Observability Criteria and Estimator Design for Stochastic Linear Hybrid Systems

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Abstract

A stochastic linear hybrid system is said to be observable if the hybrid state of the system is uniquely determined from the output. In this paper, we derive the conditions for the observability of stochastic linear hybrid systems by exploiting the information obtained from system noise characteristics. Having established the necessary criteria for observability, we study the effect of these conditions on estimator design, and also find bounds on the switching times of the system to achieve guaranteed estimator performance. We then apply these results to the estimation of a two-mode aircraft trajectory.

1 Introduction

The tracking of aircraft trajectories is a problem that has been approached with some success using hybrid systems models [1]. Related problems of interest to us are the ability to estimate the hybrid states of such systems from their outputs, and also the design of estimators for such systems.

The problem of observability, namely, the ability to estimate or reconstruct the actual state of a system given its output, is well-known and has been studied extensively, both for continuous systems [2] as well as for discrete ones [3, 4]. More recently, several researchers have approached the problem of observability of hybrid systems. A practical problem that has received increasing research attention recently is the extension of these concepts to stochastic hybrid systems. In this paper, we address the issue of observability of a class of stochastic hybrid systems – systems where the continuous dynamics are affected by white Gaussian noise.

Alessandri and Coletta [5] proposed a Luenberger observer design methodology for deterministic linear hybrid systems, and proved that the error converges if the discrete state evolution is known. Balluchi et al. [6] developed a method of combining location observers for discrete state estimation with Luenberger observers for continuous state estimation for linear systems, such that they can guarantee the exponential convergence of the estimation error. Bemporad et al. [7] defined the concept of incremental observability of continuous-time linear hybrid systems, using the solutions of a mixed-integer linear program. Recently, Vidal et al. [8] derived observability conditions for linear hybrid systems with continuous-time continuous-state dynamics, given in the form of rank conditions similar to those for continuous-time linear system observability.

For stochastic systems, the definition of observability in its classical form, as proposed by Kalman for systems with no noise, fails; we therefore need to find a meaningful interpretation of observability for systems with random noise. Baram and Kailath [9] proposed the concept of *estimability* as a better criterion to gauge stochastic linear systems. While this is one way of approaching the problem, we try to extend the definition of observability to include stochastic hybrid systems.

An important class of problems associated with applications in multi-target tracking [1] and speech recognition [10] pertains to the estimation of discrete-time Markov jump linear systems. Cost and do Val [11] analyzed such systems with finite Markov states and deterministic continuous dynamics, and derived the observability condition that the solution to the coupled Riccati equation associated with the quadratic control problem has a stabilizing solution. Vidal et al. [12] derived observability conditions for jump linear systems based on rank tests similar to those of deterministic linear hybrid systems.

The first part of this study is motivated by the results of Vidal et al. [12]. They proposed the notion of *indistinguishability* as: two initial states are indistinguishable if the corresponding outputs in free evolution are equal. This approach results in elegant rank tests for the observability of stochastic jump linear systems. Since in the design of estimators for aircraft tracking we have knowledge of not just the system dynamics, but also the noise covariances, we try to exploit this additional knowledge to improve our ability to differentiate between state trajectories. Since the output sequences of stochastic systems might be different from the same initial condition, we extend the notion of indistinguishability [12] for such systems, and based on our definition, we derive conditions for the observability of discrete-time stochastic linear hybrid systems. The latter part of this paper applies the approach of Balluchi et al. [6], so far used in the design of hybrid observers for deterministic hybrid systems with continuous-time state evolution, to discrete-time stochastic hybrid systems and estimator design.

This paper is organized as follows: Section 2 presents the observability conditions of discrete-time stochastic jump linear systems. In Section 3, we obtain conditions on the system parameters that would guarantee the exponential convergence of hybrid estimators. Examples and conclusions are presented in Sections 4 and 5 respectively.

2 Observability of discrete-time stochastic linear hybrid systems

In this section, inspired by Vidal et al. [12], we extend the concepts of indistinguishability, observability of the hybrid initial state, and discrete transition times as defined in [12] and derive more general observability conditions for discrete-time stochastic linear hybrid systems using the knowledge of noise covariances.

We consider a discrete-time stochastic linear hybrid system

$$H : \left\{ \begin{array}{rcl} x_{k+1} & = & A(q_k)x_k + w_k(q_k) \\ y_k & = & C(q_k)x_k + v_k(q_k) \\ q_{k+1} & = & \delta(q_k, \gamma_k) \end{array} \right. , \ k \in \{0, 1, \cdots\} \tag{1}$$

where k is a non-negative integer ($k \in \mathbb{N}$); $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^p$ are the continuous state and output variables respectively; $q_k \in \{1, 2, \cdots, N\}$ is the discrete state, $\gamma_k \in \{\gamma^1, \cdots, \gamma^m\}$ is a discrete control input, and $\delta(\cdot, \cdot)$ is a deterministic discrete transition relation which governs the discrete state evolution. We assume the event time at which a discrete transition occurs is unknown. The system parameters $A(q_k) \in \mathbb{R}^{n \times n}$ and $C(q_k) \in \mathbb{R}^{p \times n}$ for $q_k \in \{1, 2, \cdots, N\}$ are real matrices. We assume that the initial state x_{k_0} is an unknown, zero-mean white Gaussian random variable with covariance $E[x_{k_0} x_{k_0}^T] = \Pi_0$ and that the process noise $w_k(q_k)$ and the measurement noise $v_k(q_k)$ are uncorrelated, zero-mean white Gaussian sequences with the covariance matrices $E[w_k(q_k)w_k(q_k)^T] = \rho(q_k)I$ and $E[v_k(q_k)v_k(q_k)^T] = \sigma(q_k)I$ respectively. These random sequences are assumed to be uncorrelated with the initial state, i.e., $E[x_{k_0} w_k(q_k)^T] = E[x_{k_0} v_k(q_k)^T] = 0$. I denotes the identity matrix. Since the state evolution of a hybrid system has continuous trajectories as well as discrete jumps, we define a hybrid time trajectory:

Definition 1 *(Hybrid time trajectory) A hybrid time trajectory is a sequence of intervals*

$$[k_0, k_1 - 1][k_1, k_2 - 1] \cdots [k_i, k_{i+1} - 1] \cdots$$

where $k_i (i \geq 1)$ is the time at which i-th discrete state transition occurs.

Before deriving the observability conditions, we review the definition of observability for discrete-time stochastic linear hybrid systems [12]:

Definition 2 *(Observability of discrete-time stochastic linear hybrid systems) A discrete-time linear hybrid system H is observable on $[k_0, k_0 + K]$ if the hybrid state (q_k, x_k) for $k \in [k_0, k_0 + K]$ is uniquely determined from the output sequence $\mathcal{Y}_K = [y_{k_0}^T \cdots y_{k_0+K}^T]^T$, where $K \in \mathbb{N}$.*

Vidal et al. [12] developed rank tests for the observability of stochastic jump linear systems of the form described by H (Eq.(1)) using the notion of indistinguishability. Since we know the noise covariances as well as the system dynamics for a stochastic system, we use this additional knowledge to obtain a more general condition. Since the output sequences of stochastic systems could be different from the same initial condition, we extend the notion of indistinguishability [12] as follows:

Definition 3 *(Indistinguishability of discrete-time stochastic linear hybrid systems) A discrete-time linear hybrid system H is indistinguishable on $[k_0, k_0+K]$ if there exist output sequences $\mathcal{Y}_K$ and $\mathcal{Y}'_K$ on $k \in [k_0, k_0+K]$ starting from any two different hybrid states (q_{k_0}, x_{k_0}) and (q'_{k_0}, x'_{k_0}), whose covariances are equal.*

2.1 Observability of the hybrid initial state

In this section, using a procedure similar to that in [12], we derive the conditions under which the hybrid initial state (q_{k_0}, x_{k_0}) can be uniquely determined from the output sequence $\{y_k\}$ on $[k_0, k_1-1]$ $(k_1 - 1 \le k_0 + K)$, i.e., before the first discrete transition occurs. We define $\kappa_i := k_{i+1} - k_i$ $(i \ge 0)$ as the sojourn time, which denotes how long the system stays in a discrete state after the i-th discrete transition. Based on **Definition** 2 and **Definition** 3, we get the following lemma:

Lemma 1 *The hybrid initial state of a discrete-time linear hybrid system H is observable if and only if it is distinguishable.*

Proof: The proof follows directly from **Definition** 2 and **Definition** 3. □

In order to check if the hybrid initial state is indistinguishable, we have to compute the covariance of output sequence $\mathcal{Y}_{\kappa_0}$ on $[k_0, k_1-1]$. The output sequence starting from the hybrid initial state (q_{k_0}, x_{k_0}) on $[k_0, k_1-1]$ is

$$\mathcal{Y}_{\kappa_0}(q_{k_0}) = \mathcal{O}_{\kappa_0}(q_{k_0})x_{k_0} + \mathcal{T}_{\kappa_0}(q_{k_0})W_{\kappa_0}(q_{k_0}) + V_{\kappa_0}(q_{k_0}) \tag{2}$$

where

$$\begin{aligned}
\mathcal{O}_{\kappa_0}(q_{k_0}) &= [C(q_{k_0})^T (C(q_{k_0})A(q_{k_0}))^T \cdots ((C(q_{k_0})A(q_{k_0}))^{k_1-1})^T]^T \\
\mathcal{T}_{\kappa_0}(q_{k_0}) &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C(q_{k_0}) & 0 & 0 & \cdots & 0 \\ C(q_{k_0})A(q_{k_0}) & C(q_{k_0}) & 0 & \cdots & 0 \\ \vdots & & & & \\ C(q_{k_0})A(q_{k_0})^{k_1-k_0-2} & C(q_{k_0})A(q_{k_0})^{k_1-k_0-3} & \cdots & C(q_{k_0}) & 0 \end{bmatrix} \\
W_{\kappa_0}(q_{k_0}) &= [w_{k_0}(q_{k_0})^T w_{k_0+1}(q_{k_0})^T \cdots w_{k_1-1}(q_{k_0})^T]^T \\
V_{\kappa_0}(q_{k_0}) &= [v_{k_0}(q_{k_0})^T v_{k_0+1}(q_{k_0})^T \cdots v_{k_1-1}(q_{k_0})^T]^T
\end{aligned}$$

$\mathcal{O}_{\kappa_i}(q_{k_i}) \in \mathbb{R}^{p\kappa_i \times n}$ is the extended observability matrix for the linear system in Eq.(1) [12] and $\mathcal{T}_{\kappa_0}(q_{k_0})$ is a Toeplitz matrix.

If $rank[\mathcal{O}_{\kappa_0}(q_{k_0})] = n$, i.e., the linear system $(A(q_{k_0}), C(q_{k_0}))$ is observable and $\kappa_0 \ge n$, then a least-squares solution (which we denote by $\hat{x}_{k_0}(q_{k_0})$) to Eq.(2) can be determined uniquely.

$$\hat{x}_{k_0}(q_{k_0}) = \mathcal{O}^{\dagger}_{\kappa_0}(q_{k_0})\mathcal{Y}_{\kappa_0}(q_{k_0}) = x_{k_0} + \mathcal{O}^{\dagger}_{\kappa_0}(q_{k_0})\mathcal{T}_{\kappa_0}(q_{k_0})W_{\kappa_0}(q_{k_0}) + \mathcal{O}^{\dagger}_{\kappa_0}(q_{k_0})V_{\kappa_0}(q_{k_0}) \tag{3}$$

where $\mathcal{O}^{\dagger}_{\kappa_0}(q_{k_0}) = (\mathcal{O}^T_{\kappa_0}(q_{k_0})\mathcal{O}_{\kappa_0}(q_{k_0}))^{-1}\mathcal{O}^T_{\kappa_0}(q_{k_0})$. The last two terms on the right hand side of Eq.(3) represent the estimation error due to the process noise and the measurement noise. Similarly, the output sequence from another hybrid initial state (q'_{k_0}, x'_{k_0}) over $[k_0, k_1-1]$ is

$$\mathcal{Y}_{\kappa_0}(q'_{k_0}) = \mathcal{O}_{\kappa_0}(q'_{k_0})x'_{k_0} + \mathcal{T}_{\kappa_0}(q'_{k_0})W_{\kappa_0}(q'_{k_0}) + V_{\kappa_0}(q'_{k_0}) \tag{4}$$

From **Lemma** 1, in order that the hybrid initial state of a discrete-time stochastic linear hybrid system be observable, it should be distinguishable, i.e., the covariances of $\mathcal{Y}_{\kappa_0}(q_{k_0})$ and $\mathcal{Y}_{\kappa_0}(q'_{k_0})$ satisfy:

$$E[\mathcal{Y}_{\kappa_0}(q_{k_0})\mathcal{Y}_{\kappa_0}(q_{k_0})^T] \ne E[\mathcal{Y}_{\kappa_0}(q'_{k_0})\mathcal{Y}_{\kappa_0}(q'_{k_0})^T] \tag{5}$$

where
$$\begin{array}{rcl} E[\mathcal{Y}_{\kappa_0}(q_{k_0})\mathcal{Y}_{\kappa_0}(q_{k_0})^T] & = & \mathcal{O}_{\kappa_0}(q_{k_0})\Pi_0\mathcal{O}_{\kappa_0}(q_{k_0})^T + \rho(q_{k_0})\mathcal{T}_{\kappa_0}(q_{k_0})\mathcal{T}_{\kappa_0}^T(q_{k_0}) + \sigma(q_{k_0})I \\ E[\mathcal{Y}_{\kappa_0}(q'_{k_0})\mathcal{Y}_{\kappa_0}(q'_{k_0})^T] & = & \mathcal{O}_{\kappa_0}(q'_{k_0})\Pi_0\mathcal{O}_{\kappa_0}(q'_{k_0})^T + \rho(q'_{k_0})\mathcal{T}_{\kappa_0}(q'_{k_0})\mathcal{T}_{\kappa_0}^T(q'_{k_0}) + \sigma(q'_{k_0})I \end{array} \tag{6}$$

Then, the discrete initial state can be uniquely determined from the covariance of the output sequence and the continuous initial state can also be uniquely determined using Eq.(3). In order to reduce the required κ_0 for observability (the sojourn time in the discrete state q_{k_0} required for observability of the hybrid initial state), we define τ as the minimum integer which satisfies $rank[\mathcal{O}_\tau(q_k)] = n(\forall q_k \in \{1, 2, \cdots, N\})$, and $\bar{\tau} = \max \tau$ (similar to the joint observability index used in [12]). Then, we have the following condition for the observability of the hybrid initial state:

Lemma 2 *(Observability of the hybrid initial state) If $(A(q_k), C(q_k))$ are observable for each $q_k \in \{1, \cdots, N\}$ and $\kappa_0 \geq \bar{\tau}$, the hybrid initial state (q_{k_0}, x_{k_0}) is observable if and only if*

$$\begin{array}{rl} & \mathcal{O}_{\bar{\tau}}(q_{k_0})\Pi_0\mathcal{O}_{\bar{\tau}}(q_{k_0})^T + \rho(q_{k_0})\mathcal{T}_{\bar{\tau}}(q_{k_0})\mathcal{T}_{\bar{\tau}}^T(q_{k_0}) + \sigma(q_{k_0})I \\ \neq & \mathcal{O}_{\bar{\tau}}(q'_{k_0})\Pi_0\mathcal{O}_{\bar{\tau}}(q'_{k_0})^T + \rho(q'_{k_0})\mathcal{T}_{\bar{\tau}}(q'_{k_0})\mathcal{T}_{\bar{\tau}}^T(q'_{k_0}) + \sigma(q'_{k_0})I \end{array}$$

for all $q_{k_0} \neq q'_{k_0} \in \{1, \cdots, N\}$.

Proof: Since the linear system in each discrete state is observable and $\kappa_0 \geq \bar{\tau}$, the initial continuous state can be uniquely determined using Eq.(3) if the initial discrete state is identified.
(if) Since the covariances of the output sequences for each discrete state are distinct, the initial discrete state is uniquely determined by checking the covariance of the output sequence.
(only if) The proof follows directly from **Definition** 3. □

We show through the following simple example how a noise free unobservable discrete-time linear hybrid system may be rendered observable, if each discrete state has different measurement noise covariances.

Example: Consider a discrete-time linear hybrid system with two discrete states

$$q_1 : \left\{ \begin{array}{rcl} x_{k+1} & = & x_k \\ y_k & = & c_1 x_k + v_1 \end{array} \right. \qquad q_2 : \left\{ \begin{array}{rcl} x_{k+1} & = & x_k \\ y_k & = & c_2 x_k + v_2 \end{array} \right.$$

where $c_1 \neq 0$, $c_2 \neq 0$, and $c_1 \neq c_2$. The covariance of the initial state is $E[x_0 x_0^T] = \pi_0 \in \mathbb{R}^+$. v_1 and v_2 are uncorrelated, zero-mean white Gaussian sequences with covariances $E[v_1 v_1^T] = \sigma_1 \neq 0$ and $E[v_2 v_2^T] = \sigma_2 \neq 0$ respectively. If $v_1 = v_2 = 0$, the hybrid system is unobservable because two different hybrid initial states (q_1, x_0) and $(q_2, \frac{c_1}{c_2} x_0)$ generate the same output sequences [8]. However, if v_1 and v_2 are not identically zero and have different covariances, then we can uniquely determine the hybrid initial state. If we consider the case in which the actual hybrid initial state is (q_1, x_0), the output and its covariance are

$$y \;=\; c_1 x_0 + v_1, \qquad E[yy^T] \;=\; \pi_0 c_1 c_1^T + \sigma_1 \tag{7}$$

Next, if the actual hybrid initial state is $(q_2, \frac{c_1}{c_2} x_0)$, the output and its covariance are

$$y \;=\; c_2(\tfrac{c_1}{c_2} x_0) + v_2, \qquad E[yy^T] \;=\; \pi_0 c_1 c_1^T + \sigma_2 \tag{8}$$

Since $\sigma_1 \neq \sigma_2$, we can determine the discrete initial state uniquely. For instance, if the output comes from q_1, then the estimate of the initial state is $\hat{x}_0 = x_0 + \frac{v_1}{c_1}$.

2.2 Observability of the discrete transition times

Lemma 2 gives the condition for the hybrid initial state to be observable, over a time interval up to, but not including the first transition. In this section, we focus without loss of generality on deriving the conditions under which the first discrete transition time k_1 can be uniquely determined from the output sequence $\mathcal{Y}_K$ on $[k_0, k_0 + K]$; the times of the ensuing transitions $k_i (i \in \{2, \dots\})$ can be computed in the same way [12]. We define observability of the first discrete transition time as follows:

Definition 4 *(Observability of the first discrete transition time) The first discrete transition time of a discrete-time linear hybrid system H is observable on $[k_0, k_0 + K]$ if it can be determined uniquely from the output sequence $\mathcal{Y}_K = [y_{k_0}^T \cdots y_{k_0+K}^T]^T$.*

If there is a discrete transition at time k_1, the output at time k_1 and its covariance are

$$\begin{array}{rcl} y_{k_1} & = & C(q_{k_1})A(q_{k_0})^{k_1-k_0}x_{k_0} + C(q_{k_1})\mathcal{F}_{\kappa_0}(q_{k_0})W_{\kappa_0}(q_{k_0}) + v_{k_1}(q_{k_1}) \\ E[y_{k_1}y_{k_1}^T] & = & C(q_{k_1})A(q_{k_0})^{k_1-k_0}\Pi_0(A(q_{k_0})^{k_1-k_0})^T C(q_{k_1})^T \\ & & +\rho(q_{k_0})C(q_{k_1})\mathcal{F}_{\kappa_0}(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})^T C(q_{k_1})^T + \sigma(q_{k_1})I \end{array} \quad (9)$$

where $\mathcal{F}_{\kappa_0}(q_{k_0}) := [A(q_{k_0})^{k_1-k_0-1}\ A(q_{k_0})^{k_1-k_0-2}\ \cdots\ I]$. If there is no state transition at time k_1, the output at time k_1 and its covariance are

$$\begin{array}{rcl} y_{k_1} & = & C(q_{k_0})A(q_{k_0})^{k_1-k_0}x_{k_0} + C(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})W_{\kappa_0}(q_{k_0}) + v_{k_1}(q_{k_0}) \\ E[y_{k_1}y_{k_1}^T] & = & C(q_{k_0})A(q_{k_0})^{k_1-k_0}\Pi_0(A(q_{k_0})^{k_1-k_0})^T C(q_{k_0})^T \\ & & +\rho(q_{k_0})C(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})^T C(q_{k_0})^T + \sigma(q_{k_0})I \end{array} \quad (10)$$

In order that the transition at time k_1 be observable, the covariances of y_{k_1}'s in Eq.(9) and Eq.(10) should be different. Thus, the observability condition of the first discrete transition time is:

Lemma 3 *(Observability of the first discrete transition time) The first discrete transition time is observable if and only if*

$$\begin{array}{rl} & C(q_{k_1})A(q_{k_0})^{k_1-k_0}\Pi_0(A(q_{k_0})^{k_1-k_0})^T C(q_{k_1})^T \\ & +\rho(q_{k_0})C(q_{k_1})\mathcal{F}_{\kappa_0}(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})^T C(q_{k_1})^T + \sigma(q_{k_1})I \\ \neq & C(q_{k_0})A(q_{k_0})^{k_1-k_0}\Pi_0(A(q_{k_0})^{k_1-k_0})^T C(q_{k_0})^T \\ & +\rho(q_{k_0})C(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})\mathcal{F}_{\kappa_0}(q_{k_0})^T C(q_{k_0})^T + \sigma(q_{k_0})I \end{array}$$

for all $q_k \neq q'_k \in \{1, \cdots, N\}$.

Proof: The proof follows by construction.

Therefore, from **Lemma** 2 and **Lemma** 3, the hybrid initial state and the first discrete transition time can be uniquely determined. The remaining state trajectories can be determined by repeating the procedure. For k_i $(i \geq 1)$, the $\hat{x}_{k_i}$ will be given from the initial state estimate [12]. Thus, we have the following observability condition:

Theorem 1 *A discrete-time linear hybrid system H is observable if and only if it satisfies **Lemma** 2 and **Lemma** 3.*

Proof: The proof follows directly from **Lemma** 2 and **Lemma** 3. □

This test needs the operations of multiplication and addition of matrices which are system parameters and noise covariances, the computation is straightforward with computational complexity depending on data size.

3 Design of estimators for stochastic hybrid systems

Having established conditions for the observability of stochastic linear hybrid systems, we would like to design estimators for those observable systems, and also quantify values of system parameters that would guarantee performance (exponential convergence, in our case). We extend the design methods proposed by Balluchi et al. [6] for hybrid systems with continuous-time, continuous state dynamics to encompass discrete-time stochastic hybrid systems.

A hybrid estimator finds estimates $\hat{q}$ and $\hat{x}$ for the current discrete state q and the continuous state x respectively. In this section, we first describe the structure of the hybrid estimator, and then analyze the continuous component of the estimator in detail to obtain bounds on the time between discrete transitions of state which would guarantee exponential convergence of our hybrid estimator. Throughout this paper, all norms, unless specified otherwise, are 2-norms.

Definition 5 *(Exponential convergence of a hybrid estimator) Given a hybrid system H with N discrete modes, we say that a hybrid estimator is exponentially convergent if its discrete state estimate $\hat{q}$ exhibits correct identification of the discrete-state transition sequence of the original system after a finite number of steps; the* *continuous state estimate at any instant has a unique mean and convergent covariance; and the mean of the estimation error, $\bar{\zeta} = E[\hat{x} - x]$ converges exponentially to the set $\|\bar{\zeta}\| \leq M_0$ with a rate of convergence μ, where M_0 is the steady-state error bound, and $|\mu| < 1$. In other words, the estimator is convergent if, for any switching time k_i,*

$$
\begin{aligned}
\hat{q}_k &= q_k, \forall k > K, K \in \mathbb{N}^+ && (11)\\
\|\bar{\zeta}_k\| &\leq \mu^{(k-k_i)}\|\bar{\zeta}_{k_i}\| + M_0, \forall k > k_i && (12)
\end{aligned}
$$

3.1 Structure of the hybrid estimator

We design the hybrid estimator as a combination of a discrete observer to detect the discrete state switches, and an estimator to estimate the continuous dynamics, as proposed in [6]. In the rest of this paper, we assume that we have a discrete observer that correctly identifies the discrete state, either immediately after a switch takes place, or with a known time delay Δ after a discrete transition. A discrete observer could be constructed using a bank of N estimators as a residual generator [1, 6] – even in this case, we could further increase the probability of correct discrete-state identification by enforcing a decision time delay Δ on the discrete observer. This would be possible only if the system were observable in the sense of a stochastic hybrid system, as explained earlier. In this section, we design a least-squares estimator in the form of N Kalman filters for the continuous state estimate. Although the underlying system in [6] is continuous-time and deterministic, the design methodology of [6] adapts well to discrete-time stochastic hybrid systems, as we show here.

3.2 Discrete-time Kalman filter

We consider a hybrid system of the form described in Eq.(1). For the sake of simplicity of notation, we replace $A(q_k)$ and $C(q_k)$ with A_l and C_l, where $l \in \{1 \dots N\}$. We can then write the equations for the least-square estimator of a linear stochastic system as

$$\hat{x}_{k+1} = (A_l - K_l C_l)\hat{x}_k + K_{P,k,l} y_k,\ k \geq 0 \tag{13}$$

where l is the estimated discrete state, and $K_{P,k,l}$ is the optimal Kalman filter gain for mode l, given by

$$K_{P,k,l} = A_l P_k C_l^T (R_l + C_l P_k C_l^T)^{-1} \tag{14}$$

and P_k satisfies the discrete Riccati recursion

$$P_{k+1} = A_l P_k A_l^T + Q_l - K_{P,k,l}(R_l + C_l P_k C_l^T) K_{P,k,l}^T,\ P(0) \triangleq \Pi_0 \tag{15}$$

Theorem 2 *([13]): The Discrete Algebraic Riccati Equation (DARE) has a stabilizing solution that is unique if and only if*

$$
\begin{aligned}
&\{A_l, C_l\} \text{ is detectable} && (16)\\
&\{A_l, Q_l^{1/2}\} \text{ is controllable on the unit circle.} && (17)
\end{aligned}
$$

Any such solution is positive definite.

If these conditions are satisfied for every discrete state $i \in \{1 \dots N\}$, we can design a bank of N steady-state, exponentially convergent Kalman filters to estimate the continuous state of the system.
We can then show that, for a given discrete state i, correctly identified,

$$
\begin{aligned}
\hat{x}_{k+1} &= (A_l - K_l C_l)\hat{x}_k + K_l y_k && (18)\\
\hat{\zeta}_{k+1} &= (A_l - K_l C_l)\hat{\zeta}_k && (19)
\end{aligned}
$$

Clearly, $\hat{\zeta}$ is exponentially convergent if

$$(A_l - K_l C_l) \text{ is stable} \tag{20}$$

3.3 Error dynamics

In this section, we follow the methodology of [6] to determine the evolution of the estimation error across the discrete transition sequence. Let us consider two consequent discrete transitions of H, occurring at times k_i and k_{i+1}. Suppose the transition at time k_{i+1} was from discrete state m to l, and was detected at time k'_{i+1} such that $k'_{i+1} - k_{i+1} \leq \Delta$. Similarly, $k'_i - k_i \leq \Delta$. This is illustrated in Fig.(1).

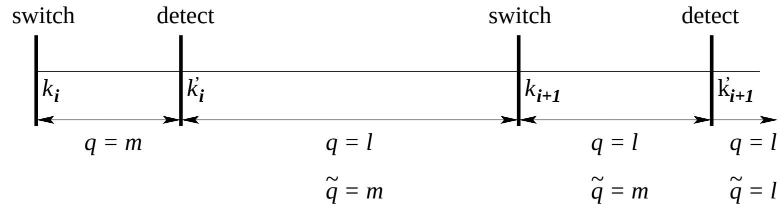


Figure 1: Illustration of the transition sequence

We are interested in the region $k \in \{k'_i, k'_i + 1, \ldots, k'_{i+1}\}$. Since we assume that by time-step k'_i the discrete state has been identified correctly, for the exponential convergence of the estimation error on k'_i to k'_{i+1} :

1. The error converges exponentially between k'_i and k_{i+1}; and
2. The error divergence between k_{i+1} and k'_{i+1} due to wrong discrete state estimation does not upset the exponential convergence of the error on k'_i to k'_{i+1}.

Following the methodology of [6], dividing the time interval between k'_i and k'_{i+1} into two regions, we get error dynamics of the form

$$\begin{aligned} \bar{\zeta}_{k+1} &= (A_m - K_m C_m)\bar{\zeta}_k \\ & \quad k \in \{k'_i, \ldots, k_{i+1} - 1\} \qquad (21) \\ \bar{\zeta}_{k+1} &= (A_m - K_m C_m)\bar{\zeta}_k + [(A_m - A_l) - K_m(C_m - C_l)]\bar{x}_k \\ & \quad k \in \{k_{i+1}, \ldots, k'_{i+1} - 1\} \qquad (22) \end{aligned}$$

where $\bar{x} = E[x]$. The second term in Eq.(22) arises because a Kalman filter designed for the discrete state m is being used to estimate the dynamics of the discrete state l. Combining Eqs. (21) and (22), we express the error dynamics by

$$\bar{\zeta}_{k+1} = (A_m - K_m C_m)\bar{\zeta}_k + u_k, \; k \in \{k'_i, \ldots, k'_{i+1} - 1\} \qquad (23)$$

where

$$u_k = \begin{cases} 0, k \in \{k'_i, \ldots, k_{i+1} - 1\} \\ ((A_m - A_l) - K_m(C_m - C_l))\bar{x}_k, \; k \in \{k_{i+1}, \ldots, k'_{i+1} - 1\} \end{cases} \qquad (24)$$

From this, we get

$$\bar{\zeta}_{k+1} = (A_m - K_m C_m)^{k+1-k'_i}\bar{\zeta}_{k'_i} + [(A_m - K_m C_m)^{k-k'_i} : \ldots I] \begin{bmatrix} u_{k'_i} \\ \vdots \\ u_k \end{bmatrix} \qquad (25)$$

$$\|\bar{\zeta}_{k+1}\| = \left\| (A_m - K_m C_m)^{k+1-k'_i}\bar{\zeta}_{k'_i} + [(A_m - K_m C_m)^{k-k'_i} : \ldots I] \begin{bmatrix} u_{k'_i} \\ \vdots \\ u_k \end{bmatrix} \right\| \qquad (26)$$

This gives us

$$\begin{aligned}
\|\bar{\zeta}_{k+1}\| &= \left\|(A_m - K_m C_m)^{k+1-k_i'}\bar{\zeta}_{k_i'} + \sum_{l=0}^{k-k_i'}(A_m - K_m C_m)^{k-k_i'-l}u_{k_i'+l}\right\|,\ k \in \{k_{i+1}, \dots, k_{i+1}'-1\} && (27)\\
&\leq \left\|(A_m - K_m C_m)^{k+1-k_i'}\bar{\zeta}_{k_i'}\right\| + \left\|\sum_{l=0}^{k-k_i'}(A_m - K_m C_m)^{k-k_i'-l}u_{k_i'+l}\right\|,\ k \in \{k_{i+1}, \dots, k_{i+1}'-1\} && (28)
\end{aligned}$$

Lemma 4 *Given a matrix $A \in \mathbb{R}^{n\times n}$ with all distinct eigenvalues,*

$$\|A^t\| \leq k(A)\alpha^t(A),\ \forall t \geq 0 \tag{29}$$

where $\alpha(A)$ is the maximal absolute value of the eigenvalues of A, and $k(A) = \|Q\|\|Q^{-1}\|$, the condition number of A under the inverse, where $Q^{-1}AQ = J$, the Jordan canonical form.

Proof: The proof follows that for the continuous-time case ([6], [14]). From this we can show that for $t \geq 0$, if m is the size of the largest Jordan block of A,

$$\|A^t\| \leq mk(A)\alpha^t(A)\max\frac{t^r}{\alpha^r(A)},\ 0 \leq r \leq m-1, \tag{30}$$

When A has all distinct eigenvalues, this reduces to Eq.(29). □

Further simplification of Eq.(28) using **Lemma** 4 gives us

$$\begin{aligned}
\|\bar{\zeta}_{k+1}\| &\leq k(A_m - C_m)[\alpha(A_m - K_m C_m)]^{k+1-k_i'}\|\bar{\zeta}_{k_i'}\| \\
&+ k(A_m - C_m)\max\|u_k\|(k - k_{i+1}),\ k \in \{k_{i+1}, \dots, k_{i+1}'\} && (31)
\end{aligned}$$

Since $k_{i+1}' - k_{i+1} \leq \Delta$, if

$$\|u_k\|_\infty \leq U = \max\|(A_m - A_l) - K_m(C_m - C_l)\|_1 X \tag{32}$$

such that $X \geq \|x\|_\infty$, $X > 0$, we can write

$$\|\bar{\zeta}_{k+1}\| \leq k(A_m - C_m)[\alpha(A_m - K_m C_m)]^{k+1-k_i'}\|\bar{\zeta}_{k_i'}\| + \sqrt{n}U\Delta k(A_m - C_m) \tag{33}$$

Lemma 5 *Consider a hybrid system with a single discrete state, in which the discrete-time evolution of the continuous state variable is given by*

$$x_{k+1} = \eta x_k,\ |\eta| < 1 \tag{34}$$

Suppose the state x is subject to resets $x(t_s) = a\eta x(t_s - 1) + b$, occurring at switching times $\{t_s\}$, with $a \geq 1$ and $b \geq 0$. Then the evolution of x can be described by

$$\begin{aligned}
x_k &= \eta^{k-t_{s-1}}x_{t_{s-1}},\ k \in \{t_{s-1}, \dots, t_s - 1\} && (35)\\
x_{t_s} &= a\eta^{t_s - t_{s-1}}x_{t_{s-1}} + b && (36)
\end{aligned}$$

Let us also assume there exists a lower bound β on the time between resets, i.e., $t_s - t_{s-1} \geq \beta \geq 1$, for all $s > 1$. Then, if $x_{t_0} > 0$ and $\mu = \eta^{(\frac{\log_\eta a}{\beta}+1)}$ such that $|\mu| < 1$, then $x(k)$ converges exponentially to the set $[0, \frac{b}{1-\eta^\beta}]$ with a rate of convergence greater than or equal to μ.

Proof: The proof is similar to the proof of **Lemma** 3 in Balluchi et al.([6]). We can show that if the above conditions are satisfied, then

$$x_{t_s} < \mu^{(t_s - t_0)}x_{t_0} + \frac{b}{1-\eta^\beta} \tag{37}$$

This implies that the state x_{t_s} after every reset is bounded above exponentially by rate μ, and converges to the set $[0, \frac{b}{1-\eta^\beta}]$. Since the inter-reset dynamics decays exponentially with rate η, and $|\eta| \leq |\mu|$, the evolution between resets is also bounded above by an exponential with rate μ. □

Using Eqs.(16), (17), (20), (32) and (33) with **Lemma** 4 and **Lemma** 5, we arrive at the following theorem:

Theorem 3 *Consider a stochastic linear hybrid system of the form in Eq.(1), a steady-state error bound* M_0 *and rate of convergence* μ, $|\mu| < 1$, $|\alpha(A_m - K_m C_m)| \leq |\mu|$ *for all* $m = 1 \ldots N$, *where* $\alpha(A)$ *is the maximal absolute value of the eigenvalues of* A. *Let* $k(A) = \|Q\| \|Q^{-1}\|$, *the condition number of* A *under the inverse, where* $Q^{-1}AQ = J$, *the Jordan canonical form. Then if the following seven conditions are satisfied:*

1. *The system is observable under the definition in Section 2*
2. $\{A_m, C_m\}$ *couples are observable for all* $m = 1 \ldots N$
3. $\{A_m, Q_m^{1/2}\}$ *couples are controllable for all* $m = 1 \ldots N$
4. $(A_m - K_m C_m)$ *is stable for all* $m = 1 \ldots N$ *with all distinct eigenvalues*
5. *There exists* $X > 0$ *such that* $\|x_k\|_\infty \leq X$, $k = 1, 2, \ldots$ *such that*
$$\|u_k\|_\infty \leq U = \max \|(A_m - A_l) - K_m(C_m - C_l)\|_1 X \tag{38}$$
6. *The discrete decision time,* Δ *satisfies the relation*
$$\Delta \leq \frac{M_0}{\sqrt{n} U \max[k(A_m - K_m C_m)]} \tag{39}$$
7. *The time between switching events,* β *satisfies the conditions*
$$\beta > \beta_{min} + \Delta, \text{ where} \tag{40}$$
$$\beta_{min} > \max[\frac{1}{|\log \mu|} \log \left| \left(1 - \frac{\sqrt{n} U \Delta k(A_m - K_m C_m)}{M_0}\right)\right|, \max \frac{\log[k(A_m - K_m C_m)]}{|\log[\alpha(A_m - K_m C_m)]|}] \tag{41}$$

we can design a hybrid estimator that converges to within the steady-state bound M_0 *with a rate of convergence greater than or equal to* μ.

Proof: The proof follows directly from the fact that the error dynamics are bounded by Eq.(33), which is in the form of Eq.(36) in **Lemma** 5. Applying **Lemma** 5 for the appropriate values of a, b and η, we can prove **Theorem** 3. Conditions (2)-(4) are needed for convergence of the estimators, while condition (1) is needed for the detection of the switch and for the design of the discrete observer. □

Corollary 1 *If Conditions (1)-(6) of* **Theorem** *3 are satisfied, then, given a steady-state error bound* M_0 *and a rate of convergence* μ, *we can design an estimator that converges exponentially to* M_0 *with a rate of at least* μ *if the time between switching events is at least* $\beta = \beta_{min} + \Delta$, *where*

$$\beta_{min} = \max\left[\frac{1}{|\log \mu|} \log \left| \left(1 - \frac{\sqrt{n} U \Delta k(A_m - K_m C_m)}{M_0}\right)\right|, \max \frac{\log[k(A_m - K_m C_m)]}{|\log[\alpha(A_m - K_m C_m)]|}\right] \tag{42}$$

Remark 1 *: An important difference between the continuous-time hybrid systems analyzed in [6] and the discrete-time hybrid systems that we consider is that we can no longer make* M_0 *arbitrarily small by simply changing the value of* Δ *such that Eq.(39) is still satisfied - the discrete nature of the system restricts* Δ *to values in* $\mathbb{N}$.

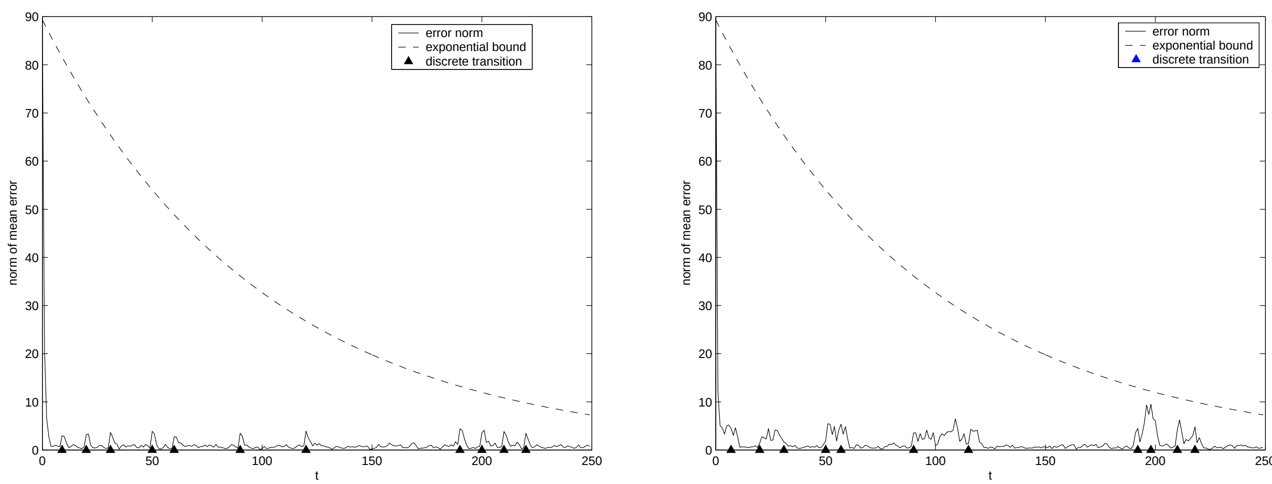
4 Example: Aircraft Trajectory

We apply the above design criteria to the design of an estimator for the switched, linearized trajectory of an aircraft. We consider two discrete states, both coordinated turns, but with different angular velocities, one with a turn rate of 2° per second, and the other with a turn rate of 5° per second, which represent aircraft trajectories composed of slow turns and sharp turns. For brevity, we only include two discrete state example in this paper but we have successfully designed hybrid estimators for aircraft trajectory tracking and conflict detection and resolution problems with multiple discrete states such as constant velocity straight flight modes with different noise characteristics and coordinated turn modes with various angular velocities. The dynamics of a coordinated turn is given by

$$x_k = \begin{bmatrix} 1 & \frac{\sin\omega T}{\omega} & 0 & -\frac{1-\cos\omega T}{\omega} \\ 0 & \cos\omega T & 0 & -\sin\omega T \\ 0 & \frac{1-\cos\omega T}{\omega} & 1 & \frac{\sin\omega T}{\omega} \\ 1 & \sin\omega T & 0 & \cos\omega T \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} u_{k-1} + w_k \tag{43}$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k + v_k \tag{44}$$

where $x = [\ x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2\ ]$ where x_1 and x_2 are the position coordinates, $u = [\ u_1 \quad u_2\ ]^T$ where u_1 and u_2 are the velocity components, ω is the turn rate, T is the sampling interval, w is the process noise, and v is the sensor noise. We choose an operating velocity of 150 knots. We find that for an instantaneous discrete decision time, the time between discrete transitions should be at least 8 seconds to guarantee exponential convergence with a rate of 0.99. The comparison of the bounds is shown in Figure (2(a)). We also note that by Lemma (5) the norm of the mean error does not have to be monotonic, but if the conditions explained above are satisfied, it will be bounded by an exponential of rate μ. This is also seen in the example.



(a) (b)

Figure 2: (a) Exponential convergence of error. (b) Convergence of error when modes have same dynamics but different noise characteristics. The triangles denote discrete transition times ($\mu = 0.99$, $M_0 = 0$, and $T = 2sec$).

As explained in the Section 2, **Lemma** 2, identical dynamics with different noise characteristics in each discrete state might still make the system observable in the stochastic hybrid context. We demonstrate this by designing an exponentially convergent hybrid estimator for a switched aircraft trajectory - the two discrete states correspond to 2° per second turns with different process noise covariances. This is shown in Figure (2(b)).

5 Conclusions

In this paper, we have extended the definition of observability to include stochastic linear hybrid systems, and have used prior knowledge of system noise characteristics to improve the observability conditions for a discrete-time stochastic linear hybrid system. We have also found bounds on the time between discrete transitions to guarantee the exponential convergence of hybrid estimators for such systems. An interesting direction for future work would be the extension of these results to hybrid systems with continuous state resets.

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