi_t) + \beta_2] \\
\dot{v}_4_t &:= 1 - 2\beta^2_3 \\
\beta_0_t &:= -\frac{\alpha_t \gamma_t \chi_t [\alpha_t \gamma_t (v_3_t + \beta_0 v_8_t) + 4\sigma^2_Y \beta_0 t \beta_2 t \chi_t]}{2\sigma^2_X \sigma^2_Y (1 - \chi_t)},
\end{align*}
$$

all of which have terminal values 0. Since the system of ODEs above is linear, it has a unique solution (given the solution to the BVP), and $v_2, v_5, v_7$ and $v_9$ are well-defined by the above formulas since $\alpha, \gamma > 0$. Furthermore, by inspection, $(\beta_0, v_3) = (0, 0)$ solve their respective ODEs. We conclude that there exists a LME.

**Proof of Theorem 2**

Let

$$
\begin{align*}
\bar{\beta}_2 &= \beta_2 / (1 - \chi); \quad \bar{v}_6 = v_6 \gamma / (1 - \chi)^2; \quad \bar{v}_8 = v_8 \gamma / (1 - \chi).
\end{align*}
$$
The boundary value problem is

\[
\dot{v}_{6t} = \gamma_t \left\{-\beta_{1t}^2 + 2\beta_{1t}\tilde{\beta}_{2t} + \tilde{\beta}_{2t}^2 + \tilde{v}_{6t} \left( \frac{\alpha_t^2}{\sigma_Y^2} + \frac{2(\hat{u}_\theta + \alpha_t)^2}{\sigma_X^2} \right) \right\} 
\]

(B.32)

\[
\dot{v}_{8t} = \gamma_t \left\{ -(2 + 4\alpha_t)\beta_{1t} - 2\tilde{\beta}_{2t} + \tilde{v}_{8t}(\hat{u}_\theta + \alpha_t)^2\chi_t - 4\beta_{1t}^2\chi_t \right\} 
\]

(B.33)

\[
\dot{\beta}_{1t} = \frac{\gamma_t}{4\sigma_X^2\sigma_Y^2(1 + \tilde{u}_\theta\chi_t)} \times \left\{ 2\sigma_X^2\alpha_t \left( \hat{u}_\theta^2 - 2\beta_{1t}^2 + \alpha_t(\hat{u}_\theta + 2\beta_{1t}) \right) 
+ \tilde{v}_{8t}(\tilde{u}_\theta + \alpha_t)^2(\hat{u}_\theta - 2\beta_{1t})\chi_t 
+ 4\beta_{1t}\chi_t \left[ \hat{u}_\theta \sigma_Y^2 + (2\hat{u}_\theta \sigma_X^2 + \sigma_Y^2) \alpha_t^2 + \hat{u}_\theta \alpha_t \left( \hat{u}_\theta \sigma_X^2 + 2\sigma_Y^2 - \sigma_X^2 \beta_{1t} \right) \right] 
- 4\sigma_Y^2(\hat{u}_\theta + \alpha_t)^2\tilde{\beta}_{2t}\chi_t + 4\sigma_Y^2(\hat{u}_\theta + \alpha_t)^2\beta_{1t}(\hat{u}_\theta - 2\tilde{\beta}_{2t})\chi_t^2 \right\} 
\]

(B.34)

\[
\dot{\beta}_{2t} = \frac{\gamma_t}{4\sigma_X^2\sigma_Y^2(1 + \tilde{u}_\theta\chi_t)} \times \left\{ 2\sigma_X^2\alpha_t \left[ \hat{u}_\theta^2 + 2\beta_{1t}^2 + \alpha_t(\hat{u}_\theta + 2\beta_{2t}) \right] 
+ \alpha_t\chi_t(\tilde{u}_\theta + \alpha_t)^2 \left[ -4\tilde{v}_{6t} + \tilde{v}_{8t}(\hat{u}_\theta - 2\tilde{\beta}_{2t}) \right] 
+ 4\alpha_t\chi_t\tilde{u}_\theta\sigma_X^2 \left[ \beta_{1t}^2 + (\hat{u}_\theta + 2\alpha_t)\tilde{\beta}_{2t} \right] 
- 4(\hat{u}_\theta + \alpha_t)^2 \left[ \hat{u}_\theta\tilde{v}_{6t}\alpha_t + \sigma_Y^2\tilde{\beta}_{2t}(\hat{u}_\theta - 2\tilde{\beta}_{2t}) \right] \chi_t^2 \right\} 
\]

(B.35)

\[
\dot{\beta}_{3t} = \frac{\gamma_t}{4\sigma_X^2\sigma_Y^2(1 + \tilde{u}_\theta\chi_t)} \times \left\{ -4\sigma_X^2\alpha_t^2\beta_{1t} 
+ 2\alpha_t\chi_t(\tilde{u}_\theta + \alpha_t) \left[ -\hat{u}_\theta \sigma_X^2 + 2\hat{u}_\theta \sigma_X^2\alpha_t - \tilde{v}_{8t}\alpha_t(\hat{u}_\theta + \alpha_t) \right] 
- 2\alpha_t\chi_t \left[ 2\hat{u}_\theta \sigma_X^2\alpha_t \beta_{1t} - 2\sigma_X^2 \beta_{1t}^2 \right] 
- \chi_t^2 \left[ \tilde{v}_{8t}\alpha_t(\hat{u}_\theta + \alpha_t)^2(\hat{u}_\theta - 2\beta_{1t}) + 4\hat{u}_\theta \sigma_X^2\alpha_t(\hat{u}_\theta + \alpha_t - \beta_{1t})\beta_{1t} \right] 
+ 4\sigma_X^2\chi_t^2(\hat{u}_\theta + \alpha_t)^2(1 + 2\alpha_t)\tilde{\beta}_{2t} + 8\sigma_Y^2(\hat{u}_\theta + \alpha_t)^2\beta_{1t}\tilde{\beta}_{2t}\chi_t^3 \right\} 
\]

(B.36)

\[
\dot{\gamma}_t = -\frac{\alpha_t^2\gamma_t}{\sigma_Y^2} 
\]

(B.37)

\[
\dot{\chi}_t = \gamma_t \left\{ \alpha_t^2(1 - \chi_t)/\sigma_Y^2 - (\hat{u}_\theta + \alpha_t)^2\chi_t^2/\sigma_X^2 \right\} . 
\]

(B.38)

with boundary conditions \((\gamma_0, \chi_0, \tilde{v}_{6T}, \tilde{v}_{8T}, \beta_{1T}, \tilde{\beta}_{2T}, \beta_{3T}) = (\gamma^0, 0, 0, 0, \frac{1+2\hat{u}_\theta}{2(1-\chi_T)}, \frac{1+2\hat{u}_\theta}{2(1-\chi_T)}, 1/2)\).

The core of the proof is to establish the existence of a solution \((\tilde{v}_6, \tilde{v}_8, \beta_1, \tilde{\beta}_2, \beta_3, \gamma, \chi)\) to the boundary value problem for all \(T < T(\gamma^0)\); from there, it is straightforward to verify that the remaining coefficients are well-defined and that the HJB equation is satisfied. We complete these steps at the end of the proof.\(^{42}\)

It is useful to introduce \(z = (\tilde{v}_6, \tilde{v}_8, \beta_1, \tilde{\beta}_2, \beta_3, \gamma, \chi)\) and write the system of ODEs (B.32)-(B.38) as \(\dot{z}_t = F(z_t)\). We write \(\tilde{z} = (z_1, z_2, \ldots, z_5)\) and \(\tilde{F}(z) = (F_1(z), F_2(z), \ldots, F_5(z))\).

\(^{42}\)In particular, we have ignored \(\beta_0\) for now since it does not appear in any of the ODEs displayed here; afterward, it is easily shown to be identically zero.
Define $B : \mathbb{R}_+^2 \to \mathbb{R}^5$ by $B(\gamma, \chi) = \left(0, 0, \frac{1+2u_0}{2(2-\chi)}, \frac{(1+2u_0)}{2(2-\chi)}, 1/2\right)$, formed by writing the terminal value of $\tilde{z}$ as a function of $(\gamma, \chi)$. Define $s_0 \in \mathbb{R}^5$ by $s_0 = B(\gamma^o, 0) = (0, 0, \frac{1+2u_0}{4}, \frac{1+2u_0}{4}, 1/2)$. For $x \in \mathbb{R}^n$, let $||x||_\infty$ denote the sup norm, $\sup_{1 \leq i \leq n} |x_i|$. For any $\rho > 0$, let $S_\rho(s_0)$ denote the $\rho$-ball around $s_0$

$$S_\rho(s_0) := \{s \in \mathbb{R}^5| ||s - s_0||_\infty \leq \rho \}.$$ 

For all $s \in S_\rho(s_0)$, let IVP-s denote the the initial value problem defined by (B.32)-(B.38) and initial conditions $(\bar{v}_{00}, \bar{v}_{80}, \beta_{10}, \tilde{\beta}_{20}, \beta_{30}, \gamma_0, \chi_0) = (s, \gamma^o, 0)$. Whenever a solution to IVP-s exists, it is unique as $F$ is of class $C^1$; denote it by $z(s)$, where $z(s) = (\tilde{z}(s), \gamma(s), \chi(s)) = (\bar{v}(s), \bar{v}(s), \beta_1(s), \tilde{\beta}_2(s), \beta_3(s), \gamma(s), \chi(s))$. Note that such a solution solves the BVP if and only if

$$\bar{z}(s) = B(\gamma_T(s), \chi_T(s)),$$

(B.39)

as the initial values $\gamma_0(s) = \gamma^o$ and $\chi_0(s) = 0$ are satisfied by construction. Note also that $\bar{z}(s) = s + \int_0^T \tilde{F}(\bar{z}_t(s)) dt$; hence (B.39) is satisfied if and only if $s$ is a fixed point of the function $g : S_\rho(s_0) \to \mathbb{R}^5$ defined by

$$g(s) := B(\gamma_T(s), \chi_T(s)) - \int_0^T \tilde{F}(\bar{z}_t(s)) dt.$$  

(B.40)

Note, moreover, that for any solution, we have by Lemma 3 that $\chi_t \in [0, \check{\chi})$ where we define $\check{\chi}$ as 1 for the purpose of this proof.

Before establishing conditions sufficient for $g(s)$ to be a continuous self-map on $S_\rho$ for a given $\rho > 0$, we establish the following result, which gives existence, uniqueness and uniform bounds of solutions to IVP-s for all $s \in S_\rho$. Specifically, for arbitrary $K > 0$, we ensure that the solution $\bar{z}_t(s)$ varies at most $K$ from its starting point $s$ for all $t \in [0, T]$, and thus by the triangle inequality, this solution varies most $\rho + K$ from $s_0$. These bounds will be used further when we turn to the self-map property.

**Lemma B.2.** Fix $\gamma^o > 0$, $\rho > 0$ and $K > 0$. Then there exists a threshold $T^{SBC}(\gamma^o; \rho, K) > 0$ such that if $T < T^{SBC}(\gamma^o; \rho, K)$, then for all $s \in S_\rho(s_0)$ a unique solution to IVP-s exists over $[0, T]$. Moreover, for all $t \in [0, T]$, $\bar{z}_t(s) \in S_{\rho + K}(s_0)$. We call this property the System Bound Condition (SBC).

**Proof.** Recall that $\tilde{F}$ is of class $C^1$, and hence given $s \in S_\rho(s_0)$, the solution $z(s)$ is unique whenever it exists. Toward the SBC, note that it suffices to ensure that for all $||\tilde{z}(s) - s||_\infty < K$, since then by the triangle inequality, $||\tilde{z}(s) - s_0||_\infty \leq ||\tilde{z}(s) - s||_\infty + ||s - s_0||_\infty < \rho + K$. 

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In what follows, we construct bounds on $\tilde{F}$ by writing $\tilde{F}(z(s)) = \tilde{F}(z(s) - s_0 + s_0)$ and using the conjectured bounds $\|\tilde{z}(s) - s_0\|_\infty < \rho + K$, $\gamma \in (0, \gamma^0]$, $\chi \in [0, \chi)$ for the solution, when it exists. Using these bounds on $\tilde{F}$, we then identify a threshold time $T^{SBC}(\gamma^0; \rho, K)$ such that at all times $t < T^{SBC}(\gamma^0; \rho, K)$ the solution to IVP-s (exists and) satisfies the conjectured bounds.

Note that the desired component-wise inequalities $|z_{i\ell}(s) - s_{i0}| < \rho + K$, $i \in \{1, 2, \ldots, 5\}$, imply the further bounds

\[
|\tilde{v}_{i\ell}|, |\tilde{v}_{i\ell}| < \rho + K
\]

\[
|\beta_{i\ell}| < \tilde{\beta}_i(\rho, K) := \frac{1 + 2\tilde{u}_\theta}{4} + \rho + K
\]

\[
|\tilde{\beta}_{2\ell}| < \tilde{\beta}_2(\rho, K) := \frac{1 + 2\tilde{u}_\theta}{4} + \rho + K
\]

\[
|\beta_{3\ell}| < \tilde{\beta}_3(\rho, K) := 1/2 + \rho + K
\]

\[
|\alpha_{i\ell}| < \tilde{\alpha}(\rho, K) := \tilde{\beta}_1(\rho, K)\chi + \tilde{\beta}_3(\rho, K).
\]

Hereafter, we suppress the dependence of $\tilde{\beta}_i$, $i \in \{1, 2, 3\}$, and $\tilde{\alpha}$ on $(\rho, K)$.

Define functions $h_i : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$ as follows:

\[
h_1(\gamma^0; \rho, K) := \gamma^0 \left\{ (\tilde{\beta}_1 + \tilde{\beta}_2)^2 + \tilde{v}_6 (\tilde{\alpha}^2/\sigma_Y^2 + 2(\tilde{u}_\theta + \tilde{\alpha})^2/\sigma_X^2) \right\}
\]

\[
h_2(\gamma^0; \rho, K) := \gamma^0 \left\{ (2 + 4\tilde{\alpha})\tilde{\beta}_1 + 2\tilde{\beta}_2 + \tilde{v}_8 (\tilde{u}_\theta + \tilde{\alpha})^2/\sigma_X^2 + 4\tilde{\beta}_1^2/\chi^2 \right\}
\]

\[
h_3(\gamma^0; \rho, K) := \frac{\gamma^0}{4\sigma_X^2/\sigma_Y^2} \times \left\{ 2\tilde{\sigma}_x^2\tilde{\alpha} \left( \tilde{u}_\theta^2 + 2\tilde{\beta}_1^2 + \tilde{\alpha}(\tilde{u}_\theta + 2\tilde{\beta}_1) \right) \right.
\]

\[
\left. +4\tilde{\beta}_1\tilde{X} \left[ \tilde{u}_\theta^2/\sigma_Y^2 + (2\tilde{u}_\theta^2 + \sigma_Y^2) \tilde{\alpha}^2 + \tilde{u}_\theta^2\tilde{\alpha} \right] \tilde{u}_\theta\tilde{\sigma}_X^2 + 2\sigma_Y^2 + \sigma_X^2\tilde{\beta}_1 \right\}
\]

\[
h_4(\gamma^0; \rho, K) := \frac{\gamma^0}{4\sigma_X^2/\sigma_Y^2} \times \left\{ 2\tilde{\sigma}_x^2\tilde{\alpha} \left( \tilde{u}_\theta^2 + 2\tilde{\beta}_1^2 + \tilde{\alpha}(\tilde{u}_\theta + 2\tilde{\beta}_2) \right) \right.
\]

\[
\left. +4\tilde{\beta}_1\tilde{X} \left[ \tilde{u}_\theta^2/\sigma_Y^2 + (2\tilde{u}_\theta^2 + \sigma_Y^2) \tilde{\alpha}^2 + \tilde{u}_\theta^2\tilde{\alpha} \right] \tilde{u}_\theta\tilde{\sigma}_X^2 + 2\sigma_Y^2 + \sigma_X^2\tilde{\beta}_2 \right\}
\]

\[
h_5(\gamma^0; \rho, K) := \frac{\gamma^0}{4\sigma_X^2/\sigma_Y^2} \times \left\{ 4\sigma_X^2\tilde{\alpha}^2\tilde{\beta}_1 + 2\tilde{\sigma}_x(\tilde{u}_\theta + \tilde{\alpha}) \left[ \tilde{u}_\theta\tilde{\sigma}_X^2 + 2\tilde{u}_\theta\tilde{\sigma}_X^2\tilde{\alpha} + \tilde{v}_8\tilde{\alpha}(\tilde{u}_\theta + \tilde{\alpha}) \right] \right.
\]

\[
\left. +2\tilde{\alpha}\tilde{X} \left[ 2\tilde{u}_\theta\tilde{\sigma}_X^2\tilde{\alpha}\tilde{\beta}_1 + 2\sigma_Y^2\tilde{\beta}_1^2 \right] \right\}
\]

\[
\tilde{X}^2 \left[ \tilde{v}_8\tilde{\alpha}(\tilde{u}_\theta + \tilde{\alpha})^2(\tilde{u}_\theta + 2\tilde{\beta}_1) \right] + 4\tilde{u}_\theta\tilde{\sigma}_X^2\tilde{\alpha}\tilde{\beta}_1(\tilde{u}_\theta + \tilde{\alpha} + \tilde{\beta}_1) \right\}
\]

\[
+4\sigma_Y^2\tilde{X}^2(\tilde{u}_\theta + \tilde{\alpha})^2(1 + 2\tilde{\alpha})\tilde{\beta}_2 + 8\sigma_Y^2(\tilde{u}_\theta + \tilde{\alpha})^2\tilde{\beta}_1\tilde{\beta}_2\tilde{X}^3 \right\}.
\]

\[\text{We use } \mathbb{R}_{++} \text{ to denote } (0, +\infty).\]
Now for arbitrary \((\rho, K) \in \mathbb{R}^2_{++}\), define

\[
T_{SBC}(\gamma^o; \rho, K) := \min_{i \in \{1, 2, \ldots, 5\}} \frac{K}{h_i(\gamma^o; \rho, K)}.
\]  

We claim that by construction, for any \(t < T_{SBC}(\gamma^o; \rho, K)\), if a solution exists at time \(t\), then \(\|\tilde{z}_i(s) - s\|_\infty < K\), \(\gamma_t \in (0, \gamma^o]\) and \(\chi_t \in [0, \tilde{\chi})\). To see this, suppose by way of contradiction that there is some \(s \in S_\rho\) and some \(t < T_{SBC}(\gamma^o; \rho, K)\) at which a solution to IVP-s exists but either \(|z_i(t) - s_i| \geq K\) for some \(i \in \{1, 2, \ldots, 5\}\), \(\gamma_t \notin (0, \gamma^o]\) or \(\chi_t \notin [0, \tilde{\chi})\); let \(\tau\) be the infimum of such times. Now by Lemma 3, it cannot be that \(\gamma_t \notin (0, \gamma^o]\) or \(\chi_t \notin [0, \tilde{\chi})\) while \(\tilde{z}_i(s)\) exists, so (by continuity of \(z(s)\) w.r.t. time) it must be that for some \(i \in \{1, 2, \ldots, 5\}\), \(|z_i(t) - s_i| \geq K\), and the bounds \(\gamma_t \in (0, \gamma^o]\) and \(\chi_t \in [0, \tilde{\chi})\) are satisfied for all \(t \in [0, \tau]\).

By construction of the \(h_i(\gamma^o; \rho, K)\), for all \(t \in [0, \tau]\) we have \(|F_i(z_i(t))| \leq h_i(\gamma^o; \rho, K)\) and thus

\[
|z_{i\tau}(s) - s_i| = \left| \int_0^{\tau} F_i(z_i(s)) \, dt \right| \\
\leq \int_0^{\tau} |F_i(z_i(s))| \, dt \\
\leq \tau \cdot h_i(\gamma^o; \rho, K) \\
< T_{SBC}^{SBC}(\gamma^o; \rho, K) h_i(\gamma^o; \rho, K) \\
\leq K,
\]

where the second to last line uses that \(\tau < T_{SBC}(\gamma^o; \rho, K)\) and the last line uses the definition of \(T_{SBC}^{SBC}(\gamma^o; \rho, K)\); but via the strict inequality, this contradicts the definition of \(\tau\), proving the claim. By the triangle inequality, it follows that \(z_i(s) \in S_{\rho + K}(s_0)\) if a solution exists at time \(t < T_{SBC}(\gamma^o; \rho, K)\). Together, these bounds imply that the solution cannot explode prior to time \(T_{SBC}(\gamma^o; \rho, K)\). In other words, a unique solution must exist over \([0, T]\) for any \(T < T_{SBC}(\gamma^o; \rho, K)\) and it satisfies the SBC.

In order to invoke a fixed point theorem, the key remaining step is to establish, through the following lemma, that \(g\) is a well-defined, continuous self-map on \(S_\rho\) when \(T\) is below a threshold \(T(\gamma^o; \rho, K)\). The expression for the latter is provided in the proof of the lemma.

**Lemma B.3.** Fix \(\gamma^o > 0\), \(\rho > 0\) and \(K > 0\). There exists \(T(\gamma^o; \rho, K) \leq T_{SBC}(\gamma^o; \rho, K)\) such that for all \(T < T(\gamma^o; \rho, K)\), \(g\) is a well-defined, continuous self-map on \(S_\rho\).

**Proof.** First, the inequality \(T(\gamma^o; \rho, K) \leq T_{SBC}(\gamma^o; \rho, K)\), which holds by construction (as carried out below), ensures that a unique solution to IVP-s exists for all \(s \in S_\rho\). Next, we
argue that \( g \) is continuous. Note that \( g(s) \) can be written as \( B(\gamma_T(s), \chi_T(s)) - [z_T(s) - s] \). Since \( F \) is of class \( C^1 \) on the domain \( S_{\rho + K} \times (0, \gamma^o) \times [0, \bar{\chi}], z_t(s) \) (which includes \( \gamma \) and \( \chi \)) is locally Lipschitz continuous in \( s \), uniformly in \( t \in [0, T] \), and \( B \) is continuous, and thus continuity of \( g \) follows readily.

To complete the proof, we show that if \( T < T(\gamma^o; \rho, K) \), \( g \) satisfies the condition

\[
||g(s) - s_0||_{\infty} \leq \rho \quad \text{for all } s \in S_{\rho},
\]

which we refer to as the Self-Map Condition (SMC).

Note that \( g(s) - s_0 = \Delta(s) - \int_0^T \tilde{F}(z_t(s))dt \), where

\[
\Delta(s) := B(\gamma_T(s), \chi_T(s)) - B(\gamma^o, 0) = \left(0, 0, \frac{1 + 2\bar{u}_\theta}{2} \left[\frac{1}{2 - \chi_T(s)} - \frac{1}{2}\right], \frac{1 + 2\bar{u}_\theta}{2} \left[\frac{1}{2 - \chi_T(s)} - \frac{1}{2}\right], 0\right).
\]

The \( h^i(\rho, K) \) constructed in the proof of the previous lemma will provide us a bound for the components of \( \int_0^T \tilde{F}(z_t(s))dt \), but we must also bound \( \Delta(s) \), and in particular, \( \Delta_3(s) \) and \( \Delta_4(s) \). Note that \( \Delta_3(s) = \Delta_4(s) \).

Recalling that \( \chi \in [0, 1] \), the ODE for \( \chi \) implies that

\[
\dot{\chi}_t \leq \gamma_t \left\{ \alpha_t^2 (1 - \chi_t)/\sigma_T^2 \right\} \leq \gamma^o \alpha^2/\sigma_T^2,
\]

which depends on \((\rho, K)\) through \( \bar{\alpha} \). Hence by the fundamental theorem of calculus, we have \( \chi_t = \int_0^t \dot{\chi}_s ds \leq (\gamma^o \alpha^2/\sigma_T^2) t \).

Hence, using \( \chi_T(s) \leq 1 \) to bound \((2 - \chi_T(s))\) in the denominators from below by 1, we have the following bound for \( \Delta_3(s) = \Delta_4(s) \):

\[
|\Delta_3(s)| = \left| \frac{1 + 2\bar{u}_\theta}{2} \left[\frac{1}{2 - \chi_T(s)} - \frac{1}{2}\right] \right| = \left| \frac{1 + 2\bar{u}_\theta}{2} \frac{\chi_T(s)}{2(2 - \chi_T(s))} \right| \leq \frac{1 + 2\bar{u}_\theta}{4} \left(\gamma^o \alpha^2/\sigma_T^2\right) T
\]

For arbitrary \((\rho, K) \in \mathbb{R}^2_+ \), define \( \bar{\Delta}_i(\gamma^o; \rho, K) = \frac{1 + 2\bar{u}_\theta}{4} (\gamma^o \alpha^2/\sigma_T^2) \) for \( i \in \{3, 4\} \) and define \( \bar{\Delta}_i(\gamma^o; \rho, K) = 0 \) for \( i \in \{1, 2, 5\} \). Note that for all \( i \in \{1, 2, 3, 4, 5\} \), \( \bar{\Delta}_i(\rho, K) \) is proportional to \( \gamma^o \), and by construction, \( |\Delta_i(s)| \leq T \bar{\Delta}_i(\gamma^o; \rho, K) \).

Now for arbitrary \((\rho, K) \in \mathbb{R}^2_+ \), define

\[
T(\gamma^o; \rho, K) := \min \left\{ T^{SBC}(\gamma^o; \rho, K), \min_{i \in \{1, 2, 3, 4, 5\}} \frac{\rho}{\bar{\Delta}_i(\gamma^o; \rho, K) + h_i(\gamma^o; \rho, K)} \right\} \quad \text{(B.42)}
\]

\[44\text{See Theorem on page 397 in Hirsch et al. (2004).}\]
To establish the SMC, it suffices to establish for each \( i \in \{1, 2, \ldots, 5\} \) that \( |g_i(s) - s_{i0}| \leq \rho \) for all \( s \in S_\rho(s_0) \).

We calculate

\[
|g_i(s) - s_{i0}| = |\mathbf{B}_i(\gamma_T(s), \chi_T(s)) - \mathbf{B}_i(\gamma^o, 0) - \int_0^T F_i(z_t(s))dt| \\
\leq |\Delta_i(s)| + \int_0^T |F_i(z_t(s))|dt \\
\leq T \Delta_i(\gamma^o; \rho, K) + Th_i(\gamma^o; \rho, K) < \rho,
\]

where (i) in the second to last line we have used the definition of \( \Delta_i(\gamma^o; \rho, K) \) and that \( |F_i(z_t(s))| \leq h_i(\gamma^o; \rho, K) \); and (ii) in the last line we have used that \( T < T(\gamma^o; \rho, K) \) by construction. Hence, for all \( i \in \{1, 2, \ldots, 5\} \) we have \( |g_i(s) - s_{i0}| \leq \rho \), completing the proof.

To complete the solution to the boundary value problem \((B.32)-(B.38)\), note that by Lemma B.3, \( g \) is a well-defined, continuous self-map on the compact set \( S_\rho \). By Brouwer’s Theorem, there exists \( s^* \) such that \( s^* = g(s^*) \), and hence the solution to IVP-\( s^* \) is a solution to the BVP. To see that \( T(\gamma^o) \in O(1/\gamma^o) \), note simply that \( \gamma^o \) appears as an outside factor in the denominators of the expressions defining \( T^{SBC}(\gamma^o; \rho, K) \) and \( T(\gamma^o; \rho, K) \). Moreover, since \( \rho, K \) have been chosen arbitrarily, we can then optimize \( T(\gamma^o; \rho, K) \) over choices of \( (\rho, K) \in \mathbb{R}^2_+ \).

We argue that \( \alpha \) is finite and that \( \gamma \) and \( \alpha \) are strictly positive. Finiteness comes directly from the definition \( \alpha = \beta_1 \chi + \beta_3 \) and the finiteness of the underlying variables. This implies that \( \gamma_t > 0 \) for all \( t \in [0, T] \). The ODE for \( \alpha \) is

\[
\dot{\alpha}_t = \frac{\alpha_t(\tilde{u}_\theta + \alpha_t)\gamma_t \chi_t}{2 \sigma_\chi^2 \sigma_\gamma^2 (1 + \tilde{u}_\theta \chi_t)} \left\{ 2 \tilde{u}_\theta \sigma_\gamma^2 \alpha_t - \tilde{u}_\theta \chi_t (\tilde{u}_\theta + \alpha_t) - 4 \sigma_\gamma^2 (\tilde{u}_\theta + \alpha_t) \tilde{t} \chi_t \right\}. \quad (B.43)
\]

By continuity of the solution to the BVP, the RHS of the equation above is locally Lipschitz continuous in \( \alpha \), uniformly in \( t \). Moreover, \( \alpha_T = \beta_1 T + \beta_3 T \chi_T = \frac{1 + \tilde{u}_\theta \chi_T}{2 - \chi_T} > 0 \). By a standard application of the comparison theorem to the backward version of the previous ODE, it must be that \( \alpha_t > 0 \) for all \( t \in [0, T] \).

Using the solution to the BVP and the facts above, we solve for the rest of the equilibrium
coefficients. First, we have directly

\[
\begin{align*}
v_{2t} &= \frac{2\sigma^2_Y \beta_{0t}}{\gamma_t \alpha_t} \\
v_{5t} &= \frac{\sigma^2_Y [\beta_{1t}(2 - \chi_t) - \beta_{3t} - \hat{u}_t]}{\gamma_t \alpha_t} \\
v_{7t} &= \frac{-2\sigma^2_Y (1 - 2\beta_{3t})}{\gamma_t \alpha_t} \\
v_{9t} &= \frac{2\sigma^2_Y [\beta_{2t} - \beta_{1t}(1 - \chi_t)]}{\gamma_t \alpha_t}.
\end{align*}
\]

The last three are clearly well-defined due to \(\alpha, \gamma > 0\).

The remaining ODEs for \(\beta_0, v_0, v_1, v_3\) and \(v_4\) are

\[
\begin{align*}
\dot{\beta}_{0t} &= -\frac{(\hat{u}_t + \alpha_t)\gamma_t \chi_t}{2\sigma^2_X \sigma^2_Y (1 - \chi_t)(1 + \hat{u}_t \chi_t)} \left\{ 4\hat{u}_t \sigma^2_Y \beta_{0t} \beta_{2t} (1 - \chi_t) \chi_t + \alpha_t^2 [\tilde{v}_{8t} \beta_{0t} (1 - \chi_t) + v_{3t} \gamma_t (1 + \hat{u}_t \chi_t)] \\
&\quad + \alpha_t [\hat{u}_t v_{3t} \gamma_t (1 + \hat{u}_t \chi_t) + \beta_{0t} (1 - \chi_t) (-2\hat{u}_t \sigma^2_X + \hat{u}_t \tilde{v}_{8t} + 4\sigma^2_Y \beta_{2t} \chi_t)] \right\}, \quad \beta_{0T} = 0, \\
\dot{v}_{0t} &= \beta_{0t}^2 + (\hat{u}_t + \alpha_t)^2 \gamma_t \chi_t \\
&\quad + \frac{(\hat{u}_t + \alpha_t)^2 \gamma_t \chi_t}{\sigma^2_X} \left[ -\tilde{v}_{8t} + \sigma^2_Y (\hat{u}_t + \alpha_t - 2\tilde{\beta}_{2t}) / \alpha_t \right], \quad v_{0T} = 0, \\
\dot{v}_{1t} &= -2\beta_{0t}, \quad v_{1T} = 0, \\
\dot{v}_{3t} &= 2\beta_{0t} (\beta_{1t} + \tilde{\beta}_{2t}) (1 - \chi_t) + \frac{v_{3t} (\hat{u}_t + \alpha_t)^2 \gamma_t \chi_t}{\sigma^2_X (1 - \chi_t)}, \quad v_{3T} = 0, \text{ and} \\
\dot{v}_{4t} &= 1 - 2\beta_{3t}^2, \quad v_{4T} = 0.
\end{align*}
\]

Observe the system for \((\beta_0, v_1, v_3)\) is uncoupled from \((v_0, v_4)\). By inspection, the former has solution \((\beta_0, v_1, v_3) = (0, 0, 0)\), and uniqueness follows from the the associated operator being locally Lipschitz continuous in \((\beta_0, v_1, v_3)\) uniformly in \(t \in [0, T]\). It follows that \(v_2 = 0\), and the solutions for \((v_0, v_4)\) can be obtained directly by integration, given their terminal values.

We conclude that a linear Markov equilibrium exists.

Appendix C: Proofs for Section 5

Proofs for Section 5.1

We first analyze the public case, then the no feedback case, and then we analyze the learning and payoff comparisons. Proposition 5 is then an immediate consequence of Lemmas C.5
and C.6.

Public Case

System of ODEs  We look for an equilibrium of the form $a_t = \beta_0 t + \beta_1 M_t + \beta_3 \theta$, where $M_t = \hat{M}_t$ is publicly known.

The (backward) system of ODEs is

\[
\begin{align*}
\dot{\beta}_0 &= -r\beta_0 \beta_3 \\
\dot{\beta}_1 &= -\beta_1 \beta_3 \left( r + \frac{\beta_3 \gamma_t}{\sigma_Y^2} \right) \\
\dot{\beta}_3 &= -\beta_3 \left[ -r + \beta_3 \left( r - \frac{\beta_1 \gamma_t}{\sigma_Y^2} \right) \right] \\
\dot{\gamma}_t &= \frac{\beta_3^2 \gamma_t^2}{\sigma_Y^2},
\end{align*}
\]

with initial conditions

\[
\beta_{00} = 0, \beta_{10} = -\frac{\psi \gamma_t}{\sigma_Y^2} \leq 0, \beta_{30} = 1 \text{ and } \gamma_0 = \gamma^F \in (0, \gamma^o).
\]

Note: for the value function written as

\[
V(\theta, M_t, t) = v_{0t} + v_{1t} \theta + v_{2t} M_t + v_{3t} \theta^2 + v_{4t} M_t^2 + v_{5t} \theta M_t,
\]

we have the (backward) system

\[
\begin{align*}
\dot{v}_{0t} &= -rv_{0t} - \frac{(v_{2t} - 4\sigma_Y^2 v_{4t}) \gamma_t^2}{(-2\sigma_Y^2 + v_{5t} \gamma_t)^2} \\
\dot{v}_{1t} &= -rv_{1t} - \frac{2v_{2t} \gamma_t}{-2\sigma_Y^2 + v_{5t} \gamma_t} \\
\dot{v}_{2t} &= -rv_{2t} - \frac{v_{2t}(4\sigma_Y^2 \gamma_t + 4v_{4t} \gamma_t^2)}{(-2\sigma_Y^2 + v_{5t} \gamma_t)^2} \\
\dot{v}_{3t} &= -1 - rv_{3t} + \frac{4\sigma_Y^4}{(-2\sigma_Y^2 + v_{5t} \gamma_t)^2} \\
\dot{v}_{4t} &= -rv_{4t} - \frac{4v_{2t} \gamma_t(2\sigma_Y^2 + v_{4t} \gamma_t)}{(-2\sigma_Y^2 + v_{5t} \gamma_t)^2} \\
\dot{v}_{5t} &= -rv_{5t} - \frac{4\gamma_t \left[ \sigma_Y^2 \left( -2v_{4t} + v_{5t} \right) + v_{4t} v_{5t} \gamma_t \right]}{(-2\sigma_Y^2 + v_{5t} \gamma_t)^2},
\end{align*}
\]

with initial conditions $v_{00} = v_{10} = v_{20} = v_{30} = v_{50} = 0$ and $v_{40} = -\psi$. 

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In terms of the β coefficients (for which existence of a solution is shown below), we have

\[
\begin{align*}
    v_{2t} &= \frac{2\sigma_Y^2 \beta_{0t}}{\beta_{3t} \gamma_t} \\
    v_{4t} &= \frac{\sigma_Y^2 \beta_{1t}}{\beta_{3t} \gamma_t} \\
    v_{5t} &= -\frac{2\sigma_Y^2 (1 - \beta_{3t})}{\beta_{3t} \gamma_t}.
\end{align*}
\]

Since \( \beta_{3t} > 0 \) and thus \( v_{5t} \gamma_t = -\frac{2\sigma_Y^2 (1 - \beta_{3t})}{\beta_{3t} \gamma_t} < 2\sigma_Y^2 \), the denominator in each ODE is bounded away from zero. Given \( v_2, v_4 \) and \( v_5 \), the ODEs for \( v_0, v_1 \) and \( v_3 \) are linear and uncoupled and thus have solutions.

**Existence of Linear Markov Equilibrium:** \( r = 0 \) case

When \( r = 0 \), the backward system simplifies to

\[
\begin{align*}
    \dot{\beta}_{0t} &= 0 \\
    \dot{\beta}_{1t} &= -\frac{\beta_{1t} \beta_{3t}^2 \gamma_t}{\sigma_Y^2} \\
    \dot{\beta}_{3t} &= \frac{\beta_{1t} \beta_{2t}^2 \gamma_t}{\sigma_Y^2} \\
    \dot{\gamma}_t &= \frac{\beta_{3t}^2 \gamma_t^2}{\sigma_Y^2},
\end{align*}
\]

with initial conditions

\[
\beta_{00} = 0, \beta_{10} = -\frac{\psi \gamma_t}{\sigma_Y^2} \leq 0, \beta_{30} = 1 \text{ and } \gamma_0 = \gamma^F \in (0, \gamma^o).
\]

Define \( \tilde{\psi} := \psi \gamma^o / \sigma_Y^2 \) and \( \tilde{T} := T \gamma^o / \sigma_Y^2 \).

**Lemma C.1.** Suppose \( r = 0 \). For all \( T > 0 \) and all \( \psi > 0 \), there exists a linear Markov equilibrium. The corresponding \( \gamma_T \in (0, \gamma^o) \) satisfies \( g^{pub}(\gamma_T / \gamma^o) = 0 \), where

\[
g^{pub}(\rho) := -\tilde{T} \tilde{\psi} \rho^2 (1 - \rho) + \rho (1 + \tilde{T}) - 1 = 0.
\]

In addition, \( \beta_3 \in (0, 1] \) is increasing and \( \beta_1 < 0 \) is decreasing, and \( \beta_1 + \beta_3 \) is constant.

**Proof.** Since \( g^{pub}(0) = -1 < 0 < g^{pub}(1) = \tilde{T} \), there exists \( \gamma^F \in (0, \gamma^o) \) as in the statement of the proposition. We now show that for any such \( \gamma^F \), there exists a solution to the backward
IVP with $\gamma_0 = \gamma^F$, and it satisfies $\gamma_T = \gamma^o$. The proof is constructive, and the solution is unique conditional on $\gamma_F$.

Note that $\dot{\beta}_1 + \dot{\beta}_3t = 0$, so $\beta_1 + \beta_3$ is constant, and

$$
\beta_1 + \beta_3t = \beta_{10} + \beta_{30} = 1 + \beta_{10}
\implies \beta_1 = 1 + \beta_{10} - \beta_{3t}.
$$

Hence, a uniformly bounded solution for $\beta_3$ exists if and only if the same holds for $\beta_1$.

Next, define $\Pi := \beta_1 \gamma$ and observe that $\dot{\Pi} \equiv 0$, so

$$
\beta_1 \gamma_{\Pi} = \beta_{10} \gamma^F
\implies \beta_1 = \beta_{10} \gamma^F / \gamma_{\Pi}
\implies \beta_{3t} = 1 + \beta_{10}(1 - \gamma^F / \gamma_{\Pi}),
$$

where $\gamma_{\Pi} \geq \gamma^F > 0$ for all $t$ over the interval of existence, since $\gamma$ is nondecreasing. Now $|\beta_{1t}| \leq |\beta_{10}|$, so both $\beta_1$ and $\beta_3$ are uniformly bounded; we now show that $\gamma$ is uniformly bounded above.

Using (C.2), the ODE for $\gamma$ is

$$
\dot{\gamma}_{\Pi} = [1 + \beta_{10}(1 - \gamma^F / \gamma_{\Pi})]^2 \gamma_{\Pi}^2 / \sigma^2_Y,
\implies [1 + \beta_{10} \gamma_{\Pi} - \beta_{10} \gamma^F]^2 / \sigma^2_Y.
$$

Integrating and using the initial condition for $\beta_{10}$ and $\gamma_0 = \gamma^F$ yields

$$
\gamma_{\Pi} = \frac{\gamma^F [\sigma^4_Y + t \psi(\gamma^F)^2]}{\sigma^4_Y - t \gamma^F (-\gamma^F \psi + \sigma^2_Y)},
$$

wherever this exists. The denominator is strictly positive if and only if

$$
0 < \sigma^4_Y \left[ 1 - \frac{t \gamma^o}{\sigma^2_Y} (-\rho^2 \psi + \rho) \right] =: \sigma^4_Y h(t).
$$

Now $h(t)$ is linear and thus bounded between $h(0) = 1 > 0$ and $h(T) = 1 - T(-\rho^2 \psi + \rho) = \frac{\rho^2 T}{1 - \rho} > 0$, where we have used the identity $g^{pub}(\rho) = 0$ to eliminate $\rho^2 \psi$. We conclude that the denominator is strictly positive for all $t$. 

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Moreover, at time $t = T$, we have
\[
\gamma_T = \frac{\gamma^F \sigma_Y^4 + T\psi(\gamma^F)^2}{h(T)} = \frac{\rho \gamma_0 \sigma_Y^4 [1 + \bar{T}\bar{\psi} \rho^2]}{\sigma_Y^4 \frac{T \rho^2}{1 - \rho}} = \frac{\gamma_0 [1 + \bar{T}\bar{\psi} \rho^2](1 - \rho)}{\bar{T} \rho} = \gamma_0,
\]
where the last equality follows from $g_{pub}(\rho) = 0$.

Returning to (C.1), we obtain that $\beta_1$ is negative and increasing (in the backward system), and applying the comparison theorem, it cannot change sign. From (C.2), we obtain that $\beta_3$ is less than 1, is decreasing in the backward system, and cannot change sign.

Define functions $\tilde{\psi}, \tilde{\psi}$: $[8, \infty) \to [27/8, \infty)$ by $\tilde{\psi}(\tilde{T}) = -\frac{1}{\tilde{T}} + \frac{5}{2} + \frac{\tilde{T}}{8} + \frac{(\tilde{T} - 8)^{3/2}}{8\sqrt{\tilde{T}}}$ and $\tilde{\psi}(\tilde{T}) = -\frac{1}{\tilde{T}} + \frac{5}{2} + \frac{\tilde{T}}{8} - \frac{(\tilde{T} - 8)^{3/2}}{8\sqrt{\tilde{T}}}$.

**Lemma C.2.** There is a unique linear Markov equilibrium if $\tilde{T} \leq 8$ or if $\tilde{T} > 8$ and $\tilde{\psi} \notin [\tilde{\psi}(\tilde{T}), \tilde{\psi}(\tilde{T})]$. For $\tilde{\psi} \leq 27/8$, there is a unique linear Markov equilibrium for all $\tilde{T}$ (and thus for all $T$). For the remaining parameter settings, there is multiplicity of equilibria.

**Proof.** We have $(g_{pub})'(\rho) = -\tilde{T}\tilde{\psi} \rho (2 - 3\rho) + 1 + \tilde{T}$. This quadratic has roots if and only if $\tilde{\psi} \geq \frac{3(1 + \tilde{T})}{T}$; since $(g_{pub})'(0) = 1 + \tilde{T} > 0$, $\tilde{\psi} \geq \frac{3(1 + \tilde{T})}{T}$ implies that $g_{pub}$ is strictly increasing in $[0, 1]$ and there is a unique solution to $g_{pub}(\rho) = 0$. For the rest of the proof, suppose that $\tilde{\psi} < \frac{3(1 + \tilde{T})}{T}$. Denote by $\underline{\rho}$ and $\bar{\rho}$ the roots of $(g_{pub})'(\rho) = 0$, with $0 < \underline{\rho} < \bar{\rho} < 1$, and note that these are continuously differentiable in $\tilde{T}$ and $\tilde{\psi}$ in the assumed domain:

$$\rho := \frac{1 - \sqrt{1 - 3(1 + \tilde{T})/(\tilde{T}\tilde{\psi})}}{3} \quad \bar{\rho} := \frac{1 + \sqrt{1 - 3(1 + \tilde{T})/(\tilde{T}\tilde{\psi})}}{3}.$$ 

Since $g_{pub}$ is increasing on $[0, \underline{\rho}) \cup (\bar{\rho}, 1]$ and decreasing on $(\underline{\rho}, \bar{\rho})$, so a necessary and sufficient condition for uniqueness is that either $0 < g_{pub}(\bar{\rho})$ or $g_{pub}(\underline{\rho}) < 0$. 

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By the envelope theorem,
\[
\frac{d}{d\tilde{\psi}} g^{\text{pub}}(\rho) = \frac{\partial}{\partial \tilde{\psi}} g^{\text{pub}}(\rho)|_{\rho=\tilde{\rho}} < 0 \quad \text{and} \quad \frac{d}{d\tilde{\psi}} g^{\text{pub}}(\tilde{\rho}) = \frac{\partial}{\partial \tilde{\psi}} g^{\text{pub}}(\rho)|_{\rho=\tilde{\rho}} < 0.
\]

By construction we have \( g^{\text{pub}}(\rho) > g^{\text{pub}}(\tilde{\rho}) \). Moreover, we have the limits
\[
\lim_{\tilde{\psi} \downarrow \frac{3(1+\tilde{T})}{3}} g^{\text{pub}}(\tilde{\rho}) = \lim_{\tilde{\psi} \downarrow \frac{3(1+\tilde{T})}{3}} g^{\text{pub}}(1/3) = \frac{\tilde{T} - 8}{9} \quad \text{and} \quad \lim_{\tilde{\psi} \uparrow \infty} g^{\text{pub}}(\rho) = -1.
\]

For \( \tilde{T} \leq 8 \), the first limit is nonpositive, and hence for all \( \tilde{\psi} \) in the domain, \( g^{\text{pub}}(\rho), g^{\text{pub}}(\tilde{\rho}) < 0 \), giving uniqueness. For \( \tilde{T} > 8 \), the first limit is positive, and thus there exist thresholds \( \tilde{\psi}, \underline{\psi} \) with \( \frac{3(1+\tilde{T})}{3} < \tilde{\psi} < \underline{\psi} < \infty \) defined implicitly by
\[
\begin{align*}
g^{\text{pub}}(\tilde{\rho}(\tilde{\psi})); \tilde{\psi} &= 0 \quad \text{(C.3)} \\
g^{\text{pub}}(\rho(\underline{\psi})); \underline{\psi} &= 0, \quad \text{(C.4)}
\end{align*}
\]

such that (i) if \( \tilde{\psi} < \underline{\psi}, g^{\text{pub}}(\tilde{\rho}) > 0 \), giving uniqueness and (ii) if \( \tilde{\psi} < \underline{\psi}, g^{\text{pub}}(\rho) > 0 \), giving uniqueness. We now identify \( \tilde{\psi} \) and \( \underline{\psi} \). Note that if either (C.3) or (C.4) holds, there exists a double root at some \( \rho_{2} \in (0, 1) \) and \( g^{\text{pub}} \) must be of the form
\[
g^{\text{pub}}(\rho) \equiv K(\rho - \rho_{1})(\rho - \rho_{2})^{2}
\]
\[
\iff -\tilde{T}\tilde{\psi}\rho^{2}(1 - \rho) + \rho(1 + \tilde{T}) - 1 \equiv K(\rho - \rho_{1})(\rho - \rho_{2})^{2}.
\]

Matching coefficients w.r.t. powers of \( \rho \) yields the system
\[
\begin{align*}
\tilde{T}\tilde{\psi} &= K \quad \text{(C.5)} \\
-\tilde{T}\tilde{\psi} &= -K(\rho_{1} + 2\rho_{2}) \quad \text{(C.6)} \\
1 + \tilde{T} &= K(\rho_{2}^{2} + 2\rho_{1}\rho_{2}) \quad \text{(C.7)} \\
-1 &= -K\rho_{1}\rho_{2}^{2} \quad \text{(C.8)}
\end{align*}
\]

in variables \( (K, \tilde{\psi}, \rho_{1}, \rho_{2}) \). Together (C.5) and (C.6) imply \( \rho_{1} = 1 - 2\rho_{2} \), with which (C.8) becomes \( 1 = K(1 - 2\rho_{2})\rho_{2}^{2} \). Dividing (C.7) by this yields \( 1 + \tilde{T} = \frac{2 - 3\rho_{2}}{\rho_{2}(1 - 2\rho_{2})} \). Solving this
yields $\rho_2 = \frac{4 + \bar{T} \pm \sqrt{(T-8)T}}{4(1+\bar{T})}$. Plugging these values of $\rho_2$ back into (C.7), solving for $K$ and then using (C.5) to solve for $\bar{\psi}$ yields

$$\bar{\psi}, \bar{\psi} \in \left\{ -\frac{16(1 + \bar{T})^3}{\bar{T} \left(4 + \bar{T} - \sqrt{\bar{T}(\bar{T} - 8)}\right) \left(4 - 5\bar{T} + 3\sqrt{\bar{T}(\bar{T} - 8)}\right)}, -\frac{32(1 + \bar{T})^3}{\bar{T} \left(-2 + \bar{T} + \sqrt{\bar{T}(\bar{T} - 8)}\right) \left(4 + \bar{T} + \sqrt{\bar{T}(\bar{T} - 8)}\right)} \right\}.$$

Multiplying each fraction by the conjugate of each factor in the denominator and simplifying yields the expressions given before the statement of the proposition.

No Feedback Case

System of ODEs  We look for an equilibrium of the form $a_t = \beta_0 \hat{m}_0 + \beta_1 M_t + \beta_3 \theta$, where $M_t = \mathbb{E}_t^1[\hat{M}_t]$.

The backward system is

$$\begin{align*}
\dot{\beta}_0 &= -\alpha_t \left[r \sigma_Y^2 \beta_0 \hat{m}_0 \beta_1^2 \gamma_t (1 - \chi_t)\right] \\
\dot{\beta}_1 &= -\frac{\alpha_t \beta_1 [r \sigma_Y^2 + \beta_3 \gamma_t - \beta_1 \gamma_t (1 - \chi_t)]}{\sigma_Y^2} \\
\dot{\beta}_3 &= \frac{\alpha_t [r \sigma_Y^2 (1 - \beta_3) + \beta_1 \beta_3 \gamma_t]}{\sigma_Y^2} \\
\dot{\alpha} &= r (1 - \alpha_t) \alpha_t \\
\dot{\gamma} &= \frac{\alpha^2 \gamma_t^2}{\sigma_Y^2},
\end{align*}$$

with initial conditions $\beta_{10} = 0, \beta_{10} = -\frac{\psi_{\gamma_0}}{\sigma_Y^2 + \psi_{\gamma_0} \chi(\gamma_0)}, \beta_{30} = 1, \alpha_0 = \frac{\sigma_Y^2}{\sigma_Y^2 + \psi_0 \chi(\gamma_0)}$ and $\gamma_0 = \gamma^F$, where $\chi(\gamma) := 1 - \gamma/\gamma^o$.

Writing the value function as

$$V(t, \theta, M_t) = v_{0t} + v_{1t} \theta + v_{2t} M_t + v_{3t} \theta^2 + v_{4t} M_t^2 + v_{5t} \theta M_t,$$
we have the (backward) system

\[
\begin{align*}
\dot{v}_{0t} &= -rv_{0t} - \frac{v_{2t} \gamma_t^2 [v_{2t} + 4 \dot{m}_0 v_{4t}(1 - \chi_t)]}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]^2} \\
\dot{v}_{1t} &= -rv_{1t} - \frac{2 \gamma_t [2 \dot{m}_0 v_{4t} v_{5t} \gamma_t (1 - \chi_t) + v_{2t} (-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t))]}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]^2} \\
\dot{v}_{2t} &= -rv_{2t} - \frac{4 \gamma_t [\sigma_{1t}^2 v_{2t} + 2 \dot{m}_0 v_{4t}^2 \gamma_t (1 - \chi_t)]}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]^2} \\
\dot{v}_{3t} &= -rv_{3t} - \frac{v_{5t} \gamma_t [-4 \sigma_{1t}^2 \gamma_t (v_{5t} + 4 v_{4t} \chi_t)]}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]^2} \\
\dot{v}_{4t} &= -rv_{4t} + \frac{4 v_{4t} \gamma_t}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]^2} \\
\dot{v}_{5t} &= -rv_{5t} - \frac{4 v_{4t} \gamma_t}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]} + \frac{4 v_{5t} \gamma_t [-\sigma_{1t}^2 \gamma_t]}{[-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t)]^2}
\end{align*}
\]

with initial conditions \(v_{10} = v_{20} = v_{30} = v_{50} = 0, v_{40} = -\psi \gamma_0 \chi_0\) and \(v_{40} = -\psi\). Given a solution to the \(\beta\) system (with existence shown in the next subsection),

\[
\begin{align*}
v_{2t} &= \frac{2 \sigma_{1t}^2}{\alpha_t \gamma_t} \\
v_{4t} &= \frac{\sigma_{1t}^2 \beta_{1t}}{\alpha_t \gamma_t} \\
v_{5t} &= \frac{-2 \sigma_{1t}^2 (1 - \beta_{3t})}{\alpha_t \gamma_t}
\end{align*}
\]

As \(\alpha\) is positive and constant (as shown in the next subsection), the coefficients \(v_2, v_4\) and \(v_5\) are well-defined in terms of the solution to the \(\beta\) system. Now in the ODEs for \(v_0, v_1\) and \(v_3\), we have in the denominators \(-2 \sigma_{1t}^2 + \gamma_t (v_{5t} + 2 v_{4t} \chi_t) = \frac{2 \sigma_{1t}^2}{\alpha_t}\) bounded away from 0. Given \(v_2, v_4\) and \(v_5\), these ODEs are linear and uncoupled and thus have solutions.
Existence of Linear Markov Equilibrium: $r = 0$ case When $r = 0$, the associated backward system is:

\[
\begin{align*}
\dot{\beta}_1 &= -\frac{\beta_1 \gamma (\beta_3 + \beta_1 \chi - \beta_1) (\beta_3 + \beta_1 \chi)}{\sigma_Y^2}, \\
\dot{\beta}_3 &= \frac{\beta_3 \beta_1 \gamma (\beta_3 + \beta_1 \chi)}{\sigma_Y^2}, \\
\dot{\beta}_0 &= -\frac{\beta_2^2 \gamma (1 - \chi)(\beta_3 + \beta_1 \chi)}{\sigma_Y^2}, \\
\dot{\gamma} &= \frac{\gamma^2 (\beta_3 + \beta_1 \chi)^2}{\sigma_Y^2}
\end{align*}
\]

where $\chi = \chi(\gamma) = 1 - \gamma/\gamma_o$, and with initial conditions

\[
\beta_{30} = 1, \beta_{00} = 0, \gamma_0 = \gamma^F \in (0, \gamma^o), \text{ and } \beta_{10} = -\frac{\psi \gamma^F}{\sigma_Y^2 + \psi \gamma^F \chi(\gamma^F)} < 0.
\]

Lemma C.3. For all $T > 0$ and $\psi > 0$, there exists a linear Markov equilibrium. The corresponding $\gamma_T \in (0, \gamma^o)$ satisfies $g^{NF}(\gamma_T/\gamma^o) = 0$ where

\[
g^{NF}(\rho) := \tilde{T} \rho - (1 - \rho)[1 + \tilde{\psi} \rho (1 - \rho)]^2, \ \rho \in [0, 1],
\]

with $\tilde{T} := T \gamma^o/\sigma_Y^2$ and $\tilde{\psi} := \psi \gamma^o/\sigma_Y^2$.

Proof. Observe first that since $g^{NF}(0) < 0$ and $g^{NF}(1) > 0$, there exists $\gamma^F \in (0, \gamma^o)$ as in the statement of the Proposition. Now fix any such solution as the initial condition for the posterior variance in the backward IVP. We will show that this problem admits a solution over $[0, T]$ with the property that $\gamma_T = \gamma^o$.

Consider the backward IVP indexed by $\gamma^F$ over its maximal interval of existence. Notice first that $\dot{\beta}_0 + \dot{\beta}_1 + \dot{\beta}_3 \equiv 0$, and so

\[
\begin{align*}
\beta_{0t} + \beta_{1t} + \beta_{3t} &= \beta_{00} + \beta_{10} + \beta_{30} = 1 - \frac{\psi \gamma^F}{\sigma_Y^2 + \psi \gamma^F \chi(\gamma^F)}.
\end{align*}
\]

Thus, as long as $\beta_1$ and $\beta_3$ exist, $\beta_0$ will too, and since $\beta_0$ does not appear in any of the other ODEs, we can ignore it from the analysis.

Similarly, $\alpha := \beta_3 + \beta_1 \chi$ satisfies $\dot{\alpha} \equiv 0$, and so

\[
\begin{align*}
\alpha_t := \beta_{3t} + \beta_{1t} \chi_t &= \bar{\alpha} := 1 - \frac{\psi \gamma^F \chi(\gamma^F)}{\sigma_Y^2 + \psi \gamma^F \chi(\gamma^F)} = \frac{\sigma_Y^2}{\sigma_Y^2 + \psi \gamma^F \chi(\gamma^F)} \in (0, 1).
\end{align*}
\]
Consider now the subsystem

\[
\begin{align*}
\dot{\beta}_1 &= -\frac{\beta_1 \gamma (\beta_3 + \beta_1 \chi - \beta_1) (\beta_3 + \beta_1 \chi)}{\sigma_Y^2} = -\frac{\beta_1 \bar{\alpha} \gamma (\bar{\alpha} - \beta_1)}{\sigma_Y^2}, \\
\dot{\beta}_3 &= \frac{\beta_3 \beta_1 \gamma (\beta_3 + \beta_1 \chi)}{\sigma_Y^2} = \frac{\beta_3 \beta_1 \gamma \bar{\alpha}}{\sigma_Y^2},
\end{align*}
\]

(C.9)

and observe that since \(\beta_{10} < 0\) and \(\beta_{30} = 1 > 0\), the same inequalities hold in a neighborhood of zero.

We claim that \(\beta_3\) and \(\beta_1\) do not change signs. First, both cannot vanish at the same time, as this would violate that \(\alpha_t = \bar{\alpha} > 0\). Now suppose \(\beta_3\) is the first to do it, say at time \(t\); then for all \(s \in [0, t]\), \(\beta_{10} < 0\) and by the comparison theorem, \(\beta_3 > 0\) for all \(s \in [0, t]\), a contradiction. Likewise, a contradiction obtains if \(\beta_1\) vanishes first. We therefore conclude that \(\beta_1\) is increasing while \(\beta_3\) is decreasing, and that they lie in \([\beta_{10}, 0]\) and \([0, 1]\) as long as they exist.

The existence of a solution of the IVP over \([0, T]\) then reduces to the existence of a solution to the \(\gamma\)-ODE when this one is driven by \(\bar{\alpha}\). As long as this one exists, straightforward integration shows that

\[
\gamma_t = \frac{\gamma^F \sigma_Y^2}{\sigma_Y^2 - \gamma^F \bar{\alpha}^2 t}.
\]

Since \(\gamma^F > 0\), the right-hand side is well defined over \([0, T]\) if \(\sigma_Y^2 - \gamma^F \bar{\alpha}^2 T > 0\). Using that \(\bar{\alpha} = \sigma_Y^2 / [\sigma_Y^2 + \psi \gamma^F (1 - \gamma^F / \gamma^o)] = 1 / [1 + \bar{\psi} \rho (1 - \rho)]\), where \(\rho = \gamma^F / \gamma^o\) and \(\bar{\psi} = \psi \gamma^o / \sigma_Y^2\), we get

\[
\sigma_Y^2 - \gamma^F \bar{\alpha}^2 T > 0 \iff 1 - \rho \bar{\alpha}^2 \bar{T} > 0 \iff [1 + \bar{\psi} \rho (1 - \rho)]^2 - \rho \bar{T} > 0.
\]

By definition of \(\gamma^F\), however, \(g^{NF}(\rho) = 0\), which implies

\[
\iff [1 + \bar{\psi} \rho (1 - \rho)]^2 - \rho \bar{T} = \rho [1 + \bar{\psi} \rho (1 - \rho)]^2 > 0.
\]

Moreover,

\[
\gamma_T = \frac{\gamma^F \sigma_Y^2}{\sigma_Y^2 - \gamma^F \bar{\alpha}^2 T} = \frac{\gamma^F}{1 - \rho \bar{\alpha}^2 T} = \frac{\gamma^F [1 + \bar{\psi} \rho (1 - \rho)]^2}{\rho [1 + \bar{\psi} \rho (1 - \rho)]^2} = \gamma^o,
\]

concluding the proof.

**Lemma C.4.** Suppose that \(\bar{\psi} \in (0, 2]\). Then \(g^{NF}\) has a unique root in \([0, 1]\), and thus if \(\bar{\psi} \in (0, 2]\), there is a unique LME.

**Proof.** To show uniqueness, we prove that, under the previous range of values of \(\psi\), the
derivative of \( g^{NF} \) is positive at any point that satisfies \( g^{NF}(\rho) = 0 \); thus, \( g^{NF} \) can only cross zero one, and hence, it does it from below.

It is easy to verify that

\[
(g^{NF})'(\rho) = \tilde{T} + [1 + \tilde{\psi}(1 - \rho)]^2 - 2\tilde{\psi}(1 - \rho)(1 - 2\rho)[1 + \tilde{\psi}(1 - \rho)].
\]

At a crossing point, however, \( \tilde{T}\rho + \rho[1 + \tilde{\psi}(1 - \rho)]^2 = [1 + \tilde{\psi}(1 - \rho)]^2 \), and so

\[
(g^{NF})'(\rho) = \frac{1}{\rho}[1 + \tilde{\psi}(1 - \rho)] \left\{ 1 + \tilde{\psi}(1 - \rho) - 2\tilde{\psi}(1 - \rho)(1 - 2\rho) \right\}.
\]

Since \( \tilde{\psi} > -4 \) and \( \rho \in (0, 1) \), \( [1 + \tilde{\psi}(1 - \rho)] > 0 \). Thus, it suffices to show that

\[
Q(\rho) := 1 - \tilde{\psi}\rho + 5\tilde{\psi}\rho^2 - 4\tilde{\psi}\rho^3 > 0, \ \text{for} \ \rho \in (0, 1).
\]

Now \( Q(\rho) = 1 - \tilde{\psi}\rho(1 - \rho)(1 - 4\rho) \), so when \( \tilde{\psi} > 0 \), the previous cubic is positive (and at least 1) over \([1/2, 1]\). Also, over \([0, 1/2]\) we have

\[
Q(\rho) = 1 - \tilde{\psi}\rho + 5\tilde{\psi}\rho^2[1 - 4\rho/5] > 1 - \tilde{\psi}\rho > 1 - \tilde{\psi}/2.
\]

Since \( \psi \leq 2 \), the result follows. \( \square \)

**Learning and Payoff Comparisons**

**Lemma C.5.** If \( \tilde{\psi} \in (0, 2] \), then there is more learning in the public case for all \( T > 0 \).

**Proof.** Let \( \rho^x = \gamma^x_T/\gamma^0 \in (0, 1) \), where \( \gamma^x_T \) is the terminal value of \( \gamma \) in the BVP of case \( x \in \{\text{public, no feedback}\} \). When \( \tilde{\psi} \in (0, 2] \), these values are the unique solutions to the equations

\[
0 = g^{NF}(\rho) := \rho\tilde{T} - (1 - \rho)[1 + \tilde{\psi}(1 - \rho)]^2 = \rho(1 + \tilde{T}) - 1 - \tilde{\psi}(1 - \rho)^2[2 + \tilde{\psi}(1 - \rho)]
\]

\[
0 = g^{pub}(\rho) := \rho(1 + \tilde{T}) - 1 - \tilde{\psi}\tilde{T}\rho^2(1 - \rho), \quad \text{(C.10)}
\]

respectively. In particular, observe that \( \rho^x > 1/(1 + \tilde{T}) \), \( x \in \{\text{public, no feedback}\} \). Our goal is to show \( \rho^{pub} < \rho^{NF} \).
Now, using that \( \rho_{\text{pub}}(1 + \tilde{T}) - 1 = \tilde{\psi}(\rho_{\text{pub}})^2(1 - \rho_{\text{pub}}) \), we get that
\[
g_{\text{NF}}(\rho_{\text{pub}}) = \tilde{\psi}(1 - \rho_{\text{pub}}) \left\{ \tilde{T}^2(\rho_{\text{pub}})^2 - (1 - \rho_{\text{pub}})[2\rho_{\text{pub}}\tilde{T} + \rho_{\text{pub}}(1 + \tilde{T}) - 1] \right\}.
\]
Thus, letting
\[
Q(\rho) := \tilde{T}^2\rho^2 - (1 - \rho)[2\rho\tilde{T} + \rho(1 + \tilde{T}) - 1] = \rho^2(\tilde{T}^2 + 3\tilde{T} + 1) - \rho(3\tilde{T} + 2) + 1,
\]
it suffices to show that \( Q(\rho_{\text{pub}}) < 0 \), as \( g_{\text{NF}}(\rho) < 0 \) if and only if \( \rho < \rho_{\text{NF}} \).

Observe that the roots of \( Q \) are given by
\[
\rho_- := \frac{(3 - \sqrt{5})\tilde{T} + 2}{2(\tilde{T}^2 + 3\tilde{T} + 1)}, \quad \text{and} \quad \rho_+ := \frac{(3 + \sqrt{5})\tilde{T} + 2}{2(\tilde{T}^2 + 3\tilde{T} + 1)},
\]
and that \( \rho_- < \frac{1}{1 + \tilde{T}} < \rho_+ \). Consequently, it suffices to show that \( g_{\text{pub}}(\rho_+) > 0 \): this ensures that \( \rho_{\text{pub}} < \rho_+ \), and since \( \rho_{\text{pub}} > \frac{1}{1 + \tilde{T}} > \rho_- \), this implies that \( Q(\rho_{\text{pub}}) < 0 \).

Straightforward algebraic manipulation yields that
\[
g_{\text{pub}}(\rho_+) > 0
\iff
4(1 + \tilde{T})[(3 + \sqrt{5})\tilde{T} + 2][\tilde{T}^2 + 3\tilde{T} + 1]^2 - 8[\tilde{T}^2 + 3\tilde{T} + 1]^3
- \psi\tilde{T}[(3 + \sqrt{5})\tilde{T} + 2]^2[2\tilde{T}^2 + (3 - \sqrt{5})\tilde{T}] > 0
\]
Since the constraint is tightest when \( \tilde{\psi} = 2 \), we aim to show that, for all \( \tilde{T} > 0 \),
\[
4(1 + \tilde{T})[(3 + \sqrt{5})\tilde{T} + 2][\tilde{T}^2 + 3\tilde{T} + 1]^2 - 8[\tilde{T}^2 + 3\tilde{T} + 1]^3
- 2\tilde{T}[(3 + \sqrt{5})\tilde{T} + 2]^2[2\tilde{T}^2 + (3 - \sqrt{5})\tilde{T}] > 0
\iff
4[(1 + \sqrt{5})\tilde{T} + (\sqrt{5} - 1)][\tilde{T}^2 + 3\tilde{T} + 1]^2 - 2[(3 + \sqrt{5})\tilde{T} + 2]^2[2\tilde{T}^2 + (3 - \sqrt{5})\tilde{T}] > 0.
The polynomial on the left-hand side can be then written as $\tilde{T}\sum_{i=0}^{5} a_i \tilde{T}^i$ where

\begin{align*}
a_5 &= 4(1 + \sqrt{5}) > 0 \\
a_4 &= 4(-9 + \sqrt{5}) < 0 \\
a_3 &= 4(-13 + 11\sqrt{5}) > 0 \\
a_2 &= 68(\sqrt{5} - 1) > 0 \\
a_1 &= 4(-11 + 9\sqrt{5}) > 0 \\
a_0 &= 4(\sqrt{5} - 1) > 0
\end{align*}

(C.11)

It is trivial to check that $a_3 + a_4 \tilde{T} + a_5 \tilde{T}^2 > 0$ for all $\tilde{T} > 0$. Since the rest of the coefficients are strictly positive, the proof is concluded. \qed

Let $V^x$ denote the ex ante payoff to player 1 in the case $x \in \{\text{public, no feedback}\}$.

It follows that

\[
V^{\text{pub}} = \mathbb{E}_0 \left[ - \int_0^T (a_t - \theta)^2 dt - \psi M_t^2 \right]
\]

\[
= - \int_0^T \mathbb{E}_0 \left[ (\beta M_t + [\beta_3 t - 1] \theta)^2 \right] dt - \psi (\tilde{m}_0^2 + \gamma^o - \gamma_T)
\]

\[
= - \int_0^T \left[ (\beta_3 t - 1)^2 \gamma^o + \beta_2^2 (\gamma^o - \gamma_t) + 2\beta_1 t (\beta_3 t - 1)(\gamma^o - \gamma_t) \right] dt - \tilde{\psi} \sigma_Y^2 (1 - \rho^{\text{pub}})
\]

Using the solutions for the coefficients and $\gamma_t$ in terms of $\gamma^F$ and carrying out the simplifications, we obtain

\[
V^{\text{pub}} = V^{\text{pub}}(\rho^{\text{pub}}),
\]

where

\[
V^{\text{pub}}(\rho) := \sigma_Y^2 \left\{ -\tilde{\psi}(1 - \rho) + \tilde{T} \tilde{\psi} \rho^2 [-\tilde{\psi}(1 - \rho) + 1] + \ln \left( \frac{1 - \rho}{\tilde{T} \rho} \right) \right\}.
\]

In the no feedback case, note that

\[
\mathbb{E}_0[M_t^2] = \mathbb{E}_0[(\chi_t \theta + (1 - \chi_t)\tilde{m}_0)^2]
\]

\[
= \mathbb{E}_0[\chi_t^2 \theta^2]
\]

\[
= \chi_t^2 \gamma^o.
\]
Hence,

\[
E_0[\hat{M}_t^2] = E_0[(\hat{M}_t - M_t)^2] + E_0[M_t^2] = \chi_t \gamma_t + \chi_t^2 \gamma^o.
\]

Using \(a_t = \beta_0 t + \beta_1 M_t + \beta_3 t \theta = \alpha_t \theta\), we now calculate

\[
V^{NF} = E_0 \left[ - \int_0^T (a_t - \theta)^2 dt - \psi (\chi_T \gamma_T + \chi_T^2 \gamma^o) \right]
\]

\[
= E_0 \left[ - \int_0^T \theta^2 (1 - \bar{\alpha})^2 dt - \psi \chi_T (\gamma_T + \chi_T \gamma^o) \right]
\]

\[
= -(1 - \bar{\alpha})^2 \gamma^o T - \psi \chi_T (\gamma_T + \chi_T \gamma^o).
\]

Expressing \(\gamma_T = 1 - \gamma_T / \gamma^o, \gamma_T\) and \(\bar{\alpha}\) in terms of \(\gamma^F = \gamma_T\), we have

\[
V^{NF} = V^{NF}(\rho^{NF}),
\]

where

\[
V^{NF}(\rho) := \sigma_Y^2 \left\{ -\tilde{\psi}^2 \rho (1 - \rho)^3 - \tilde{\psi} (1 - \rho) \right\}.
\]

**Lemma C.6.** For \(\tilde{\psi} \in (0, 1]\), the long-run player is better off in the no feedback case than in the public case for all \(T > 0\).

**Proof.** We show that (i) \(V^{pub}(\rho^{pub}) < V^{NF}(\rho^{pub})\) and (ii) \(V^{NF}(\rho)\) is increasing for \(\rho \geq \rho^{pub}\), so that \(V^{pub}(\rho^{pub}) < V^{NF}(\rho^{NF})\).

Toward establishing (i), define

\[
\tilde{V}(\rho) := V^{pub}(\rho) - V^{NF}(\rho)
\]

\[
= \sigma_Y^2 \left\{ T \tilde{\psi}^2 \rho^2 [-\tilde{\psi} (1 - \rho) + 1] + \ln \left( \frac{1 - \rho}{T \rho} \right) + \tilde{\psi}^2 \rho (1 - \rho)^3 \right\},
\]

so that our first goal is to show \(\tilde{V}(\rho^{pub}) < 0\). Using the inequality \(\ln(x) < x - 1\) for \(x > 0\),
we have

\[ \tilde{V}(\rho) < \sigma^2 \tilde{V}_2(\rho), \]

where \( \tilde{V}_2(\rho) := \tilde{T}^2 \tilde{\psi} \rho^2 [1 - \tilde{\psi}(1 - \rho)] + 1 - \rho(1 + \tilde{T}) + \tilde{T} \tilde{\psi}^2 \rho^2 (1 - \rho)^3, \) and so it suffices to show \( \tilde{V}_2(\rho_{pub}) < 0. \)

Now the equation \( g_{pub}(\rho_{pub}) = 0 \) is equivalent to \( \tilde{\psi} = -\frac{1 - (1 + \tilde{T}) \rho}{T \rho^2 (1 - \rho)} |_{\rho = \rho_{pub}}; \) using this to eliminate \( \tilde{\psi} \) and simplifying, we obtain

\[ \tilde{V}_2(\rho_{pub}) = -\frac{[\rho(1 + \tilde{T}) - 1]^3}{\tilde{T} \rho^2} |_{\rho = \rho_{pub}}, \]

which is strictly negative as \( \rho_{pub} > \frac{1}{1+\tilde{T}}, \) establishing claim (i).

Toward claim (ii), differentiate

\[ \frac{d}{d\rho} V^{NF}(\rho) = \sigma^2 \tilde{V}_3 \left\{-\tilde{\psi}^2 [3\rho(1 - \rho)^2 + (1 - \rho)^3] + \tilde{\psi} \right\} \]

\[ = \sigma^2 \tilde{V}_3 \left\{-\tilde{\psi}(1 - \rho)^2 (1 - 4\rho) + 1 \right\}. \]

The expression in braces is positive iff \( h(\rho) := (1 - \rho)^2 (1 - 4\rho) < \frac{1}{\tilde{\psi}}, \) Now for \( \rho \in [0, 1], \) \( h(\rho) \) attains its maximum value of 1 at \( \rho = 0. \) Hence, the expression is positive for all \( \rho \in (0, 1) \) if \( \tilde{\psi} \leq 1. \) We conclude that \( V^{NF}(\rho) \) is increasing for all \( \rho \in (0, 1), \) and hence for all \( \rho \geq \rho_{pub}, \)

if \( \tilde{\psi} \leq 1. \)

Combining parts (i) and (ii) yields \( V^{pub}(\rho_{pub}) < V^{NF}(\rho_{pub}) < V^{NF}(\rho^{NF}) \) as desired.  

**Proofs for Section 5.2**

**Proof of Proposition 6.** The proof is by contradiction; suppose that a linear Markov equilibrium exists in which the long-run player plays \( a_t = \beta_{0t} + \beta_{1t} M_{2t} + \beta_{2t} L_t + \beta_{3t} \theta. \) Define \( \alpha_{0t} := \beta_{0t}, \alpha_{2t} := \beta_{2t} + \beta_{1t} (1 - \chi_t) \) and \( \alpha_{3t} = \beta_{3t} + \beta_{1t} \chi_t. \) We derive a collection of necessary conditions for equilibrium and show that there is no real value of \( \alpha_{30} \) for which they can be satisfied.

From the long-run player’s perspective, given an action profile \( (a_t)_{t \geq 0}, \) the state variables
\(M_t\) and \(L_t\) evolve according to
\[
\begin{align*}
    dL_t &= \mu_L(a_t)dt + \sigma_LdZ^X_t, \\
    dM_t &= \mu_M(a_t)dt + \sigma_MdZ^X_t,
\end{align*}
\]
where
\[
\begin{align*}
    \mu_L(a) &= \hat{B}_t \left[ a + \xi(m-l) - \alpha_0 t - (\alpha_2 + \alpha_3)l \right] / (1 - \chi_t) \\
    \mu_M(a) &= \sum \alpha_3 t \gamma_t \left[ a - \alpha_0 t - \alpha_2 l - \alpha_3 m \right] \\
    \sigma_L &= \hat{B}_t \sigma_X / (1 - \chi_t) \\
    \sigma_M &= \hat{B}_t \sigma_X \\
    \hat{B}_t &= \gamma_t (\alpha_3 + \xi \chi_t) \sigma_X^2 / \sigma_Y^2 \\
    \Sigma &= \frac{1}{\sigma_X^2} + \frac{1}{\sigma_Y^2},
\end{align*}
\]
and where the learning ODEs are
\[
\begin{align*}
    \dot{\gamma}_t &= -\gamma_t^2 \alpha_3 \Sigma, \quad \text{(C.12)} \\
    \dot{\chi}_t &= \gamma_t \alpha_3 \Sigma (1 - \chi_t) - (\alpha_3 + \xi \chi_t) \hat{B}_t. \quad \text{(C.13)}
\end{align*}
\]
In any conjectured equilibrium, given \((t, \theta, M_t, L_t) = (t, \theta, m, l)\), the long-run player’s expected continuation payoff from following the equilibrium strategy is \(\mathbb{E}_t \left[ \int_t^T (\theta - L_s) a_s \, ds \right]\), which is of the form \(v_{0t} + v_{1t} \theta + v_{2t} m + v_{3t} l + v_{4t} \theta^2 + v_{5t} m^2 + v_{6t} \theta l + v_{7t} \theta m + v_{8t} \theta l + v_{9t} m l\).

By optimality, the long-run player’s value function must satisfy the HJB equation
\[
0 = \sup_a \left\{ (\theta - L) a + V_t + \mu_L(a) V_L + \mu_M(a) V_M + \frac{1}{2} \sigma_M^2 V_{MM} + \frac{1}{2} \sigma_L^2 V_{LL} + \sigma_L \sigma_M V_{LM} \right\},
\]
and since this is linear in \(a\), the following indifference condition must hold:
\[
0 = (\theta - l) + \frac{\hat{B}_t}{1 - \chi_t} V_L + \alpha_3 t \gamma_t \Sigma V_M. \quad \text{(C.14)}
\]
Since (C.14) must hold for all values of \(\theta, m\) and \(l\), we match coefficients to obtain the
following four equations:

\[
\begin{align*}
\text{constant:} & \quad 0 = \gamma_t \left[ \sum v_{2t} \alpha_{3t} + \frac{v_{3t}(\alpha_{3t} + \chi_1)}{\sigma^2_X (1 - \chi_t)} \right] \quad (C.15) \\
\theta: & \quad 0 = 1 + \sum v_{7t} \alpha_{3t} \gamma_t + \frac{v_{8t} \gamma_t (\alpha_{3t} + \chi_1)}{\sigma^2_X (1 - \chi_t)} \quad (C.16) \\
m: & \quad 0 = \gamma_t \left[ 2\sum v_{5t} \alpha_{3t} + \frac{v_{9t} (\alpha_{3t} + \chi_1)}{\sigma^2_X (1 - \chi_t)} \right] \quad (C.17) \\
l: & \quad 0 = -1 + \sum v_{9t} \alpha_{3t} \gamma_t + \frac{2v_{10t} \gamma_t (\alpha_{3t} + \chi_1)}{\sigma^2_X (1 - \chi_t)}. \quad (C.18)
\end{align*}
\]

Note that \( \gamma_0 = \gamma^o > 0 \), and since \( \chi_0 = 0 \), (C.16) (or (C.18)) implies \( \alpha_{30} \neq 0 \). Hence, for sufficiently small \( t \), we can solve (C.17)-(C.18) to obtain

\[
\begin{align*}
v_{5t} &= -\frac{\sigma^2_X v_{9t} (\alpha_{3t} + \chi_1)}{2(\sigma^2_X + \sigma^2_Y) \alpha_{3t} (1 - \chi_t)} \quad (C.19) \\
v_{6t} &= \frac{[\sigma^2_X \sigma^2_Y - (\sigma^2_X + \sigma^2_Y) v_{9t} \alpha_{3t} \gamma_t] (1 - \chi_t)}{2 \sigma^2_Y \gamma_t (\alpha_{3t} + \chi_1)} \quad (C.20).
\end{align*}
\]

Differentiate \( v_{5t} \), use (C.13) to replace \( \dot{\chi}_t \) and evaluate at time \( t = 0 \) to obtain

\[
\dot{v}_{50} = -\frac{v_{90} \alpha_{30} (\xi + \alpha_{30}) \gamma_0 + \sigma^2_Y \dot{v}_{90}}{2(\sigma^2_X + \sigma^2_Y)}. \quad (C.21)
\]

On the other hand, the indifference condition reduces the HJB equation to

\[
0 = V_t + \mu_L(0)V_L + \mu_M(0)V_M + \frac{1}{2} \sigma^2_M V_{MM} + \frac{1}{2} \sigma^2_L V_{LL} + \sigma_L \sigma_M V_{LM}. \quad (C.22)
\]

Now (C.22) must hold for all \( \theta, m \) and \( l \), and by matching coefficients, we obtain a set of 10 equations. Evaluating the coefficients on \( m^2 \) and \( ml \) at \( t = 0 \), and using (C.19), (C.20) and (C.21) to eliminate \( v_5 \), \( v_6 \) and \( \dot{v}_5 \), we obtain

\[
\begin{align*}
m^2: & \quad 0 = \frac{(\sigma^2_X + 2\sigma^2_Y)v_{90} \alpha_{30}(\xi + \alpha_{30}) \gamma_0 - \sigma^2_X \sigma^2_Y \dot{v}_{90}}{\sigma^2_X (\sigma^2_X + \sigma^2_Y)} \quad (C.23) \\
& \implies \dot{v}_{90} = \frac{(\sigma^2_X + 2\sigma^2_Y)v_{90} \alpha_{30}(\xi + \alpha_{30}) \gamma_0}{\sigma^2_X \sigma^2_Y} \\
ml: & \quad 0 = \frac{\sigma^2_X}{\sigma^2_Y} - \frac{v_{90} \alpha_{30}(\xi + \alpha_{30}) \gamma_0}{\sigma^2_Y} + \dot{v}_{90} \\
& \implies \dot{v}_{90} = -\frac{\sigma^2_X}{\sigma^2_Y} + \frac{(\sigma^2_X + 2\sigma^2_Y)v_{90} \alpha_{30}(\xi + \alpha_{30}) \gamma_0}{\sigma^2_X \sigma^2_Y} \quad (C.24)
\end{align*}
\]
Clearly, (C.23) and (C.24) cannot hold simultaneously, which gives us the desired contradiction. (Note that if \( v_{90} \neq 0 \) and \( \alpha_{30} > 0 \), equality (of the form \( +\infty = +\infty \)) between the right hand sides of (C.23) and (C.24) can be achieved if \( \alpha_{30} = +\infty \), supporting the interpretation that the long-run player would trade away all information in the first instant.)

References


