Steven M Bergantino
14.382
Handout #2

Distribution Theory:

(1) Univariate Results for Normal Distribution

(T1)

- $X \sim N(\mu, \sigma^2) \Rightarrow (X-\mu)/\sigma \sim N(0,1)$
- $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2_1$
- $X_1, \ldots, X_n$ independent $\chi^2_{r_1}, \ldots, \chi^2_{r_n}$ random variables $\Rightarrow \sum_{i=1}^n X_i \sim \chi^2_{\sum_i r_i}$
- $Z_1, \ldots, Z_n$ independent $N(0,1)$ random variables $\Rightarrow \sum_{i=1}^n Z_i^2 \sim \chi^2_n$
- $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ independently $\Rightarrow F = \dfrac{X_1/n_1}{X_2/n_2} \sim \boxed{F(n_1, n_2) = \dfrac{\chi^2_{n_1}/n_1}{\chi^2_{n_2}/n_2}}$
- $Z \sim N(0,1)$ and $X \sim \chi^2_n$ independently $\Rightarrow t = \dfrac{Z}{\sqrt{X/n}} \sim t_n$ $\qquad \boxed{t(n) = \dfrac{N(0,1)}{\sqrt{\chi^2(n)/n}}}$
- $t \sim t_n \Rightarrow \boxed{t^2 \sim F(1,n)}$
- $\boxed{\chi^2_n/n \xrightarrow{p} 1}$
- $\boxed{t_n \xrightarrow{d} N(0,1)}$
- $\boxed{n_1 F(n_1, n_2) \xrightarrow[n_2 \to \infty]{d} \chi^2_{n_1}}$

2. Expectations and Variances of Random Vectors

X: $K \times 1$ random vector $\qquad Y = L \times 1$ random vector

$$\underset{(K\times 1)}{EX} \equiv \begin{bmatrix} EX_1 \\ \vdots \\ EX_K \end{bmatrix} \qquad \underset{(K\times K)}{V(X)} \equiv E(X-EX)(X-EX)' \qquad \underset{(K\times L)}{Cov(X,Y)} \equiv E(X-EX)(Y-EY)'$$

Observe that

$$\underbrace{\mathrm{Cov}(\underset{(k\times 1)}{X},\underset{(\ell\times 1)}{Y})}_{(k\times \ell)} = E\begin{bmatrix} X_1-EX_1\\ \vdots \\ X_k-EX_k\end{bmatrix}\left[(Y_1-EY_1)\ \cdots\ (Y_L-EY_L)\right]$$

Not symmetric in general (even when $K=L$)

$$=\begin{bmatrix} E(X_1-EX_1)(Y_1-EY_1) & \cdots & E(X_1-EX_1)(Y_L-EY_L)\\ \vdots & & \vdots \\ E(X_k-EX_k)(Y_1-EY_1) & \cdots & E(X_k-EX_k)(Y_L-EY_L)\end{bmatrix} = \begin{bmatrix} \mathrm{Cov}(X_1,Y_1) & \cdots & \mathrm{Cov}(X_1,Y_L)\\ \vdots & & \vdots \\ \mathrm{Cov}(X_k,Y_1) & \cdots & \mathrm{Cov}(X_k,Y_L)\end{bmatrix}$$

$$\Rightarrow V(X) = \mathrm{Cov}(X,X) = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \cdots & \mathrm{Cov}(X_1,X_k)\\ \mathrm{Cov}(X_2,X_1) & \ddots & & \vdots \\ \vdots & & \ddots & \mathrm{Cov}(X_{k-1},X_k) \\ \mathrm{Cov}(X_k,X_1) & \cdots & \mathrm{Cov}(X_k,X_{k-1}) & \mathrm{Var}(X_k)\end{bmatrix} \leftarrow \text{symmetric}$$

(T2) **Properties of $E()$, $V()$ and $\mathrm{Cov}()$**

- $Ea = a$
- $EX' = (EX)'$
- $E(AX) = AEX$

$\left.\right\} \Rightarrow EX'A' = (EX')A'$

- $E(X+Y) = EX+EY$ if X, Y have same dimension
- $\mathrm{Cov}(AX+a, BY+b) = A\,\mathrm{Cov}(X,Y)B'$
- $\Rightarrow$ $\boxed{V(AX+a) = AV(X)A'}$
- $\mathrm{Cov}(Y,X) = [\mathrm{Cov}(X,Y)]'$
- $\Rightarrow$ $V(X)$ is symmetric
- $V(X)$ is PSD (PD unless $\exists\, c \neq 0$ s.t. $c'X = 0$ w/ probability 1) proof
- $E(g(x)f(y)\mid x) = g(x)E(f(y)\mid X)$ $\forall$ functions $f(), g()$
- $EY = E\,E(Y\mid X)$ Law of Iterated Expectations

These are fairly straightforward generalizations of the univariate results so we will only prove a couple of the properties:

- $$\begin{aligned}\mathrm{Cov}(AX+a, BY+b) &= E[AX+a-E(AX+a)][BY+b-E(BY+b)]' \\ &= E[A(X-EX)(B(Y-EY))'] \\ &= E[A(X-EX)(Y-EY)'B'] \\ &= A\,E[(X-EX)(Y-EY)'B'] \\ &= A\,E[(X-EX)(Y-EY)']B' \\ &= A\,\mathrm{Cov}(X,Y)B'\end{aligned}$$

- $\mathrm{Cov}(Y,X) = E(Y-EY)(X-EX)' = E[(X-EX)(Y-EY)']' = [E(X-EX)(Y-EY)']' = [\mathrm{Cov}(X,Y)]'$

- $\forall C \in \mathbb{R}^k$, $\underbrace{C'X}_{scalar}$ is a random variable. Thus, we have

$$\left.\begin{array}{l} 0 \le \underset{\underset{E(C'X - C'EX)^2}{\parallel}}{V(C'X)} = C'V(X)(C')' = C'V(X)C \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\vee}{0}\ \forall C \end{array}\right\} \Rightarrow V(X) \text{ PSD}$$

Further, unless $\exists\, C \neq 0$ s.t. $C'X = 0$ w/ probability 1, $V(C'X) > 0$ and $V(X)$ is PD.

3. Multivariate Results for the Normal Distribution

$(X: n \times 1)$ $\quad X \sim N(\mu, \Sigma) \Rightarrow f(x) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp[-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)]$

(T3) **Theorem:** Let X_1 and X_2 be jointly normally distributed

$$\underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{X} \sim N\left(\underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}}_{\mu}, \underbrace{\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}}_{\Sigma}\right)$$

then

$X_1 \sim N(\mu_1, \Sigma_{11})$ and $X_1|_{X_2} \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) \mid X_2$

$X_2 \sim N(\mu_2, \Sigma_{22})$ and $X_2|_{X_1} \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}) \mid X$

(T4) **Theorem:** $X \sim N(\mu, \Sigma) \Rightarrow Y = AX + a \sim N(A\mu + a, A\Sigma A')$

Proof:
Y is a linear combination of normal random variables and so is normal w/
$EY = E(AX + a) = AEX + a = A\mu + a$
$V(Y) = V(AX + a) = AV(X)A' = A\Sigma A'$ //

(C4) **Corollary:** $Z \sim N(0, I)$ and C square w/ $C'C = I \Rightarrow C'Z \sim N(0, I)$

(T5) **Theorem:** $Z \sim N(0, I)$ and A idempotent, symmetric w/ $\text{rank}(A) = J$
$\Rightarrow Z'AZ \sim \chi^2_J$.

Proof:
A symmetric $\Rightarrow A = C\Lambda C'$ w/ $C'C = I$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$
A idempotent, symmetric $\Rightarrow \lambda_i \in \{0, 1\}\ \forall i$, and since $\text{rank}(A) = J$, $\lambda_i = 1$

for exactly J eigenvalues. We can order the eigenvalues so that $\lambda_1,\dots,\lambda_J$ are all 1 and $\lambda_i = 0 \ \forall i > J$. Then, we have

$$Z'AZ = Z'C\Lambda C'Z = y'\Lambda y = \sum_{i=1}^n \lambda_i y_i^2 = \sum_{i=1}^J y_i^2 \sim \chi^2_J$$

since $y = C'Z \sim N(0,I)$ ~~(by~~ the previous theorem) implies that $\sum_1^J y_i^2$ is the sum of J independent $N(0,1)$ random variables which is itself a χ^2_J.

Note: $Z \sim N(0,I)$ $(n\times 1)$ $\Rightarrow$ $Z'Z \sim \chi^2_n$
$= \sum Z_i^2$

(C5) Corollary: $X_i \sim N(\mu,\sigma^2)$ iid $i=1,\dots,n \Rightarrow \sum(X_i-\bar{X})^2/\sigma^2 \sim \chi^2_{n-1}$

$Z = \frac{X-\mu}{\sigma} \sim N(0,I)$

Proof:

$$(n-1)s^2/\sigma^2 = \sum_{i=1}^n (X_i-\bar{X})^2/\sigma^2 = \sum_{i=1}^n [(X_i-\mu)-(\bar{X}-\mu)]^2/\sigma^2 = \sum_{i=1}^n (Z_i-\bar{Z})^2 = Z'Q_0Z$$

where $Z_i = (X_i-\mu)/\sigma \Rightarrow Z \sim N(0,I)$ and where $\boxed{Q_0 = I - i(i'i)^{-1}i'}$ is symmetric, idempotent w/ rank equal to its trace given by:

$$\text{rank}(Q_0) = tr(Q_0) = tr[I - i\underbrace{(i'i)^{-1}}_{1/n}i'] = tr(I) - tr(ii'/n) = n - \underbrace{tr(ii')}_{=\, tr\, i'i = n}/n$$

$= n - n/n = n - 1$ // Note: i is an $n\times 1$ vector of 1's

$ii' = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix}$

(T6) Theorem: $X \sim N(\mu,\Sigma) \Rightarrow Z \equiv \Sigma^{-1/2}(X-\mu) \sim N(0,I)$

Proof:

Σ symmetric, PD $\Rightarrow$ $\Sigma = C\Lambda C'$ and $\boxed{\Sigma^{-1/2} = C\Lambda^{-1/2}C'}$ Notice that

$$\Sigma^{-1/2}\Sigma\Sigma^{-1/2} = C\Lambda^{-1/2}\underbrace{C'C}_{I}\Lambda\underbrace{C'C}_{I}\Lambda^{-1/2}C' = C\underbrace{\Lambda^{-1/2}\Lambda\Lambda^{-1/2}}_{I}C' = CC' = I$$

Thus, we have

$EZ = \Sigma^{-1/2}E(X-\mu) = 0$

$V(Z) = \Sigma^{-1/2}V(X-\mu)\Sigma^{-1/2} = \Sigma^{-1/2}V(X)\Sigma^{-1/2} = \Sigma^{-1/2}\Sigma\Sigma^{-1/2} = I$ //

$\boxed{\Sigma^{-1} = C\Lambda^{-1}C' = C\Lambda^{-1/2}\underbrace{C'C}_{I}\Lambda^{-1/2}C' = \Sigma^{-1/2}\Sigma^{-1/2}}$ $(\Sigma^{-1/2})' = \Sigma^{-1/2}$

(T7) Theorem: $X \sim N(\mu,\Sigma) \Rightarrow (X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2_n$

Proof:
$(X-\mu)'\Sigma^{-1}(X-\mu) = Z'Z$ where $Z = \Sigma^{-1/2}(X-\mu) \sim N(0,I)$ //

(T8) Theorem: $Z \sim N(0,I)$ and A, B symmetric, idempotent, then $Z'AZ$, $Z'BZ$ independent $\Leftrightarrow AB=0$

Proof:
$Z'AZ = (AZ)'(AZ)$ and $Z'BZ = (BZ)'(BZ)$ since A, B symmetric, idempotent. Thus, $Z'AZ$ and $Z'BZ$ are independent $\Leftrightarrow$ AZ and BZ are are independent. Further AZ and BZ are normal random vectors and so are independent $\Leftrightarrow \mathrm{Cov}(AZ, BZ) = 0$. But:

$$\mathrm{Cov}(AZ, BZ) = A\,\mathrm{Cov}(Z,Z)B' = AV(Z)B' = AIB' = AB' = AB \; //$$

(For F-tests)

(C8) Corollary: $Z \sim N(0,I)$ and A, B symmetric, idempotent w/ $AB=0$ and $\mathrm{rank}(A) = r_a$ and $\mathrm{rank}(B) = r_b \Rightarrow$

$$F = \frac{Z'AZ/r_a}{Z'BZ/r_b} \sim F(r_a, r_b)$$

Proof:
$Z'AZ \sim \chi^2_{r_a}$ and $Z'BZ \sim \chi^2_{r_b}$ independently since $AB=0$ //

(For t-tests)

(T9) Theorem: $Z \sim N(0,I)$ and A symmetric, idempotent, then LZ and $Z'AZ$ are independent $\Leftrightarrow LA = 0$ for L any conformable nonstochastic matrix

Proof: (same as above)
$$\mathrm{Cov}(LZ, AZ) = L\,\mathrm{Cov}(Z,Z)A' = LV(Z)A' = LIA' = LA' = LA$$