On the Roles of Smoothing in Planning of Informative Paths

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Abstract—This paper investigates the roles of smoothing in planning of continuous information-gathering paths for mobile sensors, when the goal is to maximize the mutual information between some variables of interest at some specific time instance and the continuous measurement history over a specified time window. Analysis of the rate of information gathering for various forms of information quantities demonstrates that smoothing enables quantification of the net rate of the information gathering by measurement, leading to a monotonically increasing property of information. Also, the ability of smoothing to quantify the correct cost-to-go function in the receding-horizon decision making is identified, and is validated with numerical examples of simplified weather forecasting, sensor scheduling, and target localization.

I. INTRODUCTION

Path planning of robotic sensors to extract information from environments or from some specified objects has been emerging decision making in the context of sensor networks and robotics [1]–[5]. One of common metrics employed to define the information reward for a sensing path is the mutual information between the measurement and the quantity of interest (e.g., target locations, weather variables over some region of interest). Most previous work [2,4] based on this notion of mutual information addressed discrete decisions that determine the finite number of waypoints for robotic sensors with providing methodology to explicitly quantify the mutual information values: [2] utilized a particle filter for computation of mutual information, while [4] suggested a computationally efficient way of computing mutual information with consideration of constrained vehicle motion.

On the contrary, previous work on continuous motion planning of sensor platforms [1] did not present an explicit way of computing the mutual information in the continuous-time domain. Instead, [1] introduced the concept of “mutual information metric,” which arguably represents the rate of mutual information, inspired by the mutual information in the continuous-time domain; however, it can be shown that this metric does not equal to the rate of mutual information. In [5], we presented two explicit ways of quantifying the mutual information between the continuous measurement history in the past and the specified verification variables: the filter form and the smoother form. While the filter form was a simple extension of the literature in the community of information theory [6,7], the smoother form was a novel expression of the mutual information in the continuous-time domain; moreover, the benefits of utilizing the smoother form were identified in the sense of the computational efficiency, robustness to modeling error, and simplification of obtaining the accumulated information on the fly.

This paper extends the brief discussions in [5] regarding the on-the-fly information and the rate of information gathering, providing new insights on the roles of smoothing in a more general setting of planning of continuous information-gathering paths. Analysis of the rate of information gathering on-the-fly and its statistical mechanical interpretation [7] demonstrates an important feature of smoothing in extracting out the pure influence of sensing on the entropy reduction of the quantity of interest at a specific time. Moreover, the role of smoothing in quantifying the correct cost-to-go function for receding-horizon approximations is identified with its implication in the information supply and dissipation process. Three representative types of informative planning examples validate the suggested smoothing-based receding-horizon formulations contrasted to the filtering-based formulations.

II. PROBLEM DEFINITIONS

A. Linear System Model

Consider the dynamics of objects/environment with finite dimensional state vector $X_t \in \mathbb{R}^n$ that is described by the following linear time-varying system:

$$\dot{X}_t = A(t)X_t + W_t$$

where $W_t \in \mathbb{R}^{nx}$ is a zero-mean Gaussian process noise with $E[W_t W'_s] = \Sigma_W \delta(t-s)$, $\Sigma_W \succeq 0$, which is independent of $X_t$. The prime sign $'$ denotes the transpose of a matrix. The initial condition of the state, $X_0$ is normally distributed as $X_0 \sim \mathcal{N}(\mu_0, P_0)$. $P_0 > 0$.

The system (1) is observed by sensors with additive Gaussian noise and admits the following measurement model for $Z_t \in \mathbb{R}^m$:

$$Z_t = C(t)X_t + N_t$$

where $N_t \in \mathbb{R}^m$ is zero-mean Gaussian with $E[N_t N'_s] = \Sigma_N \delta(t-s)$, $\Sigma_N \succ 0$, which is independent of $X_t$ and $W_s$, $\forall s$. Also, a measurement history over the time window $[t_1, t_2]$ is defined as

$$Z_{[t_1, t_2]} = \{Z_t : t \in [t_1, t_2]\}.$$  

The verification variables are a subset of the state variables that are of interest to define the performance measure, and are defined as

$$V_t = M_V X_t \in \mathbb{R}^p$$

where $M_V \in \{0, 1\}^{p \times n}$, $p < n$ with every row-sum of $M_V$ being unity. Although this work is specifically interested in

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the case entries of $M_V$ are zero or one, the results can be easily extended to a general $M_V \in \mathbb{R}^{p \times n}$.

B. Planning of Informative Paths

The planning problem of informative paths addresses the design of sensing paths for mobile sensors over a fixed time window $[0, \tau]$ to reduce the uncertainty in the verification variables. Depending on the time when one is interested in the value of the verification variables, two types of planning problems are considered: the tracking problem and the forecasting problem. The objective of the tracking problem is to minimize the uncertainty in the verification variables at time $\tau$, i.e. at the end of the planning window, while the forecasting problem is concerned with the uncertainty in the verification variables at time $T \gg \tau$ in the far future.

Using entropy as the metric of uncertainty, the uncertainty reduction of the verification variables by measurements taken along the path of the sensors can be represented by the mutual information.

The optimal tracking problem can then be written as the following optimization:

$$\max_{Z_{[0,\tau]}} I(V_{\tau}; Z_{[0,\tau]})$$

(OTP)

where $I(Y_1; Y_2)$ represents the mutual information between two random quantities $Y_1$ and $Y_2$ (e.g. random variables, random processes, random functions). In other words, (OTP) finds the best (continuous) measurement history over $[0, \tau]$ that represents the largest mutual information achieved between the verification variables at $\tau$.

In (OTP), it is assumed that the measurement at a given time $t$ can be represented as a function of the location of the sensors at that time: i.e. $Z_t = Z_t(r(t))$ where $r(t)$ denotes the locations of sensors at $t$. Given the measurement model in (2), this represents the cases where either the observation matrix $C(t)$ is a function of sensor locations or the measurement noise variance $\Sigma_N$ is a function of sensor locations. This work specifically considers the first case without loss of generality. Since the motion of sensor platforms are constrained, the resulting path planning problem is to maximize the objective function in (OTP) subject to various types of constraints including vehicle motion and other waypoints constraints.

Likewise, the optimal forecasting problem is expressed as

$$\max_{Z_{[0,\tau]}} I(V_{\tau}; Z_{[0,\tau]})$$

(OFP)

with $T > \tau$, and a variety of constraints can be associated with this decision. On the basis of the optimal path planning formulations in this section, the receding-horizon approximations of (OTP) will be discussed in Section V with further emphasis.

III. Quantification of Mutual Information

This section summarizes the formulae presented in [5] for computing the mutual information in (OTP) and (OFP). The expressions given in this section will be used for the main discussions on the on-the-fly information quantities in sections IV and V.

A. Mutual Information for Tracking

For the linear system in this work, the objective value for (OTP) can be computed as

$$I(V_{\tau}; Z_{[0,\tau]}) = \frac{1}{2} \text{ldet } P_V(\tau) - \frac{1}{2} \text{ldet } Q_V(\tau)$$

(5)

where $\text{ldet}$ stands for $\log \det$, $P_V(\tau) \triangleq \text{Cov}[V_\tau]$ and $Q_V(\tau) \triangleq \text{Cov}[V_\tau | Z_{[0,\tau]}]$; these two matrices can be written as $P_V = M_V P_X M_V'$ and $Q_V = M_V Q_X M_V'$ where $P_X$ and $Q_X$ are obtained by integrating (forward in time) the following matrix differential equations:

$$\dot{P}_X(t) = A(t) P_X(t) + P_X(t) A(t)^T + \Sigma_W$$

(6)

$$\dot{Q}_X(t) = A(t) Q_X(t) + Q_X(t) A(t)^T + \Sigma_W - Q_X(t) C(t)^T \Sigma^{-1}_N C Q_X(t)$$

(7)

with initial condition, $P_X(0) = Q_X(0) = P_0$. Notice that (6) is Lyapunov equation and (7) is the Riccati equation for Kalman-Bucy filters.

Since $P_X(\tau)$ does not depend on the measurement, (OTP) is equivalent to minimize the posterior entropy $\text{ldet}(M_V Q_X(\tau) M_V')$. Specifically, in case $M_V = I$, this becomes equivalent to maximize the $\text{ldet}$ of the Fisher information matrix at time $\tau$ addressed in [1].

B. Mutual Information for Forecasting

1) Filter Form: By defining the measurement history over $(0, T]$ such that $Z_{[0,\tau]} \triangleq Z_{[0,\tau]} \cup \emptyset_{[\tau,T]}$ where $\emptyset_{[\tau,T]}$ means null measurement during $(\tau, T]$, the mutual information for the forecasting problem can be expressed as

$$I(V_T; Z_{[0,\tau]}) = I(V_T; Z_{[0,\tau]}|V_T)$$

$$= \frac{1}{2} \text{ldet } P_V(T) - \frac{1}{2} \text{ldet } Q_V(T)$$

(8)

with the Riccati equation in (7) being replaced by

$$\dot{Q}_X = AQ_X + Q_X A^T + \Sigma_W - I_{\{0,\tau\}} Q_X C(t)^T \Sigma^{-1}_N C Q_X$$

(9)

where $I_{\{0,\tau\}}(t) : \mathbb{R}_+ \mapsto \{0, 1\}$ is the indicator function that is unity for $t \in [0, \tau]$ and zero elsewhere.

2) Smoother Form: Exploiting conditional independence, the mutual information for forecasting, $I(V_T; Z_{[0,\tau]})$, can be equivalently quantified as the difference between the unconditioned and the conditioned mutual information for tracking:

$$I(V_T; Z_{[0,\tau]}) = I(X_T; Z_{[0,\tau]}) - I(X_T; Z_{[0,\tau]}|V_T).$$

(10)

With (10), the smoother form of the mutual information for forecasting is derived as

$$I(V_T; Z_{[0,\tau]}) = I(X_T; Z_{[0,\tau]}) - I(X_T; Z_{[0,\tau]}|V_T)$$

$$= \mathcal{J}_0(\tau) - \frac{1}{2} \text{ldet}(I + Q_X(\tau) \Delta_S(\tau))$$

(11)

with $\mathcal{J}_0 \triangleq \frac{1}{2} \text{ldet} S_{X|V} - \frac{1}{2} \text{ldet} S_X$ and $\Delta_S \triangleq S_{X|V} - S_X$. The matrices $S_X(\tau)$, $S_{X|V}(\tau)$, and $Q_X(\tau)$ are determined by the following matrix differential equations:

$$\dot{S}_X = -S_X A - A^T S_X - S_X \Sigma_W S_X$$

(12)

$$\dot{S}_{X|V} = -S_{X|V} (A + \Sigma_W S_X) - (A + \Sigma_W S_X)^T S_{X|V}$$

$$+ S_X \Sigma_W S_{X|V}$$

(13)

$$\dot{Q}_X = AQ_X + Q_X A^T + \Sigma_W - Q_X C(t)^T \Sigma^{-1}_N C Q_X.$$
with initial conditions \( S_X(0) = P_0^{-1} \), \( S_X|V(0) = P_0^{-1} \), and \( Q_X(0) = P_0 \); \( P_0|V \triangleq \text{Cov}(X_0|V) > 0 \) can be calculated in advance by a fixed-point smoothing process. It was demonstrated in [5] that the smoother form is preferred to the filter form in terms of the computational efficiency and accessibility of the on-the-fly knowledge of the accumulated information.

IV. ON-THE-FLY INFORMATION AND ITS RATE

One benefit discussed in [5] is that it simplifies the process of quantifying the accumulated information gathered by a partial history of measurement. This section provides further insight on that point by extending the discussions to the tracking problems, which will be exploited in the receding-horizon formulations in section V. In addition, this section also extends the brief discussion on the time derivative of the on-the-fly information quantities in [5] by providing more detailed analysis of the mutual information rate for the four on-the-fly information quantities in [5] by providing more important features.

A. Filter-Form On-the-fly Information (FOI)

1) Information: Since the Lyapunov equation in (6) and the Riccati equation in (7) and (9) are integrated forward from time 0, the available matrix values at time \( t < \tau \) are \( P_X(t) \) and \( Q_X(t) \). With these, the mutual information between the current state variables and the measurement thus far can be evaluated as

\[
\mathcal{I}(X_t; Z_{[0,t]}) = \frac{1}{2} \text{ldet} P_X(t) - \frac{1}{2} \text{ldet} Q_X(t).
\]

2) Time Derivative: The expression of the time derivative of FOI was first presented in [6], and [7] provided its interpretation as information supply and dissipation. The rate of FOI can be derived as

\[
\frac{d}{dt} \mathcal{I}(X_t; Z_{[0,t]}) = \frac{1}{2} \text{ldet} P_X(t) - \frac{1}{2} \text{ldet} Q_X(t)
\]

\[
= \frac{1}{2} \text{tr} \left\{ P_X^{-1} \dot{P}_X - Q_X^{-1} \dot{Q}_X \right\}
\]

\[
= \frac{1}{2} \text{tr} \left\{ \Sigma_N^{-1} C Q_X C' \right\} - \frac{1}{2} \text{tr} \left\{ \Sigma_W \left( Q_{X}^{-1} - P_{X}^{-1} \right) \right\}
\]

where every matrix is evaluated at \( t \). The first term depends on the measurement and represents the rate of information supply, while the second term depends on the process noise and represents the rate of information dissipation [7]. It can be proven that the signs of the supply and the dissipation term are non-negative:

\[
\text{tr} \left\{ \Sigma_N^{-1} C Q_X C' \right\} \geq 0, \quad \text{tr} \left\{ \Sigma_W \left( Q_{X}^{-1} - P_{X}^{-1} \right) \right\} \geq 0,
\]

since \( C Q_X C' \geq 0, \ Q_{X}^{-1} - P_{X}^{-1} \geq 0 \), and trace of the product of two symmetric positive definite matrices is non-negative [8]. Thus, measurement tends to increase FOI while the process noise tends to decrease it; FOI can be decreasing over time if the information dissipation dominates the information supply.

Note that previous work [1] derived a quantity identical to the information supply term in developing some information potential field. The author of [1] considered the entropy of the current state, \( H(X_t) \), which is \( -\frac{1}{2} \text{ldet} J_X(t) + \frac{1}{2} \log(2\pi e) \), where \( J_X(t) \triangleq Q_X^{-1}(t) \) is the Fisher information matrix at time \( t \). Then,

\[
- \frac{d}{dt} H(X_t|Z_{[0,t]}) = - \frac{1}{2} \text{ldet} J_X(t)
\]

\[
= \frac{1}{2} \text{tr} \left\{ J_X^{-1} \left( -J_X A - A' J_X - J_X \Sigma_W J_X + C' \Sigma_N^{-1} C \right) \right\}
\]

\[
= \frac{1}{2} \text{tr} \left\{ J_X^{-1} C' \Sigma_N^{-1} C \right\} - \frac{1}{2} \text{tr} \left\{ A + J_X^{-1} A' J_X + \Sigma_W J_X \right\}
\]

\[
= \frac{1}{2} \text{tr} \left\{ \Sigma_N^{-1} C Q_X C' \right\} + \alpha(t)
\]

where \( \alpha(t) \) consists of terms that are not dependent on the observation matrix \( C \). Note that \( \alpha(t) \neq \frac{1}{2} \text{tr} (Q_{X}^{-1} - P_{X}^{-1}) \); thus, the procedure given in [1] does not provide the expression of the rate of FOI.

Note that the dissipation term in (16) becomes zero, if \( P_X(t) = Q_X(t) \), which corresponds to the case where no measurement is ever taken up to time \( t \). Thus, the information supply term represents the rate of information accumulation assuming no measurement having been taken up to the current time. Consider the case no measurement is taken over \([0, \tau]\), i.e., \( Z_{[0,\tau]} = 0 \); then, at some \( t < \tau \), the rate of FOI equals to zero regardless of the process noise. However, the remainder term \( \alpha(t) \) derived from the information form Riccati equation is positive in case \( \Sigma_W > 0 \), representing the increase of the entropy due to the process noise.

B. Projected Filter-Form On-the-fly Information (PFOI)

1) Information: Similar to FOI, the mutual information between the current verification variables and the measurement thus far can also be computed on the fly, while computing the filter form mutual information:

\[
\mathcal{I}(V_t; Z_{[0,t]}) = \frac{1}{2} \text{ldet} P_V(t) - \frac{1}{2} \text{ldet} Q_V(t).
\]

2) Time Derivative: The time derivative of PFOI can also be expressed in terms of \( P_X(t) \) and \( Q_X(t) \) as follows.

\[
\frac{d}{dt} \mathcal{I}(V_t; Z_{t}) = \frac{1}{2} \text{tr} \left\{ P_V^{-1} \dot{P}_V - Q_V^{-1} \dot{Q}_V \right\}
\]

\[
= \frac{1}{2} \text{tr} \left\{ \Sigma_N^{-1} C Q_X M_V' Q_V^{-1} M_V Q_X C' \right\} + \beta(t),
\]

where \( \beta(t) \) represents all the remaining terms that do not depend on the observation matrix \( C \). The first term, under-braced as “Direct Supply” represents the immediate influence the measurement on the current verification variables; the remaining term \( \beta(t) \) captures all the correlated effect due to coupling in dynamics on the information supply/dissipation. The sign of \( \beta(t) \) is indefinite, while the direct supply term is non-negative as \( C Q_X M_V' Q_V^{-1} M_V Q_X C' \geq 0 \). Although \( \beta(t) \) is indefinite in general, it is zero if \( P_X(t) = Q_X(t) \) and no measurement is taken at \( t \). Thus, in case no measurement is taken over \([0, \tau]\), the value of PFOI is constantly zero in the time interval \([0, \tau]\) regardless of the process noise.
C. Smoother-Form On-the-fly Information for Forecasting (SOIF)

1) Information: The smoother form can quantify the information accumulated by the measurement thus far on the fly. In the smoother form framework, the mutual information between the future verification variables $V_T$ and the measurement up to the current time $t$ can be calculated as

$$I(V_T; Z_{[0,t]}) = I(X_t; Z_{[0,t]}) - I(X_t; Z_{[0,t]}|V_T)$$

$$= J_0(t) - \det(I + Q(t)\Delta_S(t)). \quad (20)$$

The values of matrices $J_0(t), Q(t),$ and $\Delta_S(t)$ are calculated in the process of the forward integration (12) through (14).

2) Time Derivative: The temporal derivative of the smoother form mutual information can be written as:

$$\frac{d}{dt} I(V_T; Z_{[0,t]}) = \frac{d}{dt} \left[ J_0(t) - \frac{1}{2} \det(I + Q_X(t)\Delta_S(t)) \right]$$

$$= \frac{1}{2} \text{tr} \left\{ S_X^{-1} \mathbf{S}_X^\prime \mathbf{S}_X - S_X^{-1} \mathbf{S}_X \right\}$$

$$- \frac{1}{2} \text{tr} \left[ (I + Q_X \Delta_S)^{-1} \left( \mathbf{Q}_X \Delta_S + Q \Delta_S \right) \right]$$

$$= \frac{1}{2} \text{tr} \left\{ \Sigma_{N}^{-1} Q_X \Delta_S (I + Q_X \Delta_S)^{-1} Q_X \right\}. \quad (21)$$

The sign of the quantity in (21) can be shown to be non-negative in two ways:

**Proposition 1**

The rate of the smoother-form-on-the-fly information for forecasting (SOIF) is non-negative:

$$\frac{d}{dt} I(V_T; Z_{[0,t]}) \geq 0 \quad (22)$$

**Proof:** One way to prove this is to utilize some properties of symmetric positive definite matrices. By the matrix inversion lemma [9], the matrix $\Pi \equiv Q_X \Delta_S (I + Q_X \Delta_S)^{-1} Q_X$ is symmetric:

$$\Pi = Q_X \Delta_S Q_X - Q_X \Delta_S (Q_X^{-1} + \Delta_S)^{-1} \Delta_S Q_X = \Pi'$$

The Wigner’s theorem [10] states that a product of multiple symmetric positive definite matrices is positive definite if the product is symmetric; thus, $\Pi \geq 0$. This leads to $\Pi \geq 0$, and finally $\text{tr} \left\{ \Sigma_{N}^{-1} \Pi \right\} \geq 0$, because the trace of two positive definite matrices is non-negative [8].

Since the influence of the future process noise has already been captured in $S_{X|V}$, the mutual information rate is non-negative regardless of the process noise. If one stops taking measurement at time $t$, the information reward stays constant. Thus, the mutual information rate for the smoother form can extract out the pure impact of sensing on the entropy reduction of the verification variables, while the rates for the filter forms depend on the process noise.

D. Smoother-Form On-the-fly Information for Tracking (SOIT)

1) Information: In the process of computing the tracking mutual information $I(X_t; Z_{[0,\tau]})$ by integrating forward the Lyapunov and Riccati equations in (6) and (7), the only two available matrix quantities are $P_X(t)$ and $Q_X(t)$. Using these, the mutual information between the current state and the measurement thus far, $I(X_t; Z_{[0,t]})$ can be calculated as discussed in section IV-A.1. However, this does not represent the information gathered by $Z_{[0,t]}$ for the final verification variables $V_T$, which is $I(V_T; Z_{[0,\tau]})$.

This accumulated information for $V_T$ can be quantified by considering a forecasting problem with replacing $T$ by $\tau$ and $\tau$ by $t$; then,

$$I(V_T; Z_{[0,t]}) = I(X_t; Z_{[0,t]}) - I(X_t; Z_{[0,t]}|V_T)$$

$$= \frac{1}{2} \left[ \det(S_X (|V_T| t)) - \det(S_X (\tau)) \right]$$

$$- \frac{1}{2} \det(I + Q_X (t))(S_X (|V_T| t) - S_X (\tau))) \quad (24)$$

where $S_X (\tau) \triangleq \text{Cov}(X_t|V_T)$, which is computed by integrating forward the following differential equation

$$X_t = -S_X (\tau) (A + \Sigma W X) - (A + \Sigma W (S_X)') S_X (\tau)$$

$$+ S_X (\tau) W S_X (\tau) \quad (25)$$

The initial condition is given $S_X (\tau) (0) = P_0^{-1}$, which can be expressed as

$$P_0 (\tau) = P_0 - P_0 \hat{\Phi}_{(\tau,0)} M_V P_V (\tau)^{-1} M_V \hat{\Phi}_{(\tau,0)} P_0$$

where $\hat{\Phi}_{(\tau,0)}$ is the state transition matrix from time 0 to $\tau$.

Note that the expression in (24) is not well-defined at $t = \tau$, because $S_{X|V,\tau} (\tau)$ is singular. However, the relation in (23) holds even at time $\tau$, and SOIT at time $\tau$ can be computed as:

$$I(V_T; Z_{[0,\tau]}) = I(X_{\tau}; Z_{[0,t]}) - I(X_{\tau}; Z_{[0,t]}|V_T)$$

$$= I(X_{\tau}; Z_{[0,t]}) - I(V_T^C; Z_{[0,t]}|V_T) \quad (26)$$

where $V_T^C \triangleq V_T \setminus V_T^C$, or equivalently, $V_T^C = M_V X_T$ for some $M_V^C$ such that $M_V^C$ spans the null space of $M_V$. Alternatively, SOIT at $\tau$ can be calculated by the value of FOI at $\tau$, because FOI and SOIT are identical at $\tau$ by definition.

2) Time Derivative: The rate of SOIT can be quantified by adopting the expression of the rate of SOIF in (21); the only change is that $S_X (\tau) (t)$ replaces $S_X (\tau)$ as follows.

$$\frac{d}{dt} I(V_T; Z_{[0,t]}) = \frac{1}{2} \text{tr} \left\{ \Sigma_{N}^{-1} \Pi (t)(\tau) \right\}. \quad (28)$$

where $\Pi \equiv Q_X (S_X (\tau) - S_X) [I + Q_X (S_X (\tau) - S_X)^{-1}] Q_X$.

Notice that, unlike the rate of FOI in (16) and the rate of PFOI in (19), the quantity in (28) is non-negative (except at $t = \tau$ where it is not defined), because the effect of the future process noise over $[t, \tau]$ is all encapsulated in $S_{X|V}$.

Thus, by employing the smoother form expression derived for the forecasting problem, the pure impact of sensing for the tracking problem can be quantified.

V. ON-THE-FLY INFORMATION FOR RECEIVING-HORIZON FORMULATIONS

The previous section analyzes the time derivatives of the on-the-fly information quantities from the perspective of the information supply and dissipation. Despite theoretically
important features, the on-the-fly information quantities are not essential if the goal is simply to solve the optimal path planning problems in (OTP) and (OFP). However, receding-horizon approximations of the original optimization problem are often used for better computational tractability and/or for better adaptation to changes in the environment. In this case the computation of on-the-fly information can be essential for this receding-horizon approximation, because the effect of a partial measurement history should be quantified. This section discusses the role of the smoother-form on-the-fly information quantities for the purpose of determining the cost-to-go functions of the receding-horizon formulations. This section discusses the receding-horizon of the tracking problem in (OTP), but the main results can easily be extended to the forecasting problem in (OFP).

Consider a tracking decision for the horizon of $[0, \sigma], \sigma < \tau$ when the ultimate goal is to maximize $\mathcal{I}(V_\tau; Z_{[0,\sigma]})$. For this problem, this work suggests the following formulation based on the smoother-form on-the-fly information:

$$\max_{Z_{[0,\sigma]}} \mathcal{I}(V_\tau; Z_{[0,\sigma]}) = \mathcal{I}(X_{\sigma}; Z_{[0,\sigma]}) - \mathcal{I}(X_{\sigma}; Z_{[0,\sigma]}|V_\tau).$$

(S-RH)

In other words, the decision for the time window $[0, \sigma]$ maximizes the SOIT at the end of the current receding-horizon, which can be calculated by evaluating (23) at $\sigma$. Since the time derivative of SOIT is non-negative over $[0, \tau]$, the objective value of (S-RH) increases as $\sigma$ increases. It is important to contrast (S-RH) to the following formulation based on the filter-form on-the-fly information:

$$\max_{Z_{[0,\sigma]}} \mathcal{I}(X_{\sigma}; Z_{[0,\sigma]}).$$

(F-RH)

In other words, (F-RH) aims to minimize the entropy of the current state $X_{\sigma}$, and the underlying premise of this formulation is that an accurate estimate of the current states tends to result in an accurate estimate of the future verification variables.

The formulation (F-RH) is equivalent to the formulation in [1] that maximizes $\det J_{\chi}(\sigma)$ for the interval $[0, \sigma]$. Note that the objective value of (F-RH) is not necessarily an increasing function of $\sigma$ for $\Sigma_N > 0$, because the rate of FOI can be negative if the information dissipation dominates the information supply, in particular, with large $\Sigma_W$.

In case $M_V = 1$ (i.e., $V_\tau = X_\tau$) and there is no process noise over $(\sigma, \tau]$, (F-RH) becomes equivalent to (S-RH), because then $\mathcal{I}(X_\sigma; Z_{[0,\sigma]}|X_\tau) = 0$ as there is no remaining uncertainty in $X_\sigma$ for a given $X_\tau$. However, in general $\mathcal{I}(X_\sigma; Z_{[0,\sigma]}|V_\tau) > 0$; therefore, the solutions to (S-RH) and (F-RH) differ. Also, the objective value of (F-RH) always overestimates that of (S-RH).

The difference between (S-RH) and (F-RH) can be explained in terms of information diffusion. It was shown in [11] that

$$\mathcal{I}(X_{[\sigma,\tau]}; Z_{[0,\sigma]}) = \mathcal{I}(X_\sigma; Z_{[0,\sigma]}), \forall s > \sigma$$

(29)

where $X_{[t_1,t_2]} \triangleq \{X_s: s \in [t_1, t_2]\}$. This is because the sufficient statistics for estimation the future state history $X_{[\sigma,\tau]}$ is $\tilde{X}_\sigma \triangleq \mathbb{E}[X_\sigma|Z_{[0,\sigma]}]$, which is identical to that for estimating the current state $X_\sigma$ based on the past measurement history $Z_{[0,\sigma]}$.

The relation in (29) specifically holds for $s \geq \tau$. In this case, the verification variables at $\tau$ become a subset of the future state history: i.e., $V_\tau \subset X_{[\sigma,\tau]}$. Thus, what the filter-form on-the-fly information quantifies is the influence of the past measurement on the entire future, while the smoother-form on-the-fly information pinpoints the impact on some specified variables at some specific instance of time in the future. In this sense, the smoothing term in (S-RH), $\mathcal{I}(X_\sigma; Z_{[0,\sigma]}|V_\tau)$ represents the portion of information gathered by $Z_{[0,\sigma]}$ but will be diffused out to the space that is orthogonal to $V_\tau$:

$$\mathcal{I}(X_\sigma; Z_{[0,\sigma]}|V_\tau) = \mathcal{I}(X_{[\sigma,\tau]}|V_\tau; Z_{[0,\sigma]}|V_\tau)$$

(30)

In case $M_V = I$, (30) specifically means the information diffused through the future process noise: $\mathcal{I}(W_{[\sigma,\tau]}; Z_{[0,\sigma]}|X_\tau)$ where $W_{[\sigma,\tau]} \triangleq \{W_s: s \in ([\sigma, \tau]) \}$. It should be noticed that $\mathcal{I}(W_{[\sigma,\tau]}; Z_{[0,\sigma]}|X_\tau)$ can be non-zero, although $Z_{[0,\sigma]}$ and $W_{[\sigma,\tau]}$ are uncorrelated each other, because conditioning on $X_\tau$ can correlate $Z_{[0,\sigma]}$ and $W_{[\sigma,\tau]}$.

The formulations (S-RH) and (F-RH) are written for the horizon starting from the initial time; extension to the decisions for a later horizon $[\sigma_k, \sigma_{k+1]}$ is straight-forward: they simply consider the conditioned mutual information conditioned on the previous measurement decision $Z_{[0,\sigma_k]}$.

VI. NUMERICAL EXAMPLES

This section presents numerical results to compare the two receding-horizon formulations discussed in section V with three illustrative examples each of which reflects important aspects in the path planning problems.

A. Weather Forecasting

The first example is a simplified weather forecast problem introduced in [5]. An important aspect of this example is $M_V \neq I$. The state variables represent some scalar weather variable (e.g. temperature or pressure) at a total of $n$ grid points. With the underlying nonlinear weather model – Lorenz-2003 chaos model with $72 \times 17$ grid space, a linear time-invariant model is obtained to represent short time-scale motion over some $4 \times 3$ local region. The simple Kriging formula [12] is employed to define the linear measurement model for observations taken off the grid points, and the process noise is introduced to represent the linearization error.

The goal of path planning is to reduce the entropy in the verification variables in 3 days, which corresponds to the weather variables at some $p$ grid points, by designing a 6-hr flight path for a single UAV sensor with constant speed $v$, whose motion is described as

$$\dot{x}_s = v \cos \theta, \ \dot{y}_s = v \sin \theta, \ \theta \in (-\pi, \pi].$$

(31)

Although [5] originally posed this problem as a forecasting problem with $T=3$days and $\tau=6$hrs), this paper treats it as
TABLE I
INFORMATION GATHERED DURING THE FIRST HORIZON: \( I(V_T ; Z_{[0,\tau]}) \)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>S-RH</th>
<th>S-RT</th>
<th>F-RH</th>
<th>F-RT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.04</td>
<td>0.85</td>
<td>0.86</td>
<td>0.79</td>
</tr>
<tr>
<td>2</td>
<td>0.69</td>
<td>0.62</td>
<td>0.20</td>
<td>0.14</td>
</tr>
</tbody>
</table>

the decision for the first horizon of the tracking problem: namely, \( \tau = 3 \) days and \( \sigma = 6 \) hrs) to simplify the discussion.

Two scenarios with different configurations of local region, correlation length scale parameters, vehicle speed, and vehicle initial location, are considered. The performance of four different strategies are compared to each other: two are the receding-horizon formulations in (S-RH) and (F-RH).

The other two strategies steer a sensor to climb up to the direction of the gradient of the information potential fields built on the basis of the smoother-form information rate in (21) and the filter-form information rate in (16), respectively. The key idea of rendering the information potential field is that the information supply terms can be written as a function of the location vector of the sensor; thus, a map of mutual information field can be obtained (See [13] for details). These two strategies based on the information field are denoted as S-RT and F-RT.

Figures 1(a) and 1(b) illustrate the sensor trajectories from the four scenarios for scenario 1 overlaid with the smoother-form and the filter-form information field at the initial time. In both information fields, a dark area represents an information-rich area. It can be seen that the shape of SIF and FIF is similar in terms of the locations of information-rich regions; this leads to reasonable performance of filter-based planning (F-RH and F-RT) compared to smoother-based (S-RH and S-RT) (Table I). In contrast, Figures 1(c) and 1(d) illustrate the large difference between SIF and FIF in scenario 2. As a consequence, the paths generated using S-RH and S-RT head southwards, while those from F-RH and F-RT head north, which leads to significant performance deterioration of the decisions based on the filter-form information (Table I).

In summary, this example demonstrates that a decision based on SOI and FOI can be very different depending on the problem. In this example, the process noise turns out not to be a dominant factor that causes the difference, but the dominating factors are the fact that \( \tau \gg \sigma \) and \( M_V \neq I \).

B. Sensor Scheduling

A sensor scheduling problem with some artificial dynamics is considered as the second example; the key comparison made with this example is on the overall performance of solutions from S-RH and F-RH. The system matrices are

\[
A = \begin{bmatrix}
0.1 & -0.01 \\
0.005 & 0.07
\end{bmatrix}, \quad P_0 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \Sigma_W = \text{diag}(0.3), \quad \Sigma_N = 1,
\]

(32)

and \( \tau = 5 \). The size of planning horizon is 1, and for each horizon a decision is made either to measure the first state or the second one. Thus, the true optimal solution (OPT) can be found by simply enumerating possible \( 32(=2^5) \) cases; S-RH and F-RH solutions can also be found by enumeration.

Fig. 2 illustrates the switching sequences for the three solutions. It is first noticed that the three solutions are very different; especially, the filter-form receding horizon solution behaves in an opposite way to the true optimal solution. Fig. 3 shows the information accumulation in the three solutions, where both the smoother-form accumulated information \( I(X_T ; Z_{[0,\tau]}^s) \) and the filter-form accumulated information \( I(X_{\tau_k} ; Z_{[0,\tau]}^f) \) for \( \tau_k = 1, 2, \ldots, 5 \) are shown for comparison. Looking at the decisions for the first horizon: OPT and S-RF choose to measure state 1, but F-RH selects state 2. While the filter-form information indicates that measuring state 2 (follow dotted red line with marker \( \times \)) is slightly larger than that of measuring state 1 (follow dotted black line with marker \( \square \)), the smoother-form information says that the reward for observing state 1 is much larger than the other. It can be seen that the difference in this first decision leads to a larger gap in the final performance.

One important characteristics of the system in (32) is that a relatively large process noise is injected only to the dynamics of the second state variable. F-RH keeps measuring the second variable to compensate for the increase in the uncertainty in state 2 by the impact of the process noise over the time interval of size 1; however, in a long term view, the dynamics of the two variables are coupled and the effect of process noise is propagated to the first state variable. This results in a situation that F-RH is far from optimal. S-RH provides relatively good performance by taking account of the effect of the future process noise in the decision for each horizon.

C. Target Localization

The third example, which is adapted from [1], considers localization of a target whose location is fixed in a nominal sense using a mobile sensor. The main purpose of this example is to validate the presented receding-horizon formulation
with consideration of nonlinearity in the measurement and replanning based on the actual measurement data.

The state vector $X_t$ is defined to represent the position of the target, $x_{tg}$ and $y_{tg}$. Since the target is assumed to be stationary, the system matrix $A = 0_{2 \times 2}$. The process noise represents the drift of the target position in $x$- and $y$-directions:

$$
\Sigma_W = \text{diag}(0.2^2, 0.01^2)
$$

is used. Note that the target is subject to a larger drift in $x$-direction than $y$-direction. The sensor measures the bearing angle between the target and itself:

$$
Z_t = \text{atan}\{(y_{tg} - y_s)/(x_{tg} - x_s)\} + N_t
$$

where the sensor’s motion is described by (31). The target location is tracked by a discrete extended Kalman filter (EKF) using the bearing measurement taken with frequency of 16Hz with noise standard deviation of 2.5deg. Although the underlying estimator is discrete, the sensor planning problem for a horizon $[\sigma_k, \sigma_{k+1}]$ is posed in a continuous domain by linearizing (33) around the state estimate at $\sigma_k$.

Then, the sensing noise intensity for this continuous planning becomes $\Sigma_N = 2.5^2(\pi/180)^2/16$. Once the plan for $[\sigma_k, \sigma_{k+1}]$ is made, the sensor executes it by moving along the planned path and taking discrete measurements every 1/16 seconds. The decision for the next horizon, $[\sigma_{k+1}, \sigma_{k+2}]$ is made by incorporating the actual measurement up to $\sigma_{k+1}$.

The total length of the planning window is 14 seconds, which is divided into 14 receding planning horizons. Also, $P_0 = 0.5^2 I$ is used.

Fig. 4 shows the trajectories for S-RH and F-RH solutions with the true target locations over $[0, \tau]$, which are very different from each other. The sensor moves mainly in $y$-direction in the S-RH solution but in $x$-direction in the F-RH solution. It can be seen in Fig. 5 that this difference in the path results in different characteristics in reducing the uncertainty of the target location. The top pictures that depict the time history of the entropy in the target’s $x$- and $y$-position, respectively, indicate that the S-RH solution mainly reduces the uncertainty in the estimate of $y_{tg}$, while F-RH reduces that of $x_{tg}$. The entropy in $x_{tg}$ is even increasing along the S-RH path up to 8 seconds, but it can be seen in the bottom picture that the overall entropy is decreasing over time. Since the target drifts largely in $x$-direction, F-RH tries to reduce the uncertainty in $x_{tg}$ due to the process noise. However, S-RH takes a different option of reducing the uncertainty in $y_{tg}$ and increasing the correlation between the estimates of $x_{tg}$ and $y_{tg}$. The thinner lines in the bottom picture represent the summation of the entropy of $x_{tg}$ and $y_{tg}$; a large gap between this summation and the overall entropy means high correlation between two position estimates. By looking at the overall entropy at $\tau = 14s$ in the bottom picture of Fig. 5, it can be also seen that S-RH provides a slightly better performance than F-RH in this example. Since both S-RH and F-RH are suboptimal strategies and the (linear) model used for planning is updated using the actual (nonlinear) measurement in this example, consideration of the modeled future process noise in S-RH does not necessarily lead to significantly superior performance to F-RH. However, it is important to notice the characteristics of the two receding-horizon formulations.

Fig. 6 illustrates the on-the-fly information (both S-Form and F-Form) used for the planning. The glitches seen every second are due to the modification of the linear model with the actual measurement values. Comparing the filter-form and the smoother-form information for the F-RH solution – blue solid line and blue dash-dotted line, it can be seen that the smoother-form information gathers a large amount of the information in the final phase (after 12s), while the filter-form assumes that it gathers almost a constant amount of information every time step after 2s. This information from
the earlier time periods will experience the process noise for a longer time, and thus tends to be discounted in the smoother-form. Since in this example, the dynamics of $x_{tg}$ and $y_{tg}$ are decoupled from each other, this discounting effect stands out.

VII. Conclusions

This work discussed the on-the-fly information quantities for planning of information-gathering paths, in the aspects of the rate of information gathering and the quantification of the objective function for receding-horizon formulations. It was shown that the quantities based on smoothing identifies the pure impact of sensing on the uncertainty reduction of the variables of interest and provides the correct cost-to-go value in the receding-horizon decisions. Numerical studies with weather forecasting, sensor scheduling, and target localization examples validated the proposed receding-horizon formulation and indicated significant roles of smoothing in case with large process noise and/or with interest in a specific subset of the state variables.

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References