Finite-time Behavior of Switching Networks

J. Cavalcanti and H. Balakrishnan

Abstract— **This paper addresses the analysis of networked systems with switching topologies modeled by Positive Markov Jump Linear Systems, focusing on finite-time stability criteria. Main results are sufficient conditions to assess finite-time stability, and establish probabilistic bounds on state amplification. In addition, connections between notions of asymptotic and finite-time stability are explored. Numerical examples illustrate how the proposed criteria can help to gain insight on air traffic delay propagation, one of many real world networked systems that are amenable to the analysis herein.**

I. Introduction

Local dynamics and nodal interconnections can result in complex network dynamics, especially in large scale networks, where the magnitude of nodes and links reach the order of hundreds and thousands. Such large-scale networks are prevalent in economics, biology, and infrastructure systems, and their intrinsically collective behavior prompts the question of how individual disruptions and shocks spread across the network [1].

A. Analysis of network dynamics

Robustness of large networks has been a topic of increased interest in the control community in recent years. This renewed interest is partly due to the belief that control theory can provide valuable insights to a problem that has been traditionally studied using graph-theoretic techniques. An example of such insights is the use of $\mathcal{H}_2$ metrics in order to quantify network fragility [2], [3]. The first of these papers considers only symmetric network matrices, and the results focus on network volatility, as well as the impact of individual edge weights on volatility. Consensus robustness is the main concern of [3]. The interplay between fragility and network dimensions is addressed in [4] and [5]. The latter examines which structural features of economical networks allow small input shocks to escalate into large aggregate outputs. On the other hand, the authors in [4] adopt the notion of harmonic instability to explore how disturbances to a single node, the leader in vehicular platoons, can be exponentially amplified with respect to the dimensions of the vehicular chain.

Recently, [6] addressed the issue of dimensionality and robustness in general networks, i.e., weighted, undirected graphs, and used the results to compare scaling properties of common network topologies, such as the star, ring, etc. There has also been an analysis of disease spread over switched networks, for the case of undirected networks with only two nodal states [7]. In contrast, [8] analyzed asymptotic and almost sure stability of weighted, directed networks with switching topologies, in the context of air traffic delays.

B. Stability of switched linear systems

The interactions within a networked system are described by links between individual nodes, and are synthesized by the network's topology. These interconnections are often fluid—the topology can vary with time—resulting in disparate macroscopic dynamics for the same network. A tractable approach to model time-varying networks is to assume a finite set of possible topologies, and allow the system to switch between them. Switching systems, a class of hybrid systems, capture such a phenomenon, where a dynamical system is liable to abrupt transitions in dynamics commanded by a switching signal [9], [10]. This kind of system provides a general framework for analyzing a vast range of applications, which in turn have spurred the development of rich literature regarding the asymptotic stability of switching systems. Finite-time stability, however, has thus far received considerably less attention. In [11], finite-time stability (FTS) of switched positive linear systems (i.e., switched linear systems guaranteed to have positive state values given positive initial conditions) was considered. In particular, the authors developed Lyapunov-based arguments to prove necessary and sufficient FTS conditions using an "inner-product" definition of FTS. [12] proposed a new definition of FTS ("stochastic τ-stability") considering the expected value of the system's "total energy", to address switching systems where mode transitions are described by Markov chains [13]. In addition to necessary and sufficient τ-SS conditions, connections between τ-MSS and τ-EMSS (mean square τ-stable and exponentially mean square τ-stable, respectively) were also studied [12]. Recently, [14] used Lyapunov arguments to provide sufficient FTS conditions for the case where Markov chains are unknown. Systems that exhibit linear dynamics in each discrete mode, and Markovian transitions between modes are also known as Markov Jump Linear Systems or MJLS; the stability of MJLS has been a topic of recent research [15], [16].

C. Contributions

In this paper, we consider the problem of quantifying both state amplification and the probability with which it will occur in networks that switch between a finite set of (possibly unstable) topologies according to a Markov process. Among other applications, this is a critical issue to the Federal Aviation Administration (FAA) and airlines, who are interested in how flight delays are going to evolve over the

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next few hours. Using empirical data, it was shown in [17], [8] that the underlying networks that drive the interactions between delays at different airports are indeed time-varying and can lead to widespread amplification (unstable behavior), with the actual topology at any time being determined by a Markov chain. It has also been shown in [8] that Markov Jump Linear Systems are very good abstractions of how such networks behave. Under these settings, however, the state-of-the-art is unable to provide probabilistic statements of how much states would be amplified in finite time. Indeed, [2], [6] considered time-invariant networks with stable topologies and how volatility properties of such networks are affected as the number of nodes increases. Since infrastructure systems are intrinsically large networks that do not experience significant dimensional variations in the short term, the issue of scalability is overshadowed by that of topology switching. On the other hand, [8] focused on asymptotic stability of MJLSs, which is inconclusive regarding the evolution of states within shorter periods of time, i.e., it only implies stability after a large amount of hours, and cannot guarantee bounds on delay amplification within a certain finite-time period. In constrast, the concept of Finite-Time Stochastic Stability (FTSS, [18]) naturally leads to stability conditions characterized both by amplification magnitude and associated probabilities. In this light, we present FTSS criteria that allow quantifying not only the amplification accross the network, but also the degree of confidence with which it will occur. In particular, we provide FTSS criteria based on the 1-norm, to account for the aggregate network states, as well as 2-norm-based criteria, to characterize the network's volatility.

II. Model

Throughout this paper, bold-faced lowercase letters represent vectors, calligraphic uppercase letters, e.g. $\mathcal{Y}$, denote linear operators, and when specific reference is omitted, $\|\cdot\|$ refers to Euclidean 2-norm. I_n is the n-dimensional identity matrix. Operation $\otimes$ denotes the Kronecker product.

A. Continuous state dynamics

Prior work [8] has shown that given a networked linear system with fixed topology, the evolution of the continuous state $\mathbf{x}$, given by a vector in $\mathbb{R}^n$, can be written as:

$$\mathbf{x}(k+1) = \Gamma\, \mathbf{x}(k), \tag{1}$$

where system matrix Γ belongs to $\mathbb{R}^{n\times n}$, and denotes the network topology.

B. Switching network topologies

This paper considers the scenario in which the networked system exhibits switches in topology. The switches, or jumps, between topologies occur in a random, yet well-defined, manner: Each of the transitions happens with a known time-varying probability, and can be modeled by a random variable. The underlying probabilistic space of the stochastic process that characterizes the transitions is described below.

Let $\mathbb{T}$ be the index set of possible network topologies $\{1, 2, \cdots, N\}$, $\mathfrak{T}$ be the σ-algebra given by the discrete topology of $\mathbb{T}$, $\mathfrak{R}$ the Borel σ-algebra of $\mathbb{R}^n$, and let

$$\Omega := \prod_{k\in\mathbb{Z}_{\geq 0}} \left(\mathbb{R}^n \times \mathbb{T}\right)$$

denote our sample space, where $\prod$ and $\times$ represent product spaces. We also define

$$\Sigma_k := \sigma\left\{ \prod_{0\leq l\leq k} \left(R_l \times T_l\right) \times \prod_{p=k+1}^{+\infty} \left(\mathbb{R}^n \times \mathbb{T}\right) \middle| R_l \in \mathfrak{R}, T_l \in \mathfrak{T} \right\}$$

such that $\Sigma_k \subset \Sigma$, where $\sigma\{S\}$ denotes a σ-algebra of set S. The stochastic basis $(\Omega, \Sigma, \{\Sigma_k\}, P)$ allows us to define a stochastic process $\{Z_k\}_{k\in\mathbb{Z}_{\geq 0}}$, with $Z_k : \Sigma \to \mathbb{T}$ a random variable such that

$$Z_k\left(\xi\right) := \tau\left(k\right),$$

where

$$\xi := \left\{ \left(\mathbf{x}\left(k\right), \tau\left(k\right)\right) \middle| \mathbf{x}\left(k\right) \in \mathbb{R}^n, \tau\left(k\right) \in \mathbb{T}, k \in \mathbb{Z}_{k\geq 0} \right\}$$

and such that probability measure P satisfies

$$P\left(Z_{k+1} = j \middle| Z_k = i\right) = \pi_{ij}\left(k\right),$$

where Z_k is a simplified notation of $Z_k(\cdot)$. Since Z_{k+1} depends only on k, regardless of past information of Z_l for l less than k, then $\{Z_k\}_{k\in\mathbb{Z}_{\geq 0}}$ defines a time-varying Markov chain with transition probability matrix

$$\mathcal{M}_k = \left[\pi_{ij}\left(k\right)\right] \in \mathbb{R}^{N\times N} \tag{2}$$

and Z_0 with initial distribution $z = \{z_1, \cdots, z_N\}$.

C. Markov Jump Linear System model

Combining the model for the continuous state dynamics for each topology given by (1) with that of the switches in (2), we obtain the following Markov Jump Linear System (MJLS) model:

$$\mathbf{x}\left(k+1\right) = \Gamma_{Z_k}\mathbf{x}\left(k\right), \tag{3a}$$

$$P\left(Z_{k+1} = j \middle| Z_k = i\right) = \pi_{ij}\left(k\right), \tag{3b}$$

where $\mathbf{x}(k)$ is a vector in $\mathbb{R}^n$ that represents the nodes's state, and Γ_{Z_k} is system matrix corresponding to the current network's topology, Z_k. The network topology reflects the discrete mode of this hybrid system, and evolves as a time-varying Markov chain. The result is a discrete-time Markov Jump Linear System model [15], [19].

1) Stochastic states: Although Z_k is the process that represents the system's uncertainty, the state vector, $\mathbf{x}(k)$, is the quantity that actually reflects how the network evolves. We therefore introduce new node states to synthesize both:

$$\mathbf{y}(k) := \begin{bmatrix} y_1(k) & \cdots & y_N(k)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^N \tag{4}$$

$$y_i(k) := \mathbb{E}\left[\mathbf{x}(k)^{\mathrm{T}}\mathbf{x}(k)\mathbf{1}_{Z_k=i}\right] \in \mathbb{R},$$

$$\mathbf{v}(k) := \begin{bmatrix} \mathbf{v}_1(k)^{\mathrm{T}} & \cdots & \mathbf{v}_N(k)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{5}$$

$$\mathbf{v}_i(k) := \mathbb{E}\left[\mathbf{x}(k)\mathbf{1}_{Z_k=i}\right] \in \mathbb{R}^n,$$

$$\mathbf{W}(k) := \begin{bmatrix} \mathbf{W}_1(k) & \cdots & \mathbf{W}_N\end{bmatrix} \in \mathbf{R}^{n\times Nn}, \tag{6}$$

$$\mathbf{W}_i(k) := \mathbb{E}\left[\mathbf{x}(k)\mathbf{x}^{\mathrm{T}}(k)\mathbf{1}_{Z_k=i}\right] \in \mathbf{R}^{n\times n},$$

where $\mathbb{E}[Z]$ denotes the expected value of the random variable Z. Definitions (5) and (6) use the indicator function $\mathbf{1}_{Z_k=j}$, which is shorthand for

$$\mathbf{1}_{Z_k=j}(\xi) := \begin{cases} 1, & \xi \in \{\zeta \in \Omega | Z_k\left(\zeta\right) = j\} \\ 0, & \xi \notin \{\zeta \in \Omega | Z_k\left(\zeta\right) = j\} \end{cases} \tag{7}$$

2) Properties of the stochastic states: States $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{W}$ are convenient because they encapsulate the stochasticity of random topology Z_k and allow rewriting (3) as a linear system. The next propositions summarize this fact.

Proposition 1: [17], [20] For all $k \geq 0$, stochastic states $\mathbf{v}$ and $\mathbf{W}$, defined by (5) and (6), respectively, have linear dynamics described by

$$\mathbf{v}(k+1) = \mathcal{V}_k \mathbf{v}(k), \tag{8}$$

$$\mathbf{W}(k+1) = \mathcal{W}_k \mathbf{W}(k), \tag{9}$$

where $\mathcal{V}_k : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ and $\mathcal{W}_k : \mathbb{R}^{n\times Nn} \to \mathbb{R}^{n\times Nn}$ represent mappings

$$\mathcal{V}_k \mathbf{v} := \left(\mathcal{M}_k^{\mathrm{T}} \otimes I_n\right) \operatorname{diag}\left(\Gamma_i\right) \mathbf{v}, \tag{10}$$

$$\mathcal{W}_k \mathbf{X} := \left[\Gamma_1 \cdots \Gamma_N\right] \operatorname{diag}\left(\mathbf{X}_i^{\mathrm{T}}\right) \operatorname{diag}\left(\Gamma_i^{\mathrm{T}}\right) \left(\mathcal{M}_k \otimes I_n\right). \tag{11}$$

Proposition 2: The element-wise dynamics of stochastic state $\mathbf{y}(k)$ are dominated by linear operator $\mathcal{Y}_k$

$$\mathbf{y}(k+1) \leq \mathcal{Y}_k \mathbf{y}(k). \tag{12}$$

Proof: From (4), and applying the total probability and the conditional expectation theorems combined with Markov property, we have that

$$\begin{aligned} y_j(k+1) &= \mathbb{E}\left[\mathbf{x}(k+1)^{\mathrm{T}} \mathbf{x}(k+1) \mathbf{1}_{Z_{k+1}=j}\right] \\ &= \sum_{i=1}^{N} \pi_{ij}(k) \mathbb{E}\left[\mathbf{x}(k) \Gamma_{Z_k}^{\mathrm{T}} \Gamma_{Z_k} \mathbf{x}(k) \middle| Z_k = i\right] \\ &\leq \sum_{i=1}^{N} \pi_{ij}(k) \lambda_{\max}\left(\Gamma_i^{\mathrm{T}} \Gamma_i\right) y_i(k) \end{aligned}$$

where $\lambda_{\max}\left(\Gamma_i^{\mathrm{T}}\Gamma_i\right)$ denotes the largest eigenvalue of $\Gamma_i^{\mathrm{T}}\Gamma_i$. Stacking these element-wise inequalities then gives

$$\begin{aligned} \mathbf{y}(k+1) &\leq \begin{bmatrix} \sum_i^N \pi_{i1}(k) \lambda_{\max}\left(\Gamma_i^{\mathrm{T}}\Gamma_i\right) y_i(k) \\ \vdots \\ \sum_i^N \pi_{iN}(k) \lambda_{\max}\left(\Gamma_i^{\mathrm{T}}\Gamma_i\right) y_i(k) \end{bmatrix} \\ &= \mathcal{M}_k^{\mathrm{T}} \operatorname{diag}\left(\lambda_{\max}\left(\Gamma_i^{\mathrm{T}}\Gamma_i\right)\right) \mathbf{y}(k) \\ &= \mathcal{Y}_k \mathbf{y}(k), \end{aligned} \tag{13}$$

where the operator $\mathcal{Y}_k : \mathbb{R}^N \to \mathbb{R}^N$ is defined as

$$\mathcal{Y}_k \mathbf{x} := \mathcal{M}_k^{\mathrm{T}} \operatorname{diag}\left(\lambda_{\max}\left(\Gamma_i^{\mathrm{T}}\Gamma_i\right)\right) \mathbf{x} \tag{14}$$

■

From Proposition 2, we have that

$$\|\mathbf{y}(k+1)\|^2 = \sum_{j=1}^{N} y_j(k+1)^2 \leq \|\mathcal{Y}_k \mathbf{y}(k)\|^2,$$

which implies $\|\mathbf{y}(k+1)\| \leq \|\mathcal{Y}_k \mathbf{y}(k)\|$.

Now, let $\mathcal{C}_k : \mathbb{R}^{Nn^2} \to \mathbb{R}^{Nn^2}$ and $\mathcal{S} : \mathbb{R}^{n\times Nn} \to \mathbb{R}^{Nn^2}$ denote operators defined as

$$\mathcal{C}_k \mathbf{X} := \left(\mathcal{M}_k \otimes I_{n^2}\right) \operatorname{diag}\left(\Gamma_i \otimes \Gamma_i\right) \mathbf{X}, \tag{15}$$

$$\mathcal{S}\left(\left[\mathbf{W}_1 \cdots \mathbf{W}_N\right]\right) := \begin{bmatrix} \operatorname{vec}\left(\mathbf{W}_1\right) \\ \vdots \\ \operatorname{vec}\left(\mathbf{W}_n\right) \end{bmatrix}, \tag{16}$$

where $\operatorname{vec}(\cdot)$ is the operator that "stacks" columns of an $n \times m$ matrix on a single column vector of dimension nm.

Proposition 3: [13] For every $\mathbf{W}$ in $\mathbb{R}^{n\times Nn}$,

$$\mathcal{S}\left(\mathcal{W}_k \mathbf{W}\right) = \mathcal{C}_k\left(\mathcal{S}\mathbf{W}\right),$$

where operators $\mathcal{S}$ and $\mathcal{W}_k$ are defined in (16) and (11).

Proof: For any matrices P, Q and R of appropriate dimensions, the Kronecker product satisfies

$$\operatorname{vec}(PQR) = \left(R^{\mathrm{T}} \otimes P\right) \operatorname{vec}(Q) \tag{17}$$

From (11) and (17), it follows that

$$\begin{aligned} \mathcal{S}\left(\mathcal{W}_k \mathbf{W}\right) &= \begin{bmatrix} \operatorname{vec}\left(\sum_{i=1}^N \pi_{i1}(k) \Gamma_i \mathbf{W}_i(k) \Gamma_i^{\mathrm{T}}\right) \\ \vdots \\ \operatorname{vec}\left(\sum_{i=1}^N \pi_{iN}(k) \Gamma_i \mathbf{W}_i(k) \Gamma_i^{\mathrm{T}}\right) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^N \pi_{i1}(k) \left(\Gamma_i \otimes \Gamma_i\right) \operatorname{vec}\left(\mathbf{W}_i(k)\right) \\ \vdots \\ \sum_{i=1}^N \pi_{iN}(k) \left(\Gamma_i \otimes \Gamma_i\right) \operatorname{vec}\left(\mathbf{W}_i(k)\right) \end{bmatrix} \\ &= \left(\mathcal{M}_k \otimes I_{n^2}\right) \operatorname{diag}\left(\Gamma_i \otimes \Gamma_i\right) \begin{bmatrix} \operatorname{vec}\left(\mathbf{W}_1(k)\right) \\ \vdots \\ \operatorname{vec}\left(\mathbf{W}_N(k)\right) \end{bmatrix} \\ &= \mathcal{C}_k\left(\mathcal{S}\mathbf{W}(k)\right) \end{aligned}$$

■

The following propostion implies that $\|\mathcal{W}_k\|$ equals $\|\mathcal{C}_k\|$.

Proposition 4: [13] $\mathcal{S}$ is a homeomorphism between $\mathbb{R}^{n\times Nn}$ and $\mathbb{R}^{Nn^2}$. In particular, for any $\mathbf{W}$ in $\mathbb{R}^{n\times Nn}$,

$$\|\mathcal{W}_k \mathbf{W}\| = \|\mathcal{S}\left(\mathcal{W}_k \mathbf{W}\right)\|.$$

Applying linear dynamics (8) and (9) recursively yields

$$\mathbf{v}(k) = \mathcal{V}_{0,k} \mathcal{Z}_0 \mathbf{v}(0),$$

$$\mathbf{W}(k) = \mathcal{W}_{0,k} \mathcal{Z}_0 \mathbf{W}(0) \mathcal{Z}_0^{\mathrm{T}}.$$

In the above, given initial and final time instants k_0 and k_f,

$$\mathcal{V}_{k_0,k_f} := \mathcal{V}_{k_f-1} \mathcal{V}_{k_f-2} \cdots \mathcal{V}_{k_0}, \tag{18}$$

$$\mathcal{W}_{k_0,k_f} := \mathcal{W}_{k_f-1} \mathcal{W}_{k_f-2} \cdots \mathcal{W}_{k_0}, \tag{19}$$

and $\mathcal{Z}_0 : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is the operator that adjusts initial conditions $\mathbf{v}_i(0)$ and $\mathbf{W}_i(0)$ by the initial topology (discrete mode) distribution, $(z_1, \cdots, z_N)$

$$\mathcal{Z}_0 \mathbf{v} = \left(\operatorname{diag}\left[z_i\right] \otimes I_n\right) \mathbf{v}.$$

Analogously, we define

$$\mathcal{Y}_{k_0,k_f} := \mathcal{Y}_{k_f-1} \mathcal{Y}_{k_f-2} \cdots \mathcal{Y}_{k_0}, \tag{20}$$

$$\mathcal{C}_{k_0,k_f} := \mathcal{C}_{k_f-1} \mathcal{C}_{k_f-2} \cdots \mathcal{C}_{k_0}, \tag{21}$$

such that

$$\|\mathbf{y}(k)\| \leq \|\mathcal{Y}_{k_0,k} \mathbf{y}(k_0)\|, \tag{22}$$

$$\mathcal{S}\left(\mathbf{W}(k)\right) := \mathcal{C}_{k_0,k} \mathcal{S}\left(\mathbf{W}(k_0)\right). \tag{23}$$

Since $\mathbf{x}(k)$ takes on only nonnegative values,

$$\begin{aligned} \|\mathbf{v}(k)\|_1 &= \sum_{j=1}^{N} \sum_{i=1}^{n} \mathbf{v}_{ij}(k) = \sum_{j=1}^{N} \sum_{i=1}^{n} \mathbb{E}\left[\mathbf{x}_i(k) \mathbf{1}_{Z_k=j}\right] \\ &= \sum_{j=1}^{N} \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{x}_i(k) \mathbf{1}_{Z_k=j}\right] = \sum_{i=1}^{n} \mathbf{x}_i(k) \\ &= \|\mathbf{x}(k)\|_1 \end{aligned} \tag{24}$$

where $\mathbf{v}_{ij}(k)$ denotes the i-th entry of $\mathbf{v}_j(k)$. A useful norm property also holds for $\mathbf{y}(k)$. Without loss of generality, suppose Z_k equals 1. Then

$$\|\mathbf{x}(k)\|^2 = y_1(k) = \sqrt{\Sigma_{i=1}^N y_i(k)^2} = \|\mathbf{y}(k)\|. \tag{25}$$

The following norm-related facts are also useful in deriving the main results in this paper.

Proposition 5: Given a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$,

$$\left\|\mathbf{x}\mathbf{x}^{\mathrm{T}}\right\| = \|\mathbf{x}\|^2$$

Proof: Since $\mathbf{x}\mathbf{x}^{\mathrm{T}}$ has rank one, exactly N-1 of its eigenvalues are zero. Now, noting that

$$\mathbf{x}\mathbf{x}^{\mathrm{T}} = \begin{bmatrix}\mathbf{x}_1\\ \vdots\\ \mathbf{x}_N\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \vdots\\ \mathbf{x}_N\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\mathbf{x}_1^2 & \cdots & \mathbf{x}_1\mathbf{x}_N\\ \vdots & \ddots & \vdots\\ \mathbf{x}_1\mathbf{x}_N & \cdots & \mathbf{x}_N^2\end{bmatrix},$$

if $\lambda_{>0}$ and $\lambda_i\left(\mathbf{x}\mathbf{x}^{\mathrm{T}}\right)$ denote the sole nonzero and the i-th eigenvalues of $\mathbf{x}\mathbf{x}^{\mathrm{T}}$, then we have

$$\lambda_{>0} = \sum_{i=1}^{N}\lambda_i\left(\mathbf{x}\mathbf{x}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{x}\mathbf{x}^{\mathrm{T}}\right) = \mathbf{x}_1^2 + \cdots + \mathbf{x}_N^2 = \|\mathbf{x}\|^2.$$

■

Proposition 6: Given network states $\mathbf{x}(k)$ and $\mathbf{W}(k)$, as in (3a) and (6), then for all nonnegative k

$$\left\|\mathbf{x}(k)\mathbf{x}(k)^{\mathrm{T}}\right\| = \|\mathbf{W}(k)\|.$$

Proof: Without loss of generality, suppose the network is in mode 1, i.e., $Z_k = 1$. It follows that

$$\begin{aligned}\|\mathbf{W}(k)\| &= \sup_{\|\mathbf{z}\|=1}\|\mathbf{W}(k)\mathbf{z}\|\\ &= \sup_{\|\mathbf{z}\|=1}\left\|\left[\mathbf{x}(k)\mathbf{x}(k)^{\mathrm{T}}\ \mathbf{0}\ \cdots\ \mathbf{0}\right]\mathbf{z}\right\|\\ &= \sup_{\|\mathbf{z}\|=1}\left\|\mathbf{x}(k)\mathbf{x}(k)^{\mathrm{T}}\mathbf{z}\right\| = \left\|\mathbf{x}(k)\mathbf{x}(k)^{\mathrm{T}}\right\|.\end{aligned}$$

■

Remark 1: Results in this section aim at representing some moment dynamics of (3) as *deterministic* linear systems, a common procedure in MJLS literature [17], [15], [21], [20]. Such description is amenable to several notions of stochastic stability, and sets the stage for the main contributions of this paper, presented in the next two sections.

III. Finite-time stability of switching networks

Definition 1: **Inclusion and exclusion probabilities.** [18] Consider the discrete-time stochastic system

$$\mathbf{x}(k+1) = f\left(\mathbf{x}(k), Z_k\right), \tag{26}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector, $\{Z_k\}_{k\in\mathbb{N}}$ is a stochastic process, and $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ is a vector function describing the dynamics of the system. The associated inclusion and exclusion probabilities with respect to $(\alpha, \beta, k_0, k_f, \|\cdot\|)$ are

$$P_{\mathrm{in}}\left(\mathbf{x}(k)|\,\alpha, \beta, k_0, k_f, \|\cdot\|\right) := P\left(\Omega_{\mathrm{in}}\right),$$
$$P_{\mathrm{ex}}\left(\mathbf{x}(k)|\,\alpha, \beta, k_0, k_f, \|\cdot\|\right) := P\left(\Omega_{\mathrm{ex}}\right),$$

where

$$\Omega_{\mathrm{in}} := \left\{\xi \in \Omega : \|\mathbf{x}(k)\| \le \beta, k_0 < k \le k_f, \|\mathbf{x}(k_0)\| \le \alpha\right\},$$
$$\Omega_{\mathrm{ex}} := \left\{\xi \in \Omega : \|\mathbf{x}(k)\| > \beta, k_0 < k \le k_f, \|\mathbf{x}(k_0)\| \le \alpha\right\}.$$

Definition 2: **Finite-time stochastic stability.** [18] The stochastic discrete-time system described by (26) is said to be *finite-time stochastic stable* (FTSS) with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$ if

$$P_{\mathrm{in}}\left(\mathbf{x}(k)|\,\alpha, \beta, k_0, k_f, \|\cdot\|\right) \ge 1 - \lambda.$$

Remark 2: Definition 1 implies $P\left(\Omega_{\mathrm{in}}\right) \ge 1 - \lambda$ and $P\left(\Omega_{\mathrm{ex}}\right) < \lambda$ are equivalent.

Theorem 1: Suppose $\alpha > 0$ and $\beta > 0$, and let $\lambda \in (0, 1)$, and k_0, k_f be nonnegative integers such that

$$\|\mathbf{x}(k_0)\|_1 \le \alpha, \text{ and } \sup_{k_0<k\le k_f}\|\mathcal{V}_{k_0,k}\|_1 < \frac{\beta}{\alpha}\lambda,$$

where $\mathcal{V}_{k_0,k}$ is given by (18). The switched system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|_1)$.

Proof: Let k_0 and k_f be nonnegative integers such that $k_0 < k_f$, and let $\alpha > 0$ be such that $\|\mathbf{x}(k_0)\|_1 \le \alpha$. Also, let $\beta > 0$ and $\lambda > 0$, with λ in $(0, 1)$. Markov's Inequality and (24) imply

$$\begin{aligned}P_{\mathrm{ex}} &= P\left(\left\{\xi \in \Omega \,\middle|\, \sup_{k_0<k\le k_f}\|\mathbf{x}(k)\|_1 > \beta, \|\mathbf{x}(k_0)\|_1 \le \alpha\right\}\right)\\ &\le \frac{1}{\beta}\mathbb{E}\left[\sup_{k_0<k\le k_f}\|\mathbf{x}(k)\|_1 \,\middle|\, \|\mathbf{x}(k_0)\|_1 \le \alpha\right]\\ &= \frac{1}{\beta}\sup_{k_0<k\le k_f}\mathbb{E}\left[\sum_{i=1}^{n}\mathbf{x}_i(k) \,\middle|\, \|\mathbf{x}(k_0)\|_1 \le \alpha\right]\\ &\le \frac{\alpha}{\beta}\sup_{k_0<k\le k_f}\|\mathcal{V}_{k_0,k}\|_1.\end{aligned}$$

Therefore, if

$$\sup_{k_0<k\le k_f}\|\mathcal{V}_{k_0,k}\|_1 < \frac{\beta}{\alpha}\lambda,$$

then $P_{\mathrm{ex}}\left(\mathbf{x}(k)\,|\alpha, \beta, k_0, k_f, \|\cdot\|_1\right) < \lambda$, and the system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|_1)$. ■

Theorem 2: Let $\alpha > 0$ and $\beta > 0$, $\lambda \in (0, 1)$, and assume k_0, k_f nonnegative integers such that

$$\|\mathbf{x}(k_0)\| \le \alpha, \text{ and } \sup_{k_0<k\le k_f}\|\mathcal{C}_{k_0,k}\| < \left(\frac{\beta}{\alpha}\right)^2\lambda,$$

with $\mathcal{C}_{k_0,k}$ given by (21). The switching system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$.

Proof: Suppose $\alpha > \left\|\mathbf{x}(k_0)\right\|$, and let $\beta > 0$, $\lambda \in (0, 1)$. First, note that because $\|\cdot\|$, α and β are nonnegative,

$$\|\mathbf{x}(k_0)\| \le \alpha, \|\mathbf{x}(k)\| < \beta \iff \|\mathbf{x}(k_0)\|^2 \le \alpha^2, \|\mathbf{x}(k)\|^2 < \beta^2.$$

To conclude the proof, we can use similar arguments to those involved in proving Theorem 1, plus Propositions 5 and 6 to replace $\|\mathbf{W}(k)\|$ for $\|\mathbf{x}(k)\|^2$. ■

Corollary 1: Suppose $\alpha \ge \|\mathbf{x}(k_0)\|$, and assume $\beta > 0$, $\lambda \in \lambda$ in $(0, 1)$. Also, assume $k_0 > 0$ and $k_f > 0$ such that

$$\sup_{k_0<k\le k_f}\prod_{j=k_0}^{k}\|\mathcal{C}_j\| < \left(\frac{\beta}{\alpha}\right)^2\lambda.$$

with $\mathcal{C}_{k_0,k}$ given by (21). The switching system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$.

Proof: Suppose $\alpha \ge \|\mathbf{x}\left(k_0\right)\|$, and consider k_0, k_f nonnegative integers such that $k_0 < k_f$. For $k > k_0$, Proposition 4 and (11) imply

$$\begin{aligned}\|\mathcal{W}_{k_0,k}\mathbf{W}(k_0)\| &= \|\mathcal{W}_{k-1}\mathcal{W}_{k-2}\cdots\mathcal{W}_{k_0}\mathbf{W}(k_0)\|\\ &\le \|\mathbf{W}(k_0)\|\prod_{j=k_0}^{k-1}\|\mathcal{C}_j\|.\end{aligned}$$

Hence, if $\sup_{k_0<k\le k_f}\prod_{j=k_0}^{k}\|\mathcal{C}_j\| < \beta^2\alpha^{-2}$, then

$$\sup_{k_0<k\le k_f}\|\mathcal{W}_{k_0,k}\| < \left(\frac{\beta}{\alpha}\right)^2\lambda.$$

Therefore, from Theorem 2, the switching system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$. ■

Theorem 3: Let α, β, and λ be positive real numbers, with

λ in $(0,1)$. Let k_0 and k_f be nonnegative integers such that

$$\|\mathbf{x}(k_0)\| \le \alpha, \text{ and } \sup_{k_0<k\le k_f} \|\mathcal{Y}_{k_0,k}\| < \left(\frac{\beta}{\alpha}\right)^2 \lambda,$$

with $\mathcal{Y}_{k_0,k}$ given by (20). Then, the switching system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$.

Proof: The statement can be proved following similar steps previously used: bounding $P(\Omega_{\text{ex}})$ via Markov's Inequality, exploring a norm identity—(25) in this case—and recursively applying the linearly-dominated dynamics of stochastic state, $\mathbf{y}(k)$, through Proposition 2. ■

Remark 3: Theorems 2 and 3 are not a restatement of Theorem 1 using the 2-norm. The different operators that naturally appear in each of the proofs stress that the results concern fundamentally different characteristics, namely, state volatility and aggregate state. Although norm equivalence could be used to write all statements using the same operator, not only the norm equivalence constants would introduce conservatism, but the statement itself would be artificial.

Remark 4: Theorems 1 to 3 show network topologies have a clear impact on system stability because, along with the Markov chain, they determine linear operators (10), (11), and (14). There is a much broader class of systems (including non-networked ones) for which our results are valid, but we explicitly adopted this general setting because the real-world networks we are interested in do not adhere to special topologies, such as ring or star, as empirical evidence shows [17]. Instead, focus is on networks previously ignored by the literature, i.e., networks with time-varying and unstable topologies.

IV. Relationship between FTSS and asymptotic stability

In this section, we explore the relationship between finite-time stability and a classical notion of asymptotic stability: mean stability.

Definition 3: **Mean Stability.** The networked system described by (3) is said to be *mean stable* (MS) if the expected value of its state converges in norm to zero, that is,

$$\lim_{k\to\infty} \mathbb{E}\left[\|\mathbf{x}(k)\|\right] = 0, \tag{27}$$

for any initial conditions $\mathbf{x}(0)$ and Z_0.

Definition 4: **Exponential Mean Stability.** The networked system described by (3) is said to be *exponentially mean stable* (EMS) if there exist positive real numbers c and r, where r belongs to $(0,1)$, such that

$$\mathbb{E}\left[\|\mathbf{x}(k)\|\right] < cr^k \|\mathbf{x}(0)\|, \tag{28}$$

for any initial conditions $\mathbf{x}(0)$ and Z_0.

An immediate consequence of the above definitions is that EMS implies MS (but not vice-versa), since in addition to convergence, the former also guarantees an exponential decay rate.

Theorem 4: Let k_0 and k_f be nonnegative integers, and assume $\alpha \ge \|\mathbf{x}(k_0)\|$. If MJLS (3) is mean stable, then there exist β and λ such that (3) is also FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$.

Proof: Suppose the networked system described by (3) is MS, i.e.,

$$\lim_{k\to\infty} \mathbb{E}\left[\|\mathbf{x}(k)\|\right] = 0. \tag{29}$$

Then, given $\epsilon > 0$, there exists an integer k_ϵ such that

$$\mathbb{E}\left[\|\mathbf{x}(k)\|\right] < \epsilon, \forall k > k_\epsilon.$$

Indeed, suppose $E\left[\|\mathbf{x}(k)\|\right]$ is unbounded. Then, for every $L > 0$, there exists some integer k_L such that $\mathbb{E}\left[\|\mathbf{x}(k_L)\|\right] > L$. Put

$$\epsilon_{\max} = \max_{0\le k\le k_\epsilon} \left\{\mathbb{E}\left[\|\mathbf{x}(k)\|\right]\right\}, \quad L = \max\left\{\epsilon, \epsilon_{\max}\right\} + 1,$$

with an associated k_L. Since L is greater than $\epsilon_{\max}$, k_L must be greater than k_ϵ. Hence, there exists some $k_L > k_\epsilon$ such that $\mathbb{E}\left[\|\mathbf{x}(k_L)\|\right] > \epsilon$, which contradicts (29). Now, suppose $\|\mathbf{x}(0)\| \le \alpha$, and since $E\left[\|\mathbf{x}(k)\|\right]$ is bounded, let $\gamma(\alpha, k_0, k_f, \|\cdot\|)$ be such that

$$\sup_{k_0<k\le k_f} \mathbb{E}\left[\|\mathbf{x}(k)\|\right] < \gamma(\alpha, k_0, k_f, \|\cdot\|).$$

Markov's Inequality yields

$$P(\Omega_{ex}) < \frac{\gamma(\alpha, k_0, k_f, \|\cdot\|)}{\beta}.$$

Therefore, by picking β, λ such that $\beta\lambda > \gamma(\alpha, k_0, k_f, \|\cdot\|)$, (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$. ■

Theorem 5: Let $\alpha > 0$, $\beta > 0$ and assume $\lambda \in (0,1)$, such that $\alpha < \beta\lambda$. If switching system (3) is MS, then there exists k_0 for which $\|\mathbf{x}(k_0)\| \le \alpha$, and such that for any k_f greater than k_0, the networked system (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$.

Proof: Assume (3) is MS. Then, there is k_0 such that $\mathbb{E}\left[\|\mathbf{x}(k)\|\right] \le \alpha$ for every $k \ge k_0$, and Markov's Inequality implies

$$P(\Omega_{ex}) \le \frac{1}{\beta} \sup_{k_0<k<\infty} \mathbb{E}\left[\|\mathbf{x}(k)\|\right] < \beta^{-1}\alpha.$$

Assuming α, β and λ are such that $\alpha < \beta\lambda$, by definition (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$. ■

A. Periodic Markovian transitions

Finite-time stochastic stability and mean stability are independent notions, so in general one does not imply the other. Nevertheless, sufficient conditions to establish both FTSS and MS can be obtained by restricting the choice of time-varying Markov transition matrices, $\mathcal{M}_k$. In particular, consider periodic Markovian transition matrices (with period K):

$$\mathcal{M}_{k+K} = \mathcal{M}_k, \forall k \in \mathbb{N}. \tag{30}$$

Lemma 1: Consider positive real numbers α, β, λ, and nonnegative integers k_0, k_f such that hypotheses from Theorem 3 are satisfied. In addition, assume $\mathcal{M}_k$ periodic with period K, $k_f \ge k_0+K$, and $\left(\beta\alpha^{-1}\right)^2 \lambda < 1$. Then, switching system (3) is both FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$ and EMS.

Proof: Let α, β, λ be positive real numbers, and let k_0, $k_f \ge k_0 + K$ be nonnegative integers such that

$$\|\mathbf{x}(k_0)\| \le \alpha, \sup_{k_0<k\le k_f} \|\mathcal{Y}_{k_0,k}\| < \left(\beta\alpha^{-1}\right)^2 \lambda < 1.$$

Under these assumptions, Theorem 3 guarantees (3) is FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$. It remains to prove it's also EMS.

Consider a nonnegative integer $k > k_0$. From (22), it follows that

$$\begin{aligned}\|\mathbf{y}(k)\| &\leq \|\mathcal{Y}_{0,k}\| \, \|\mathbf{y}(0)\| \\ &\leq \|\mathcal{Y}_{0,k_0}\| \, \|\mathcal{Y}_{k_0,k_0+K}\| \cdots \|\mathcal{Y}_{k_0+nK,k}\| \, \|\mathbf{y}(0)\| \\ &< \left[\left(\beta\alpha^{-1}\sqrt{\lambda}\right)^{2\frac{n+1}{k}}\right]^k \|\mathcal{Y}_{0,k_0}\| \, \|\mathbf{y}(0)\| \,.\end{aligned}$$

Because $\|\mathcal{Y}_{0,k_0}\|$ is constant and, by hypothesis, $\left(\beta\alpha^{-1}\right)^2\lambda < 1$, the (3) is both FTSS with respect to $(\alpha, \beta, k_0, k_f, \lambda, \|\cdot\|)$ and EMS. ■

The assumption the Markov chains are periodic enables us to reconcile the notions of FTSS and EMS, but satisfying both conditions requires an excessively conservative criterion that compromise the advantages of considering each notion separately. From the FTSS point of view, limiting the "amplification-probability" product $(\beta\alpha^{-1})^2\lambda$ to unity imposes a severe constraint on the spectral radius of operators $\mathcal{Y}_{k_0,k}$. On the other hand, considering the supremum over all time window $[k_0, k_f]$ is unnecessarily restrictive for EMS, since $\|\mathcal{Y}_{k_0,k_0+K}\|$ being unity-bounded is sufficient to prove exponential decay.

V. NUMERICAL EXAMPLES

Theorems 1 to 3 look fairly similar, making it difficult to compare, offhand, their performance and computational properties. The following numerical examples highlight the differences between the criteria, and show how they provide insight on how airport flight delays propagate.

A. *Stable/unstable mode networks*

Consider a simple network consisting of two nodes whose topology alternates between two randomly generated modes:

$$\Gamma_1 = \begin{bmatrix} 0.55 & 0.24 \\ 0.28 & 1.02 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.37 & 0.05 \\ 0.33 & 0.42 \end{bmatrix},$$

the first of which is unstable ($\|\Gamma_1\| = 1.13$), whereas the second is stable ($\|\Gamma_2\| = 0.61$). To assess finite-time stability of this system, we compute the evolution of $\|\mathcal{V}_{0,k}\|_1$ and $\|\mathcal{Y}_{0,k}\|$ for k ranging from 1 to 24 when the Markov transitions are time-varying, taking different randomly generated values at each time-step.

As depicted in Figure 1, switching between the two modes prevents one operator norm from bounding the other. The switches also preclude operator norm monotonicity: the stable mode eventually dominates the dynamics, driving $\|\mathcal{V}_{0,k}\|_1$ and $\|\mathcal{Y}_{0,k}\|$ to zero, but each peaks at different time steps. Figure 1 also shows that as the stable mode prevails, $\|\mathcal{Y}_{0,k}\|$ consistently produces smaller values than $\|\mathcal{V}_{0,k}\|_1$ does. To see if there is numerical evidence that supports this trend, consider another randomly generated network

$$\Gamma_1 = \begin{bmatrix} 0.19 & 1.05 \\ 0.27 & 0.73 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.32 & 0.11 \\ 0.03 & 0.39 \end{bmatrix},$$

again consisting in an unstable mode ($\|\Gamma_1\| = 1.32$), and a stable one ($\|\Gamma_2\| = 0.44$).

Figure 2 shows $\|\mathcal{V}_{0,k}\|_1$ has a lower initial value, but then exceeds the maximum value of $\|\mathcal{Y}_{0,k}\|$, which then upper bounds $\|\mathcal{V}_{0,k}\|_1$ for the rest of the series. Therefore,

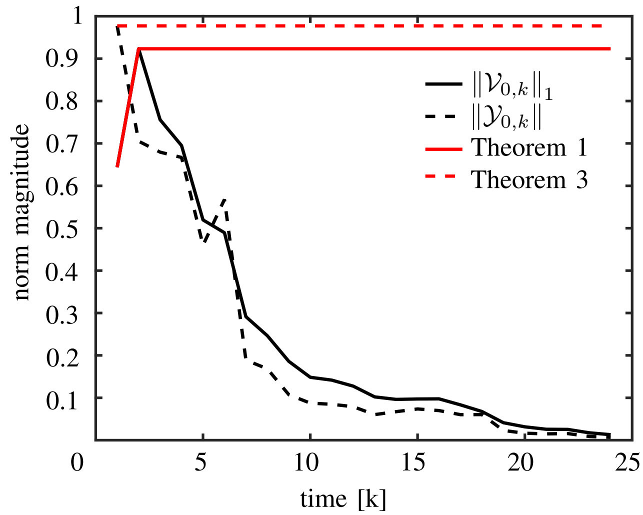

Fig. 1. Evolution of $\|\mathcal{V}_{0,k}\|_1$ and $\|\mathcal{Y}_{0,k}\|$.

under different conditions Theorem 1 may be more or less conservative than Theorem 3, because amplification ratios (or the amplification-probability product) are different for each theorem: $\beta\alpha^{-1}$ and $\left(\beta\alpha^{-1}\right)^2$, respectively. In the first example, Theorem 1 is less conservative than Theorem 3; the amplification ratio must be greater than 0.923 for the former, as opposed to 0.998 for the latter. On the other hand, Theorem 1 is more conservative in the second case; a minimum amplification ratio of 1.246 is required, compared with 1.110 for Theorem 3. Similar remarks apply replacing Theorem 3 with Theorem 2, for which reason we omit numerical examples.

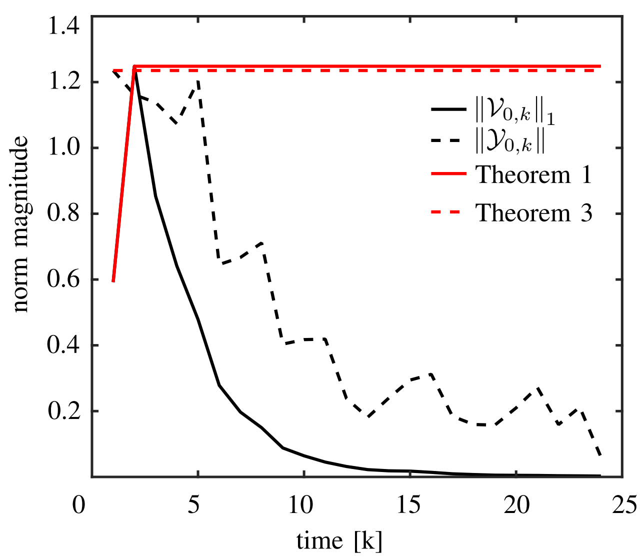

Fig. 2. Evolution of $\|\mathcal{V}_{0,k}\|_1$ and $\|\mathcal{Y}_{0,k}\|$.

B. *Large-scale, air traffic delay network example*

To compare computational properties and performance of conditions from theorems 1-3, we consider a networked system that models the propagation of air traffic delays at the thirty largest airports in the US. The associated topologies, modes and Markov transition matrices were identified in [17], [8] using data provided by the Bureau of Transportation Statistics from the years 2011-2012. We ignore resets in continuous state that can occur during discrete mode transitions [17].

Several factors impact the operations of these airports, producing various dynamics throughout the day [17], described by six delay modes. Each mode is then divided according to whether delays are decreasing or increasing, resulting in twelve discrete modes, each representing a network topology. In short, the air traffic delay network can be modeled by an MJLS, and the model parameters learned from operational data is [17]:

$$\mathbf{x}\left(k+1\right) = \Gamma_{Z_k}\mathbf{x}\left(k\right) \tag{31}$$

$$P\left(\left. Z_{k+1} = j\right| Z_k = i\right) = \pi_{ij}\left(k\right), \tag{32}$$

where $\mathbf{x}(k)$ is a vector in $\mathbb{R}^{60}$, representing inbound and outbound delays, Γ_{Z_k} is a matrix in $\mathbb{R}^{60\times 60}$ that denotes the network topology, and Z_k is the random variable taking values on $\{1, \cdots, 12\}$ that determines the current network topology.

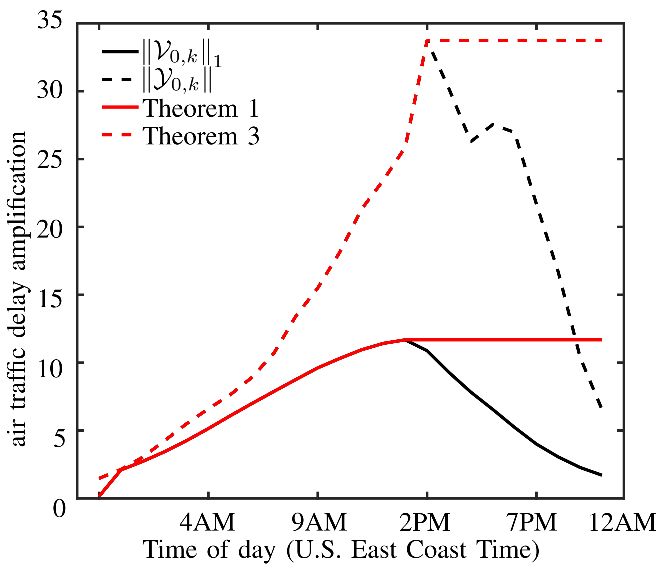

Fig. 3. Comparison between FTSS conditions from Theorems 1 and 3.

Air traffic delays typically increase in the morning hours, when weather and congestion impacts can result in the network switching to unstable modes, build up over the course of the day, reaching peak values late in the evening, until the volume of flights decreases, and airports start absorbing accumulated delays.

This behavior is illustrated by Figure 3, which shows the numerical values of the conditions from Theorems 1 and 3, considering model (31)-(32). Both $\sup_{k_0<k\le k_f}\|\mathcal{V}_{k_0,k}\|_1$ and $\sup_{k_0<k\le k_f}\|\mathcal{Y}_{k_0,k}\|$ rise until 6PM and 7PM Eastern Time, respectively, when the values start decreasing. The peak value obtained via Theorem 1 is approximately 11.7, whereas Theorem 3 saturates at 33.7.

Despite the significantly smaller (by a factor of three) upper bound obtained using Theorem 1, in reality, Theorem 3 is less conservative. Since both upper bounds depend linearly on λ, the quadratic amplification factor makes is favorable to the stability criteria determined by Theorem 3. For example, for $\lambda = 0.5$, $\beta\alpha^{-1}$ must be greater than 23.4 for system (31)-(32) to be FTSS according to Theorem 1, whereas $\beta\alpha^{-1}$ greater than 8.22 is sufficient to guarantee FTSS using Theorem 3.

Results concerning Theorem 2 were again omitted, but for a different reason. The derivation of Theorem 2 benefits from the linear dynamics of state $\mathbf{W}(k)$, in contrast with $\mathbf{y}(k)$, which is only upper bounded by linear dynamics. Thus, Theorem 2 is less conservative. The associated computational effort, however, is substantially higher. In the last example, calculations with Theorem 3 manipulate 12-by-12 matrices and require determining the spectral radius of twelve 60-by-60 matrices, which needs to be done only once, whereas Theorem 2 requires dealing with 43200-by-43200 matrices. Thus, even for moderate-sized networks, conditions from Theorem 2 become untractable.

VI. Final Remarks

The notion of finite-time stochastic stability enables the amplification of an initial state in a limited amount of time to be seamlessly combined with uncertainty in dynamics, when analyzing Markov Jump Linear Systems. Motivated by this observation, we presented sufficient FTSS conditions that are well-suited for assessing the volatility of networked systems that randomly switch between different topologies. By relaxing some of the assumptions made in prior literature (e.g., considering stable networks, unweighted/undirected networks, and fixed topologies), we generalized the class of networks that can be considered. We determined an approach to estimate how much initial conditions can be amplified, and also the probability with which this can occur. Connections between finite-time and asymptotic stability were derived, and we presented conditions under which mean stability or exponential mean stability implies finite-time stochastic stability, and criteria that simultaneously guarantee both forms of stability.

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