DNS-based Multi-Modal Decomposition of VIV

D. Lucor\textsuperscript{1}, Harish Mukundan\textsuperscript{2} & M. Triantafyllou\textsuperscript{2}

\textsuperscript{1} Laboratoire de Modélisation en Mécanique, UPMC Paris VI.

\textsuperscript{2} Department of Mechanical Engineering, MIT.

\* Supported by BP & ONR
Modal Reconstruction from CFD or Full-Scale/Experimental Data

Basic Idea:

1. Calculate/measure riser response at several locations \( (j=1, 2, \ldots J) \) - sufficient to resolve all modes - and for several time steps \( (k=1, 2, \ldots K) \).

2. Use the experimental data to accurately calculate all frequencies and modal amplitudes and phases along the riser in the presence of noise and systematic and random errors.
Formulation

\[ y(t_k, z_j) = \text{Re}\left\{ \sum_{n=1}^{N} e^{i\omega_n t_k} \phi_n(z_j) \right\} \]

The mode function \( \phi_n(z_j) \) is a complex number so we can rewrite the previous expression as

\[ y(t_k, z_j) = \text{Re}\left\{ \sum_{n=1}^{N} e^{i\omega_n t_k} (\phi_{n_{\text{Re}}}(z_j) + i \phi_{n_{\text{Im}}}(z_j)) \right\} \]

The unknowns here are \( \phi_{n_{\text{Re}}}(z_j) \) and \( \phi_{n_{\text{Im}}}(z_j) \) for each mode \( n \) and each point \( j \) in the domain

\[ y(t_k, z_j) = \text{Re}\left\{ \Delta_k^T \phi_j \right\} \]

\[ \phi_j = \begin{bmatrix} \phi_1(z_j) \\ \phi_2(z_j) \\ \vdots \\ \phi_N(z_j) \end{bmatrix} \quad \text{and} \quad \Delta_k = \begin{bmatrix} e^{\omega_1 t_k} \\ e^{\omega_2 t_k} \\ \vdots \\ e^{\omega_N t_k} \end{bmatrix} \]
Formulation (continued)

\[
y_k = \begin{bmatrix} y(t_k, z_1) \\ y(t_k, z_2) \\ \vdots \\ y(t_k, z_M) \end{bmatrix} = \text{Re} \begin{bmatrix} \Delta_k^T \phi_1 \\ \Delta_k^T \phi_2 \\ \vdots \\ \Delta_k^T \phi_M \end{bmatrix} = \text{Re} \{ \hat{\Delta}_k \hat{\phi} \}
\]

\[
\hat{\Delta}_k = \begin{bmatrix} \Delta_k^T & \Delta_k^T & \cdots & 0 \\ \Delta_k^T & \Delta_k^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_k^T \end{bmatrix}
\quad \text{and} \quad \hat{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_M \end{bmatrix}
\]

\[
Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_P \end{bmatrix} = \text{Re} \{ \begin{bmatrix} \hat{\Delta}_1 \hat{\phi} \\ \hat{\Delta}_2 \hat{\phi} \\ \vdots \\ \hat{\Delta}_P \hat{\phi} \end{bmatrix} \} = \text{Re} \{ \Delta \hat{\phi} \} = \text{Re} \{ \Delta \} \quad \text{Re} \{ \hat{\phi} \} - \text{Im} \{ \Delta \} \quad \text{Im} \{ \hat{\phi} \}
\]
Formulation (continued)

\[
\Delta = \begin{bmatrix}
\hat{\Delta}_0 \\
\hat{\Delta}_1 \\
\vdots \\
\hat{\Delta}_P
\end{bmatrix} = \begin{bmatrix}
\Delta_0^T & \cdots & 0 \\
\Delta_0^T & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & \cdots & \Delta_0^T \\
\Delta_1^T & \cdots & 0 \\
\Delta_1^T & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \Delta_1^T \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \Delta_P^T \\
\end{bmatrix}
\]

The matrix \( \Delta \) is of size \((P \times M)\) blocks, where \( P \) is the total number of time samples and \( M \) is the total number of points along the span. Each subblock is of size \((M \times N)\) where \( N \) is the total number of modes. The total size of the matrix is \((PM \times 2MN)\), the factor 2 coming from the complex numbers storage for real and imaginary parts. This system of equations is overdetermined if \( P > 2N \) and must be solved in a least squares sense.

Once, we have solved the system and obtained \( \phi_{nRe}(z_j) \) and \( \phi_{nIm}(z_j) \) for each \( n \) and each \( j \) in the domain, we can compute the modal amplitude \( |\phi_n(z_j)| = \sqrt{\phi_{nRe}^2(z_j) + \phi_{nIm}^2(z_j)} \) and the phase angle given by \( \theta_n(z_j) = \tan^{-1} \frac{\phi_{nIm}(z_j)}{\phi_{nRe}(z_j)} \).
Example 1: Traveling monochromatic wave of given $\omega$

$$y(t_k, z_j) = \text{Re}\left\{ \sum_{n=0}^{N} e^{i\omega n t_k} \phi_n(z_j) \right\}$$
\[ y(t_k, z_j) = \Re \left\{ \sum_{n=0}^{N} e^{i\omega_n t_k} \phi_n(z_j) \right\} \]

**Modal amplitude:**
\[ |\phi_n(z_j)| = \sqrt{\phi_{n Re}^2(z_j) + \phi_{n Im}^2(z_j)} \]

**Phase angle:**
\[ \theta_n(z_j) = \tan^{-1} \frac{\phi_{n Im}(z_j)}{\phi_{n Re}(z_j)} \]
Modal amplitude (first 19 odd modes)

\[ |\phi_n(z_j)| = \sqrt{\phi_{nRe}^2(z_j) + \phi_{nIm}^2(z_j)} \]

Unwrapped Phase angle (first 19 odd modes)

\[ \theta_n(z_j) = \tan^{-1} \frac{\phi_{nIm}(z_j)}{\phi_{nRe}(z_j)} \]
Formulation

\[
\begin{align*}
\dot{y}(t, z_j) &= \text{Re}\left\{ \sum_{n=1}^{N} i \omega_n e^{i \omega_n t} \phi_n(z_j) \right\} \\
\ddot{y}(t, z_j) &= \text{Re}\left\{ \sum_{n=1}^{N} -\omega_n^2 e^{i \omega_n t} \phi_n(z_j) \right\}
\end{align*}
\]

The total lift in phase with velocity \( C_{lv} \) is defined as:

\[
C_{lv}(z_j) = \frac{\frac{1}{T} \int_{0}^{T} \dot{y}(t, z_j) l(t, z_j) dt}{\sqrt{\frac{1}{T} \int_{0}^{T} \dot{y}^2(t, z_j) dt}}
\]

Let us consider the numerator first:

\[
\frac{1}{T} \int_{0}^{T} \dot{y}(t, z_j) l(t, z_j) dt = \frac{1}{T} \int_{0}^{T} \text{Re}\left\{ \sum_{n=1}^{N} i \omega_n e^{i \omega_n t} \phi_n(z_j) \right\} \text{Re}\left\{ \sum_{m=1}^{N} e^{i \omega_m t} \psi_m(z_j) \right\} dt
\]

Making use of the following equalities for the complex numbers \((z_1, z_2, z_3)\):

\[
\text{Re}\{z_1\} \text{Re}\{z_2\} = \frac{1}{4}(z_3 + \overline{z_3}) \quad \text{with} \quad z_3 = z_1z_2 + z_1 \overline{z_2}
\]

\[
\text{Re}\{z_1\} = \frac{1}{2}(z_1 + \overline{z_1})
\]

and dropping the \( e^{i(\omega_n + \omega_m)t} \) terms which become negligible for large enough \( T \), we are left

\[
\frac{1}{T} \int_{0}^{T} \dot{y}(t, z_j) l(t, z_j) dt = \frac{1}{4T} \int_{0}^{T} \left[ \sum_{n=1}^{N} \sum_{m=1}^{N} i \omega_n e^{i \omega_n t} \phi_n(z_j) \right] \sum_{m=1}^{N} e^{-i \omega_m t} \psi_m(z_j) dt
\]

\[
+ \sum_{n=1}^{N} i \omega_n e^{i \omega_n t} \phi_n(z_j) \sum_{m=1}^{N} e^{-i \omega_m t} \psi_m(z_j) \right] dt
\]

\[
= \frac{1}{2T} \int_{0}^{T} \text{Re}\left\{ \sum_{n=1}^{N} i \omega_n e^{i \omega_n t} \phi_n(z_j) \sum_{m=1}^{N} e^{-i \omega_m t} \psi_m(z_j) \right\} dt
\]

All terms vanish except when \( m = n \). In this case:

\[
\frac{1}{T} \int_{0}^{T} \dot{y}(t, z_j) l(t, z_j) dt = \frac{1}{2T} \int_{0}^{T} \text{Re}\left\{ \sum_{n=1}^{N} i \omega_n \phi_n(z_j) \overline{\psi_n(z_j)} \right\} dt
\]

\[
= \frac{1}{2} \sum_{n=1}^{N} \text{Re}\left\{ i \omega_n \phi_n(z_j) \psi_n(z_j) \right\}
\]

\[
= \frac{1}{2} \sum_{n=1}^{N} \omega_n |\phi_n(z_j)||\psi_n(z_j)| \text{Re}\left\{ i e^{i(\theta_n(z_j) - \delta_n(z_j))} \right\}
\]

\[
= \frac{1}{2} \sum_{n=1}^{N} \omega_n |\phi_n(z_j)||\psi_n(z_j)| \sin(\delta_n(z_j) - \theta_n(z_j))
\]

where \(|\psi_n(z_j)| = \sqrt{\frac{\psi_{nRe}^2(z_j) + \psi_{nIm}^2(z_j)}{\psi_{nRe}^2(z_j) + \psi_{nIm}^2(z_j)}} \) and \( \delta_n(z_j) = \tan^{-1} \frac{\psi_{nIm}(z_j)}{\psi_{nRe}(z_j)} \).
Formulation (continued)

Therefore, the total $C_{lv}$ becomes:

$$C_{lv}(z_j) = \frac{1}{\sqrt{2}} \sum_{n=1}^{N} \frac{\omega_n |\phi_n(z_j)||\psi_n(z_j)| \sin (\delta_n(z_j) - \theta_n(z_j))}{\sqrt{\sum_{n=1}^{N} \omega_n^2 |\phi_n(z_j)|^2}}.$$  \hfill (22)

We now define the component $C_{lvn}$ associated with the mode $n$ such that:

$$C_{lvn}(\omega_n, z_j) = \frac{\omega_n |\phi_n(z_j)||\psi_n(z_j)| \sin (\delta_n(z_j) - \theta_n(z_j))}{\omega_n |\phi_n(z_j)|} = |\psi_n(z_j)| \sin (\delta_n(z_j) - \theta_n(z_j))$$  \hfill (23)

The total lift in phase with acceleration $C_{la}$ is defined as follows:

$$C_{la}(z_j) = \frac{1}{T} \int_{0}^{T} \ddot{y}(t, z_j)l(t, z_j)dt \sqrt{\frac{1}{T} \int_{0}^{T} \ddot{y}(t, z_j)dt}$$  \hfill (24)

Following a very similar derivation as previously described, we express the total $C_{la}$ as:

$$C_{la}(z_j) = \frac{1}{\sqrt{2}} \sum_{n=1}^{N} -\omega_n^2 |\phi_n(z_j)||\psi_n(z_j)| \cos (\delta_n(z_j) - \theta_n(z_j)) \sqrt{\sum_{n=1}^{N} \omega_n^4 |\phi_n(z_j)|^2}.$$  \hfill (25)

We now define the component $C_{lan}$ associated with the mode $n$ such that:

$$C_{lan}(\omega_n, z_j) = \frac{-\omega_n^2 |\phi_n(z_j)||\psi_n(z_j)| \cos (\delta_n(z_j) - \theta_n(z_j))}{\omega_n^2 |\phi_n(z_j)|} = -|\psi_n(z_j)| \cos (\delta_n(z_j) - \theta_n(z_j))$$  \hfill (26)
Fluid-Structure Model used in NEKTAR

Flow: 3D Incompressible Navier Stokes
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla p + \frac{1}{\text{Re}} \nabla^2 u \]
\[ \nabla \cdot u = 0 \]

Structure: Linear model for beam and/or cable
\[ \frac{\partial^2 \xi}{\partial t^2} - c^2 \frac{\partial^2 \xi}{\partial z^2} + \gamma^2 \frac{\partial^4 \xi}{\partial z^4} + \frac{4\pi \xi}{U_r} \frac{\partial \xi}{\partial t} + \left( \frac{2\pi}{U_r} \right)^2 \xi = \frac{1}{2m} C_f(z,t) \]
\[ \xi(z,t) = (\xi(z,t), \eta(z,t)) \]
\[ c = \sqrt{\frac{T}{\rho_s U^2}} \sqrt{\frac{EI}{\rho_s U^2 d^2}} \quad \text{and} \quad U_r = U/fd \]
Flow and structural parameters used in the runs

<table>
<thead>
<tr>
<th>Flow and Structural Parameters</th>
<th>Re=1000</th>
<th>Re=1000</th>
<th>Re=1000</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FLOW</strong></td>
<td>uniform cases: Case1-3</td>
<td>linear shear: Case1-3</td>
<td>exponential shear: Case1-3</td>
</tr>
<tr>
<td>Flow Regime</td>
<td>Turbulent Regime</td>
<td>Turbulent Regime</td>
<td>Turbulent Regime</td>
</tr>
<tr>
<td>Inflow profile</td>
<td>Flat</td>
<td>Linear</td>
<td>Exponential with long tails</td>
</tr>
<tr>
<td><strong>STRUCTURE</strong></td>
<td>Beam/Cable</td>
<td>Beam/Cable</td>
<td>Beam/Cable</td>
</tr>
<tr>
<td>Boundary conditions</td>
<td>Pinned and hinged ends</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constraints</td>
<td>No streamwise motion (x-direction)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Free Transverse motion (y-direction)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aspect Ratio: L/d</td>
<td>L/d = 2028</td>
<td>L/d = 2028</td>
<td>L/d = 2028</td>
</tr>
<tr>
<td>Mass Ratio: m</td>
<td>m = 2</td>
<td>m = 2</td>
<td>m = 2</td>
</tr>
<tr>
<td>Cable Phase Velocity: c</td>
<td>c = 35</td>
<td>c = 35</td>
<td>c = 35</td>
</tr>
<tr>
<td>Beam Phase Velocity: γ</td>
<td>γ = 450</td>
<td>γ = 450</td>
<td>γ = 450</td>
</tr>
<tr>
<td>Damping: ζ</td>
<td>ζ = 0.0%</td>
<td>ζ = 0.0%</td>
<td>ζ = 0.0%</td>
</tr>
</tbody>
</table>

$U_{min}/U_{max} = 1$  $U_{min}/U_{max} = 30\%$  $U_{min}/U_{max} = 10\%$
Uniform velocity profile:

Chosen set of frequencies
\[ \omega_n = 2\pi(0.2148, 0.2295, 0.249, 0.293, 0.317, 0.332) \]

Modal amplitude crossflow displacement

Modal amplitude lift coefficient

Distribution of total Cl_v
- Full basis (256 modes)
- Incomplete basis (≤10 modes)
Linear shear velocity profile:

Chosen set of frequencies
\[ \omega_n = 2\pi(0.183 \ 0.193 \ 0.203 \ 0.212 \ 0.222 \ 0.230 \ 0.238) \]

Modal amplitude crossflow displacement

Modal amplitude lift coefficient

Distribution of total Cl\(_V\)
- Full basis (256 modes)
- Incomplete basis (≤10 modes)
Exponential shear velocity profile:

Chosen set of frequencies
\( \omega_n = 2\pi (0.146 \ 0.167 \ 0.179 \ 0.187 \ 0.199 \ 0.204 \ 0.21 \ 0.217 \ 0.225 \ 0.236) \)

Modal amplitude crossflow displacement

Modal amplitude lift coefficient

Distribution of total Cl\(_v\)
- Full basis (256 modes)
- Incomplete basis (\(\leq 10\) modes)
RMS values of crossflow response

- Full basis (256 modes)
- Incomplete basis (≤10 modes)
Clₐ modal decomposition

Total Clₐ

No Modes overlap!

Total Clₐ

Modes overlap!

Velocity profile

Span

Span

Velocity profile

Frequency

Span

Mode number n
Summary

We have used 3D DNS data of the crossflow response of long flexible cylinders subject to VIV with various inflow velocity conditions. These test cases were chosen for their complex multi-frequency response that is difficult to predict \textit{a priori} and can not be represented by standing waves.

We have proposed a modal decomposition technique that provides the spanwise distributions of the amplitude and the phase of the response. This representation allows the energy to be carried along the length in the structure in the form of traveling waves.

We have tempted to show that a sparse decomposition with a few modes but carrying most of the energy is possible.

Our ultimate goal is to obtain \textit{Gopalkrishnan-like} maps of lift and added mass coefficients versus modal amplitude and modal frequency.