14.383

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Handout #4

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Full Information Maximum Likelihood (FIML)

$$\underset{T\times m}{Y}\underset{m\times m}{B} + \underset{T\times k}{Z}\underset{k\times m}{\Gamma} = \underset{T\times m}{U}, \quad U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{pmatrix}, \quad u_i = (u_{i1} \cdots u_{im})$$

$$u_i' \sim N(0, \Sigma)$$

u_i iid across observations

We give the density for u_i' while we observe Y_i'. To do maximum likelihood, we need to write the density for y_i.

<u>Statistical Fact</u>: If $u_i \sim f_u(\cdot)$ and u_i is a function of y_i, $u_i = g(y_i)$, then the density of y_i is

$$f_y(y_i) = \underbrace{f_u(g(y_i))}_{\substack{\text{density of } u \\ \text{evaluated at } g(y_i)}} \cdot \underbrace{\left| \det\left(\left. \frac{\partial g(y)}{\partial y} \right|_{y=y_i} \right) \right|}_{\substack{\text{absolute value of the} \\ \text{determinant of the} \\ \text{jacobian } \frac{\partial g(y)}{\partial y} \text{ evaluated} \\ \text{at } g(y_i).}}$$

Here we have $\underset{1\times m}{y_i}\,\underset{m\times m}{B} + \underset{1\times k}{z_i}\,\underset{k\times m}{\Gamma} = \underset{1\times m}{u_i}$

$\Rightarrow \quad u_i' = B' y_i' + \Gamma' z_i'$

So, $\dfrac{\partial u_i'}{\partial y_i'} = B' \equiv$ jacobian of the function $g(y_i')$

If $u_i' \sim N(0, \Sigma)$ then

$$f_u(u_i') = \left(\frac{1}{2\pi}\right)^{M/2}\left(\frac{1}{|\det \Sigma|}\right)^{1/2} e^{\left\{-\frac{1}{2}\overbrace{(y_iB + z_i\Gamma)}^{u_i\,(\text{row})}\,\Sigma^{-1}\,\overbrace{(y_iB+z_i\Gamma)'}^{u_i'\,(\text{column})}\right\}}$$

Using our statistical fact, we have:

$$f_Y(y_i') = \left(\frac{1}{2\pi}\right)^{M/2}\left(\frac{1}{|\det \Sigma|}\right)^{1/2} e^{\left\{-\frac{1}{2}(y_iB+z_i\Gamma)\Sigma^{-1}(y_iB+z_i\Gamma)'\right\}} \cdot |\det(B)|$$

So, the log likelihood of a single observation is given by:

$$\log f_Y(y_i') = \text{Const} - \tfrac{1}{2}\log|\det(\Sigma)|$$
$$- \tfrac{1}{2}(y_iB + z_i\Gamma)\Sigma^{-1}(y_iB + z_i\Gamma)' + \log|\det(B)|$$

$$\mathcal{L}(B, \Gamma, \Sigma^{-1}) = \sum_{i=1}^{T} \log f_Y(y_i')$$

- Note $-\tfrac{1}{2}\log|\det(\Sigma)| = \tfrac{1}{2}\log|\det(\Sigma)^{-1}|$

$$= \tfrac{1}{2}\log|\det(\Sigma^{-1})|$$

• Also Note that:

$$-\tfrac{1}{2}\sum_{i=1}^{T}(y_i B + z_i \Gamma)\Sigma^{-1}(y_i B + z_i \Gamma)'$$

$$= -\tfrac{1}{2}\sum_{i=1}^{T} tr\left[(y_i B + z_i \Gamma)\Sigma^{-1}(y_i B + z_i \Gamma)'\right]$$

(since it is 1×1)

$$= -\tfrac{1}{2}\sum_{i=1}^{T} tr\left[\Sigma^{-1}(y_i B + z_i \Gamma)'(y_i B + z_i \Gamma)\right]$$

(since $tr(ABC) = tr(BCA)$)

$$= -\tfrac{1}{2}\, tr\left(\sum_{i=1}^{T}\Sigma^{-1}(y_i B + z_i \Gamma)'(y_i B + z_i \Gamma)\right)$$

(since sum of trace = trace of sum)

$$= -\tfrac{1}{2}\, tr\left(\Sigma^{-1}\sum_{i=1}^{T}(y_i B + z_i \Gamma)'(y_i B + z_i \Gamma)\right)$$

$$= -\tfrac{1}{2}\, tr\left(\Sigma^{-1}\sum_{i=1}^{T} u_i' u_i\right)$$

$$= -\tfrac{1}{2}\, tr\left(\Sigma^{-1} U'U\right) \quad (\text{where } U = YB + Z\Gamma)$$

So, we have:

$$\mathcal{L}(B, \Gamma, \Sigma^{-1}) = Const. + T \log|\det(B)| + \tfrac{T}{2}\log|\det(\Sigma^{-1})| - \tfrac{1}{2} tr(\Sigma^{-1}U'U)$$

We let the superscript "u" denote the unrestricted elements in (B, Γ, Σ).

Let's find the FOC's:

$$B^u : \quad \frac{\partial \mathcal{L}}{\partial B^u} = T \frac{\partial \log |\det(B)|}{\partial B^u} - \frac{1}{2} \frac{\partial tr(\Sigma^{-1} U' U)}{\partial B^u}$$

- By (Green, P. 53), $\dfrac{\partial \log |\det(B)|}{\partial B^u} = B^{-1\prime}$

- $$\frac{\partial tr(\Sigma^{-1} U' U)}{\partial B^u} = \frac{\partial tr(U \Sigma^{-1} U')}{\partial B^u} = 2 \frac{\partial U'}{\partial B^u} (U \Sigma^{-1})$$

$\left\{ \text{using } \dfrac{\partial tr(AB)}{\partial A} = B \right\}$

$$= 2 \frac{\partial (B'Y' + \Gamma' Z')}{\partial B^u} (U \Sigma^{-1})$$

$$= 2 Y' U \Sigma^{-1}$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial B^u} = T B^{-1\prime} - Y' U \Sigma^{-1} = 0} \qquad (1)$$

$$\Gamma^u : \quad \frac{\partial \mathcal{L}}{\partial \Gamma^u} = -\frac{1}{2} \frac{\partial tr(\Sigma^{-1} U' U)}{\partial B^u}$$

$$= -\frac{1}{2} \cdot 2 Z' U \Sigma^{-1}$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \Gamma^u} = - Z' U \Sigma^{-1} = 0} \qquad (2)$$

(5)

$$\Sigma^{-1}: \quad \frac{\partial \mathcal{L}}{\partial \Sigma^{-1}} = \frac{T}{2} \frac{\partial \log |\det(\Sigma^{-1})|}{\partial \Sigma^{-1}} - \frac{1}{2} \frac{\partial tr(\Sigma^{-1} U'U)}{\partial \Sigma^{-1}}$$

$$= \frac{T}{2} \Sigma - \tfrac{1}{2} U'U$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \Sigma^{-1}} = \frac{T}{2} \Sigma - \frac{1}{2} U'U = 0$$

$$\Rightarrow \boxed{\Sigma = \frac{U'U}{T}} \quad (3)$$

Let's go back to the first FOC:

$$T B^{-1\prime} - Y'U\Sigma^{-1} = 0$$

Post multiply $B^{-1\prime}$ by $\frac{U'U}{T}\Sigma^{-1} = I$ (by (3))

$$B^{-1\prime} U'U \Sigma^{-1} - Y'U\Sigma^{-1} = 0$$

$$[(UB^{-1})' - Y'] U \Sigma^{-1} = 0$$

Note that $UB^{-1} = (YB + Z\Gamma)B^{-1} = Y + Z\Gamma B^{-1} = Y - Z\pi$

$$[(Y' - \pi' Z') - Y'] U \Sigma^{-1} = 0$$

$$\boxed{\pi' Z' U \Sigma^{-1} = 0} \quad (1')$$

Consider Eq (1'). If the (i,j) element of B, given by $[B]_{ij}$, is unrestricted, then the (i,j) element of $\frac{\partial \mathcal{L}}{\partial B^u}$ is $\left[\frac{\partial \mathcal{L}}{\partial B^u}\right]_{i,j} = \underset{1\times K}{\Pi_{(i)}'} \underset{K\times m}{(Z'U)} \underset{m\times 1}{\Sigma_{(j)}^{-1}} = \underset{1\times 1}{0}$,

where $\Pi_{(i)} = i^{th}$ column of $\underset{K\times m}{\Pi}$

$\Sigma_{(j)}^{-1} = j^{th}$ column of $\underset{m\times m}{\Sigma^{-1}}$

Consider the j^{th} equation in

$$\underset{T\times m}{Y}\,\underset{m\times m}{B} + \underset{T\times K}{Z}\,\underset{K\times m}{\Gamma} = \underset{T\times m}{U}$$

The coefficients on the endogenous variables corresponding to the j^{th} equation are found in the j^{th} column of B: $B_{(j)} = \begin{pmatrix} B_{1j} \\ \vdots \\ B_{mj} \end{pmatrix}$. Suppose a subset $\begin{pmatrix} B_{r_1,j} \\ \vdots \\ B_{r_\ell,j} \end{pmatrix}$ are unrestricted.

$$\begin{aligned} \Pi_{(r_1)}' (Z'U) \Sigma_{(j)}^{-1} &= 0 \\ \Pi_{(r_2)}' (Z'U) \Sigma_{(j)}^{-1} &= 0 \\ &\vdots \\ \Pi_{(r_\ell)}' (Z'U) \Sigma_{(j)}^{-1} &= 0 \end{aligned}$$

Then these are the FOC's for the ℓ_j unrestricted coefficients on the endogenous variables in the j^{th} equation.

$$\boxed{\Pi_j' (Z'U) \Sigma_{(j)}^{-1} = 0} \qquad (1j)$$

Equation (1_j) gives the FOC for the unrestricted coefficients on the endogenous variables in the j^{th} equation.

$\underset{k \times l_j}{\Pi_j}$ selects the l_j columns of Π that correspond to the l_j unrestricted B coefficients in the j^{th} equation. l_j varies across equations.

$$\underset{l_j \times k}{\Pi'} \underset{k \times m}{(Z'U)} \underset{m \times 1}{\Sigma^{-1}_{(\,)}} = \underset{l_j \times 1}{O}$$

$\underset{l_j \times 1}{O}$ is an l_j-vector and we have the l_j restriction imposed on the l_j unrestricted B coefficents in the j^{th} equation.

<u>Ex 1</u>: $y_1 = \beta_1 y_2 + \gamma_1 z_1 + u_1$

$y_2 = \beta_2 y_1 + \gamma_2 z_2 + \gamma_3 z_3 + u_2$

$y_3 = \beta_3 y_1 + \beta_4 y_2 + \gamma_4 z_4 + u_3$

I. $YB + Z\Gamma = U$ form,

$$B = \begin{pmatrix} 1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 1 & -\beta_4 \\ 0 & 0 & 1 \end{pmatrix}$$

For Eq. 2, only $[B]_{12}$ is unrestricted

so $\underset{4 \times 1}{\Pi_2} = \underset{4 \times 1}{\Pi_{(1)}}$, 1st col. of Π.

For Eq. 3, $[B]_{13}$, $[B]_{23}$ are unrestricted

$\Pi_3 = [\Pi_{(1)} \quad \Pi_{(2)}]$ ∎

Now recall Eq (2), the FOC's from maximizing $\mathcal{L}$ w.r.t. the unrestricted elements of Γ:

$$- Z' U \Sigma^{-1} = 0$$

The same analysis applied to Eqn (1') tells us that if $[\Gamma]_{ij}$ is unrestricted,

$$Z_{(i)}' U \Sigma_{(j)}^{-1} = 0$$

Stack these as we did earlier, assuming

$$\Gamma_{(j)} = \begin{pmatrix} \gamma_{1j} \\ \vdots \\ \gamma_{kj} \end{pmatrix} \text{ has a subset } \begin{pmatrix} \gamma_{r_1, j} \\ \vdots \\ \gamma_{r_s, j} \end{pmatrix} \text{ of } S_j$$

coefficients that are unrestricted.

$$\begin{aligned} Z_{(r_1)}' U \Sigma_{(j)}^{-1} &= 0 \\ Z_{(r_2)}' U \Sigma_{(j)}^{-1} &= 0 \\ &\vdots \\ Z_{(r_s)}' U \Sigma_{(j)}^{-1} &= 0 \end{aligned}$$

These are the FOC's for the S_j unrestricted coefficients on the exogenous variables in the j^{th} equation

$$\boxed{Z_j' U \Sigma_{(j)}^{-1} = 0} \qquad (2j)$$

(9)

In Eqn. (2 j), $\underset{T\times S_j}{Z_j}$ contains the S_j columns of Z that correspond to the S_j unrestricted Γ coefficients in the j^{th} equation.

$$\left.\underset{S_j\times T}{Z_j'}\ \underset{T\times m}{U}\ \underset{m\times 1}{\Sigma_{(j)}^{-1}} = \underset{S_j\times 1}{0}\ \right\}\ S_j \text{ restrictions.}$$

Now, we introduce a selection matrix C_j for equation j where

$$\underset{T\times S_j}{Z_j} = \underset{T\times K}{Z}\ \underset{K\times S_j}{C_j}$$

C_j selects out the S_j columns of Z found in Z_j.

<u>Ex 2</u>: $$\underset{T\times 3}{Z_j} = \left[\underset{T\times 1}{Z_{(1)}}\ \underset{T\times 1}{Z_{(4)}}\ \underset{T\times 1}{Z_{(6)}}\right]$$

$$\underset{T\times 6}{Z} = \left[\underset{T\times 1}{Z_{(1)}}\ \underset{T\times 1}{Z_{(2)}}\ \underset{T\times 1}{Z_{(3)}}\ \underset{T\times 1}{Z_{(4)}}\ \underset{T\times 1}{Z_{(5)}}\ \underset{T\times 1}{Z_{(6)}}\right]$$

$$\underset{6\times 3}{C_j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{h}$

Plugging in $Z_j = Z C_j$ into Eq (2_j)

gives $C_j' Z' U \Sigma_{(j)}^{-1} = 0$.

Stack Eq (1_j) and Eq (2_j):

$$\Pi_j'(Z'U)\Sigma_{(j)}^{-1} = 0$$
$$C_j'(Z'U)\Sigma_{(j)}^{-1} = 0$$

Let $D_j = [\Pi_j \quad C_j]$. Then we have:

$$\boxed{D_j'(Z'U)\Sigma_{(j)}^{-1} = 0}$$

This gives the FOC for the unrestricted elements of (B, Γ) for the j^{th} equation.

Stacking these for the m equations,

$$\underbrace{\begin{bmatrix} D_1' & & 0 \\ & D_2' & \\ & & \ddots \\ 0 & & & D_m' \end{bmatrix}}_{\bar{D}} \begin{bmatrix} Z'U\Sigma_{(1)}^{-1} \\ \vdots \\ \vdots \\ Z'U\Sigma_{(m)}^{-1} \end{bmatrix} = 0$$

Before we can procede, lets introduce the vec( ) function. vec(A) "vectorizes" A.

If $\underset{m\times n}{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} A_{(1)} & \cdots & A_{(n)} \end{pmatrix}$, then

$$\text{vec}(A) = \underset{mn\times 1}{\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(n)} \end{pmatrix}}$$

<u>Prop</u>: $\text{Vec}(ABC) = (C' \otimes A)\,\text{Vec}(B)$

We make use of this proposition as follows:

$$\begin{bmatrix} Z'U\,\Sigma^{-1}_{(1)} \\ \vdots \\ Z'U\,\Sigma^{-1}_{(m)} \end{bmatrix} = \text{Vec}(Z'U\Sigma^{-1}) = (\Sigma^{-1} \otimes Z')\,\text{Vec}(U)$$

What is Vec(U)? Recall our stacked model in doing 3SLS:

$$\underset{Tm\times 1}{\begin{bmatrix} y^1 \\ \vdots \\ y^m \end{bmatrix}} = \underset{Tm\times \sum_{i=1}^{m} K_i}{\begin{bmatrix} X^1 & & 0 \\ & X^2 & \\ & & \ddots & \\ 0 & & & X^m \end{bmatrix}} \underset{\sum_{i=1}^{m} K_i \times 1}{\begin{bmatrix} \delta^1 \\ \vdots \\ \delta^m \end{bmatrix}} + \underset{Tm\times 1}{\begin{bmatrix} u^1 \\ \vdots \\ u^m \end{bmatrix}}$$

$$\bar{y} = \bar{X}\,\bar{\delta} + \bar{u}$$

$$\text{Vec}(U) = \bar{u} = \bar{y} - \bar{X}\,\bar{\delta}$$

Plugging everything in, we have

$$\boxed{\bar{D}(\Sigma^{-1}\otimes Z')(\bar{X}\bar{\delta}-\bar{y}) = 0} \qquad (4)$$

This is a set of nonlinear equations in (B,Γ). The solution to this problem is the (B,Γ,Σ) MLE.

Recall where δ comes from.

$$\underset{T\times 1}{y^j} = \underset{T\times r_j}{Y^j}\beta^j + \underset{T\times s_j}{Z^j}\gamma^j + u$$

$$= [Y^j \; Z^j]\begin{bmatrix}\beta^j \\ \gamma^j\end{bmatrix} \qquad \bar{\delta} = \begin{pmatrix}\delta^1 \\ \vdots \\ \delta^m\end{pmatrix}$$

$$\underset{T\times 1}{y^j} = \underset{T\times k_j}{X^j}\,\underset{k_j\times 1}{\delta^j} + \underset{T\times 1}{u^j} \qquad \left(\sum_{j=1}^{m} k_j \times 1\right)$$

The FIML estimator of $\bar{\delta}$ is:

$$\boxed{\hat{\bar{\delta}}_{FIML} = \left(\bar{D}(\Sigma^{-1}\otimes Z')\bar{X}\right)^{-1}\left(\bar{D}(\Sigma^{-1}\otimes Z')\bar{y}\right)}$$

The 3SLS estimator of $\bar{\delta}$ is:

$$\boxed{\hat{\bar{\delta}}_{3SLS} = \left(\bar{X}'(\Sigma^{-1}\otimes P_Z)\bar{X}\right)^{-1}\left(\bar{X}'(\Sigma^{-1}\otimes P_Z)\bar{y}\right)}$$

(13)

Comparing FIML, 3SLS

$$\hat{\hat{\delta}}_{3SLS} = \left(\bar{X}'(\Sigma^{-1}\otimes P_z)\bar{X}\right)^{-1}\left(\bar{X}'(\Sigma^{-1}\otimes P_z)\bar{y}\right)$$

$$\Rightarrow \quad 0 = \left(\bar{X}'(\Sigma^{-1}\otimes P_z)\bar{X}\right)\hat{\hat{\delta}}_{3SLS} - \bar{X}(\Sigma^{-1}\otimes P_z)\bar{y}$$

$$0 = \bar{X}'(\Sigma^{-1}\otimes P_z)(\bar{X}\hat{\hat{\delta}}_{3SLS} - \bar{y})$$

$\{$use $(A\otimes B)(C\otimes D) = AC\otimes BD\}$

(5) $$\boxed{0 = \bar{X}'(I\otimes Z'(Z'Z)^{-1})(\Sigma^{-1}\otimes Z')(\bar{X}\hat{\hat{\delta}}_{3SLS} - \bar{y})}$$

Compare this to:

(4) $$\boxed{0 = \bar{D}(\Sigma^{-1}\otimes Z')(\bar{X}\hat{\hat{\delta}}_{FIML} - \bar{y})}$$

3SLS and FIML both use a matrix of weights to form instrumental variables

$$\hat{\delta} = (\bar{W}'\bar{X})^{-1}(\bar{W}'\bar{y})$$

$$\bar{W}_{FIML} = (I\otimes Z)\underbrace{(\Sigma^{-1}\otimes I)\bar{D}}_{\hat{A}_{FIML}}$$

$$\bar{W}_{3SLS} = (I\otimes Z)\underbrace{(\Sigma^{-1}\otimes I)(I\otimes(Z'Z)^{-1}Z')\bar{X}}_{\hat{A}_{3SLS}}$$

Look at the plim of $(I \otimes (Z'Z)^{-1} Z') \bar{X}$:

$$\text{plim}\ (I \otimes (Z'Z)^{-1}Z')\bar{X} = \text{plim} \begin{bmatrix} \left(\frac{Z'Z}{T}\right)^{-1}\left(\frac{Z'X^1}{T}\right) & & 0 \\ & \ddots & \\ 0 & & \left(\frac{Z'Z}{T}\right)^{-1}\left(\frac{Z'X^m}{T}\right) \end{bmatrix}$$

Now, $$\text{plim}\ \frac{Z'X^i}{T} = \text{plim}\ \frac{Z'[Y^i, Z^i]}{T}$$
$$= \text{plim}\ \frac{Z'[Z\Pi_i + V_i,\ \ ZC_i]}{T}$$
$$= \text{plim}\left(\frac{Z'Z}{T}\right)[\Pi_i\ ,\ C_i]$$
$$= \text{plim}\left(\frac{Z'Z}{T}\right) D_i$$

$$\Rightarrow \text{plim}\ (I \otimes (Z'Z)^{-1}Z')\bar{X} = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_m \end{bmatrix} = \bar{D}$$

This shows that the weighting matrices $\bar{W}_{FIML}$, $\bar{W}_{3SLS}$ are asymptotically the same.

$\Rightarrow$ FIML, 3SLS are asymptotically equivalent.